Index theory on singular manifolds I

Index theory on manifolds with corners: "Generalized Gauss-Bonnet formulas"

Paul Loya

Outline of talk: Four main points

I. The Gauss-Bonnet formula

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I. The Gauss-Bonnet formula

II. Index version of Gauss-Bonnet

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III. The Atiyah-(Patodi-)Singer index formula

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I. The Gauss-Bonnet formula

II. Index version of Gauss-Bonnet

III. The Atiyah-(Patodi-)Singer index formula

IV. Index formulas on mwcs

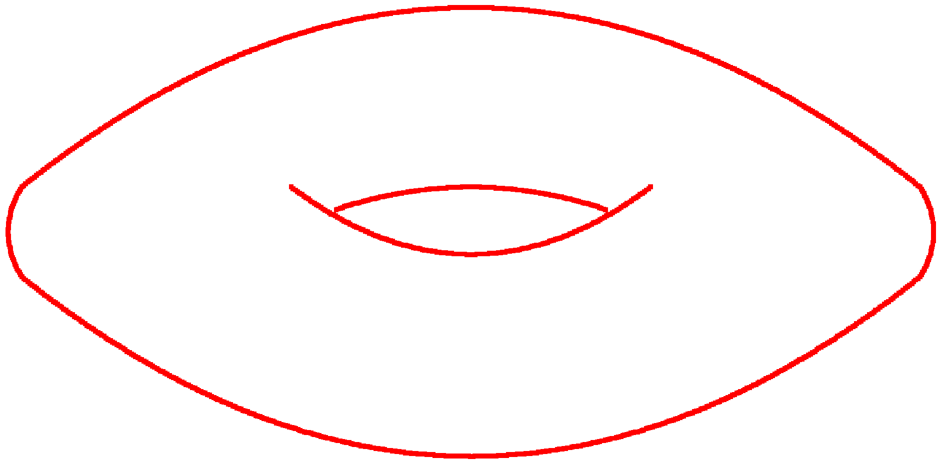
Preview of Part I

- The Gauss-Bonnet formula for a mwc ($=$ manifold with corners) involves *topology*, *geometry*, and *linear algebra*.

- The interior, smooth boundary components, and the corners all contribute to the G-B formula.

I. The Gauss-Bonnet formula

- Euler Characteristic.



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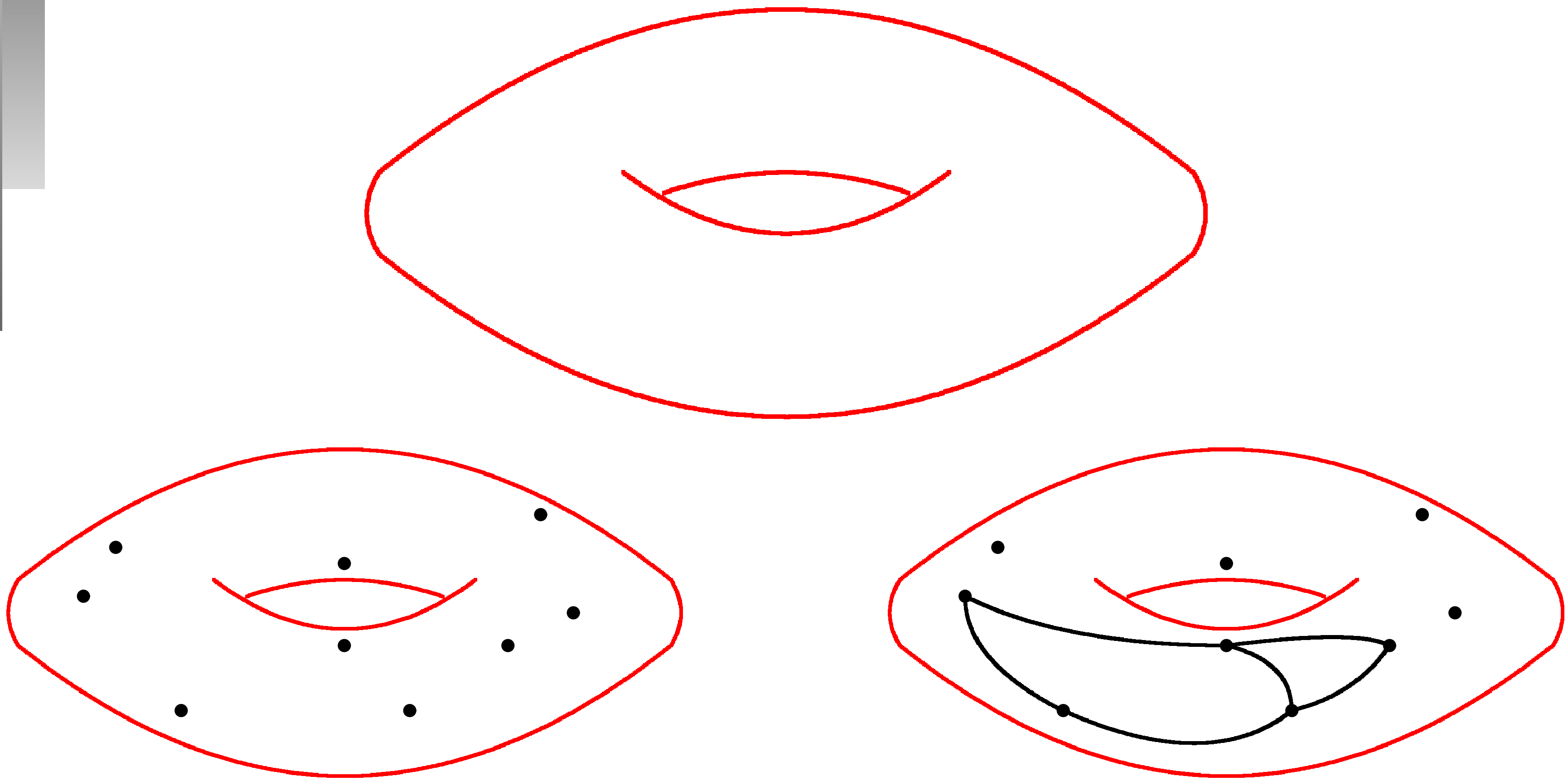

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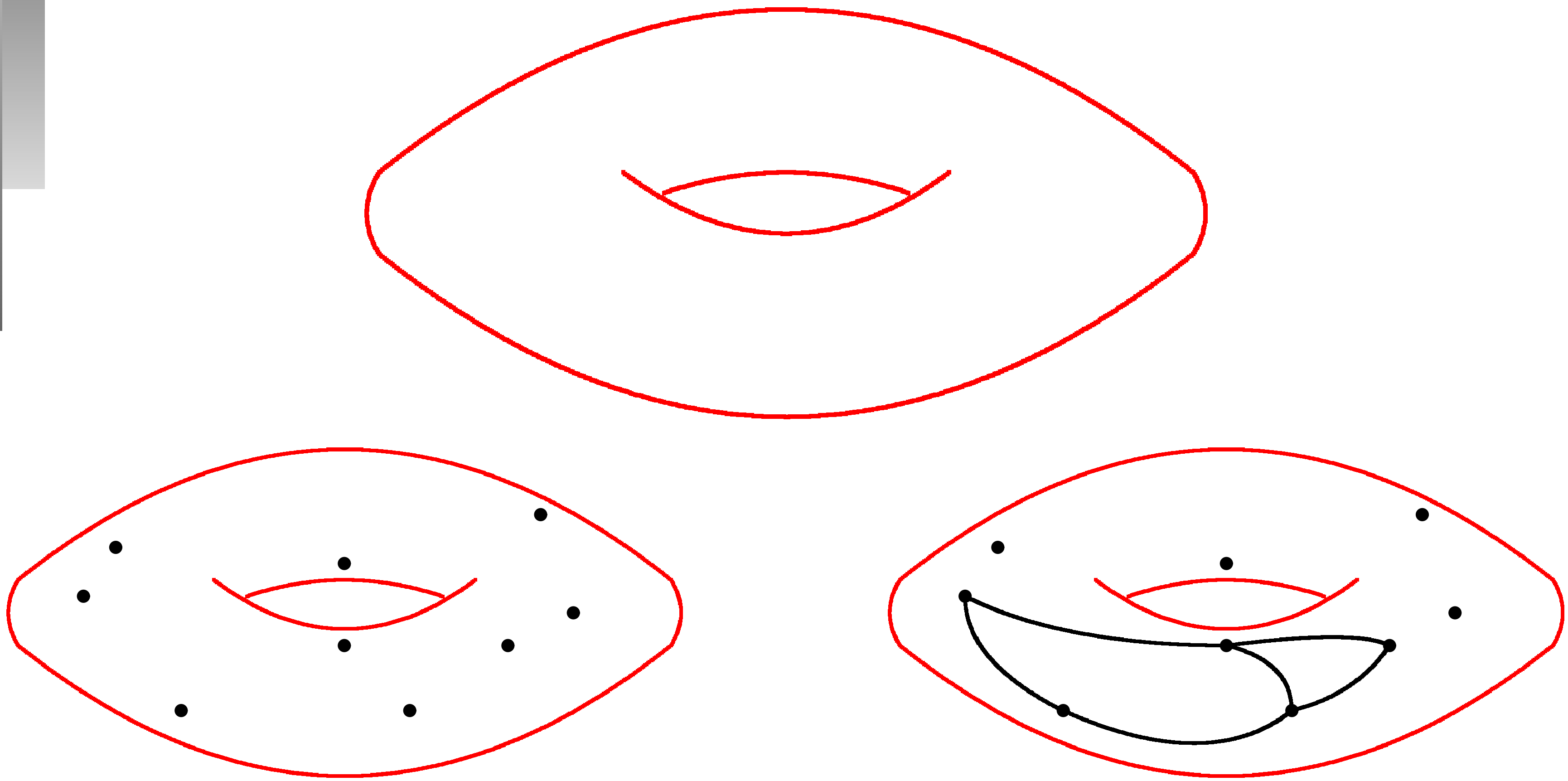
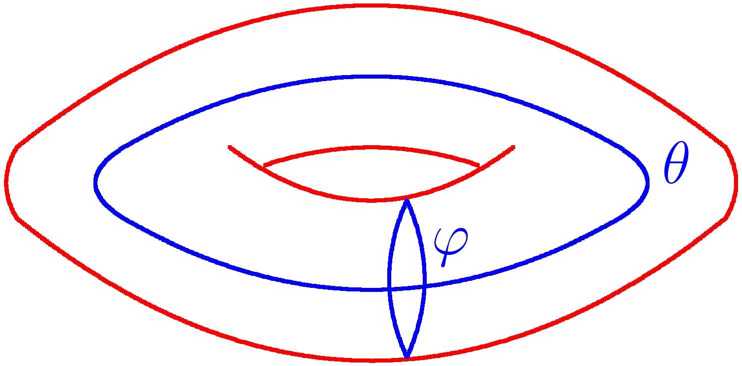


$$\chi(\mathbb{T}) = v - e + f = 0.$$

No matter how many dots you mark and how you connect them, you get 0.

- Curvature$=$ deviation of the metric from being a Euclidean metric.

Let's make the torus flat:



$$\mathbb{T} = \mathbb{S}^1_\theta \times \mathbb{S}^1_\varphi$$
$$g = d\theta^2 + d\varphi^2$$

$$\therefore K = 0$$

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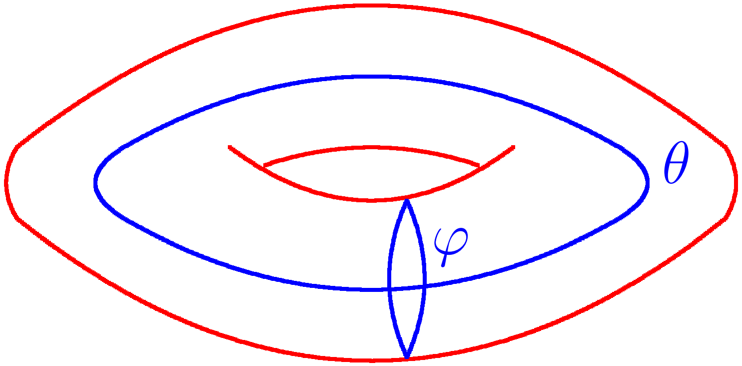

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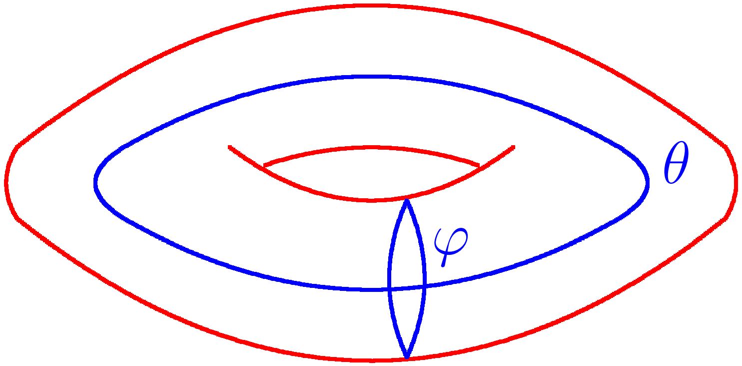

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$$0 = 0 \quad \Longrightarrow \quad \boxed{\chi(\mathbb{T}) = \int_{\mathbb{T}} K.}$$

I. The Gauss-Bonnet formula

- **Gauss-Bonnet theorem**


(G-B I) Given an oriented, compact, 2-dim., Riemannian manifold M without boundary, we have

$$\chi(M) \quad = \quad \frac{1}{2\pi}\int_M K$$

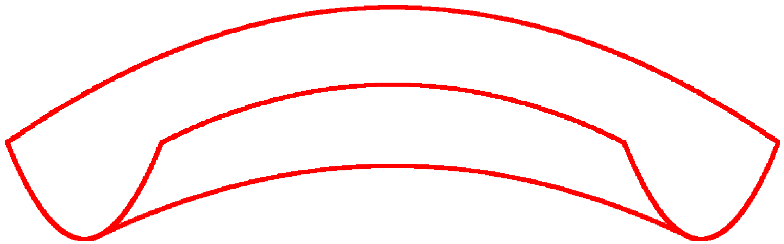
$$\uparrow \qquad\qquad\qquad \uparrow$$

topological   geometrical

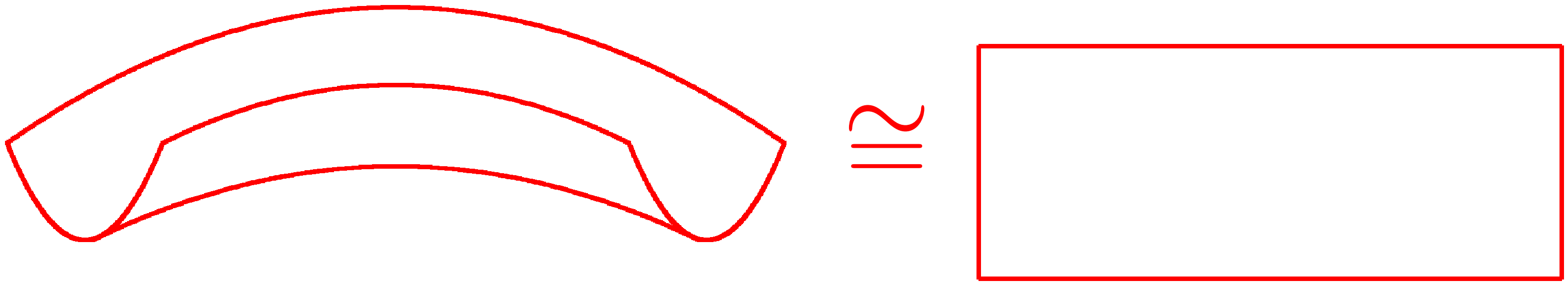
What happens when M has corners?

I. The Gauss-Bonnet formula

Chop up the torus:



Chop up the torus:



$$\therefore \quad \chi(M) = 1.$$

$$1 \neq 0 \quad \Longrightarrow \quad \chi(M) \neq \frac{1}{2\pi}\int_M K.$$

I. The Gauss-Bonnet formula

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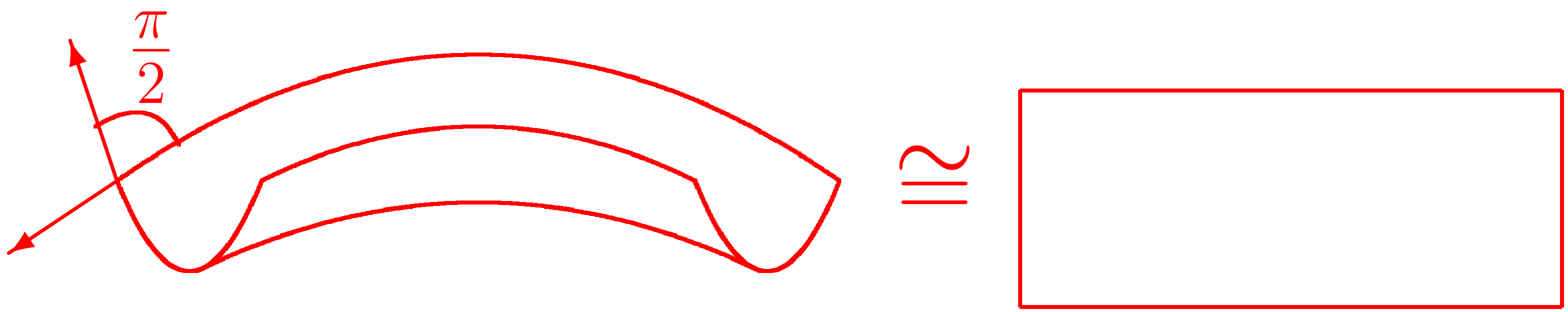
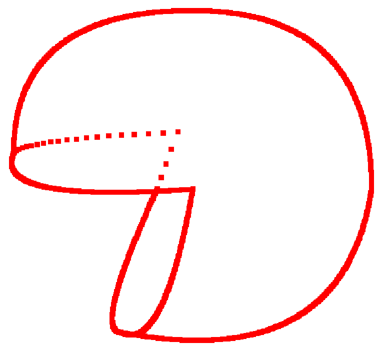


$$\therefore \quad \chi(M) = 1.$$

$$1 \neq 0 \quad \Longrightarrow \quad \chi(M) \neq \frac{1}{2\pi}\int_M K.$$

However,

$$\chi(M) = \frac{1}{2\pi}\int_M K + \frac{1}{2\pi}\Big(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}\Big).$$



(G-B II) Given an oriented, compact, 2-dim., Riemannian manifold M with corners, we have

$$\chi(M) = \frac{1}{2\pi}\int_M K + \frac{1}{2\pi}(\text{total geodesic curvature of } \partial X)$$

$$+\frac{1}{2\pi}(\text{sum of exterior angles})$$

The G-B formula bridges three areas of math: topology, diff. geometry, and linear algebra.

Summary of Part I

- **Gauss-Bonnet for smooth case:** Given an oriented, compact, 2-dim., Riemannian manifold M without boundary,

$$\underset{\text{topological}}{\underset{\uparrow}{\chi(M)}} \quad = \quad \underset{\text{geometrical}}{\underset{\uparrow}{\frac{1}{2\pi}\int_M K}}$$

When corners are present, we have $\ldots$

Summary of Part I

- **Gauss-Bonnet for singular case:** Given an oriented, compact, 2-dimensional, Riemannian manifold M with corners,

$$\chi(M) \quad = \quad \frac{1}{2\pi}\int_M K \quad + \quad \frac{1}{2\pi}(\text{curv of } \partial M)$$

$$\uparrow \qquad\qquad\qquad \uparrow \qquad\qquad\qquad \uparrow$$

topological   geometrical   boundary correction

$$+\frac{1}{2\pi}(\text{sum of exterior angles})$$

$$\uparrow$$

linear algebra, correction from corners

Upshot: Smooth boundary components and corners give new contributions to the G-B formula.

II. Index version of Gauss-Bonnet

Preview of Part II

- We shall see how to interpret the Gauss-Bonnet formula as an index formula.

II. Index version of Gauss-Bonnet

- Top/Geo Data: Let M be a compact, oriented, 2-dimensional Riemannian manifold without boundary.

II. Index version of Gauss-Bonnet

- Top/Geo Data: Let M be a compact, oriented, 2-dimensional Riemannian manifold without boundary.

- Function spaces: $C^\infty(M, \Lambda^k)$.

Let (x, y) be local coordinates on M.

$$C^\infty(M) = C^\infty(M, \Lambda^0) = \text{0-forms}$$

$$C^\infty(M, \Lambda^1) = \text{1-forms} \quad f\,dx + g\,dy$$

$$C^\infty(M, \Lambda^2) = \text{2-forms} \quad f\,dx \wedge dy$$

(There are no 3-forms on M.)

II. Index version of Gauss-Bonnet

- Operators.

The **exterior derivative**

$$d : C^\infty(M, \Lambda^k) \to C^\infty(M, \Lambda^{k+1}).$$

0-forms: $df = \partial_x f\, dx + \partial_y f\, dy$ “gradient”

1-forms: $d(f\, dx + g\, dy) = (\partial_x g - \partial_y f)\, dx \wedge dy$ “curl”

2-forms: $d = 0.$

II. Index version of Gauss-Bonnet

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2-forms: $d = 0.$

d has an adjoint:

$$d^* : C^\infty(M, \Lambda^{k+1}) \to C^\infty(M, \Lambda^k).$$

II. Index version of Gauss-Bonnet

- Operators.

The **Gauss-Bonnet operator** is

$$D_{GB} = d + d^* : C^\infty(M, \Lambda^{ev}) \to C^\infty(M, \Lambda^{odd}).$$

Facts:

1) D_{GB} is elliptic.
2) $\sigma(D_{GB}^* D_{GB})(\xi) = |\xi|^2$ ($=$ the Riemannian metric).

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Facts:

1) D_{GB} is elliptic.
2) $\sigma(D_{GB}^* D_{GB})(\xi) = |\xi|^2$ ($=$ the Riemannian metric).

Note:

$$D_{GB}^* D_{GB} = (d^* + d)(d + d^*) = (d + d^*)^2 =: \Delta = \text{Laplacian}.$$

Therefore D_{GB} is a “square root” of the Laplacian.
D_{GB} is called a “Dirac operator”.

II. Index version of Gauss-Bonnet

- Two theorems:

Theorem 1: $D_{GB} : C^\infty(M, \Lambda^{ev}) \to C^\infty(M, \Lambda^{odd})$ is **Fredholm**:

1) $\dim \ker D_{BG} < \infty$.
2) $\dim \big(C^\infty(M, \Lambda^{odd}) / \operatorname{Im} D_{GB}\big) < \infty$.

$$\therefore \qquad \operatorname{ind} D_{GB} := \dim \ker D_{GB} - \dim \operatorname{coker} D_{GB} \in \mathbb{Z}.$$

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$$\mathrm{ind}\, D_{GB} = \chi(M)$$

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Theorem 2:

$$\mathrm{ind}\, D_{GB} = \chi(M)$$

$$= \frac{1}{2\pi}\int_M K.$$

II. Index version of Gauss-Bonnet

Conclusion:

$$\boxed{\operatorname{ind} D_{GB} = \frac{1}{2\pi}\int_M K}$$

The index version of Gauss-Bonnet.

Summary of Part II

Gauss-Bonnet: index version If M is an oriented, compact, 2-dim., Riem. manifold without boundary, then

$$D_{GB} : C^\infty(M, \Lambda^{ev}) \to C^\infty(M, \Lambda^{odd})$$

is Fredholm, and

$$\begin{array}{ccc} \operatorname{ind} D_{\mathrm{GB}} & = & \frac{1}{2\pi} \int_M K \\ \uparrow & & \uparrow \\ \text{analytical} & & \text{geometrical} \end{array}$$

The index formula interpretation of Gauss-Bonnet.

How does the A(P)S index formula generalize the "index" Gauss-Bonnet formula?

Preview of Part III

- The Atiyah-Singer index formula is a higher dimensional version of the index Gauss-Bonnet formula.
- The Atiyah-Patodi-Singer index formula extends the Atiyah-Singer formula to manifolds with boundary.
- As expected from Part I on the Gauss-Bonnet formula, the A-S and A-P-S formulas differ by a boundary term.

- Top/Geo Data: Let E, F be Hermitian vector bundles over an even-dimensional, compact, oriented, Riemannian manifold M without boundary.
- Functional Ana. Data: Let

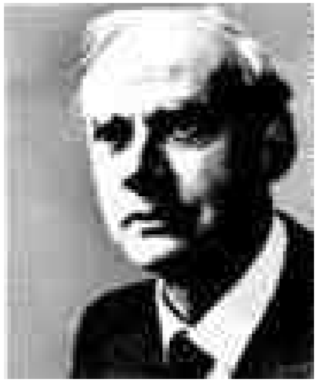
$$D : C^\infty(M, E) \to C^\infty(M, F)$$

be a **Dirac-type operator:**

1) D is elliptic.
2) $\sigma(D^*D)(\xi) = |\xi|^2$ ($=$ the Riemannian metric).

Thus, $D^*D = \Delta$ (at the principal symbol level). Roughly speaking, D is a “square root” of the Laplacian.

Why “Dirac” operator?



Paul Dirac (1902–1984), recipient of 1933 Nobel prize.

In developing quantum theory in the 1920’s he factorized the Laplacian as a square of a first order operator.

Abuse of terminology: We’ll say “Dirac operator” instead of “Dirac-type” operator.

III. The A(P)S theorem

- Examples of $D : C^\infty(M, E) \to C^\infty(M, F)$

Ex 1: Let $E = \Lambda^{ev}$, $F = \Lambda^{odd}$, and $D_{GB} = d + d^*$.

III. The A(P)S theorem

- Examples of $D : C^\infty(M,E) \to C^\infty(M,F)$

Ex 1: Let $E = \Lambda^{ev}$, $F = \Lambda^{odd}$, and $D_{GB} = d + d^*$.

Ex 2: $M = \mathbb{R}^2$, $E = F = \mathbb{C}$, and

$$D_{CR} = \partial_x + i\partial_y : C^\infty(\mathbb{R}^2, \mathbb{C}) \to C^\infty(\mathbb{R}^2, \mathbb{C}).$$

III. The A(P)S theorem

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$$D_{CR} = \partial_x + i\partial_y : C^\infty(\mathbb{R}^2, \mathbb{C}) \to C^\infty(\mathbb{R}^2, \mathbb{C}).$$

Observe:

$$\begin{aligned} D_{CR}^* D_{CR} = (-\partial_x + i\partial_y)(\partial_x + i\partial_y) &= -\partial_x^2 - \partial_y^2 \\ &= -(\partial_x^2 + \partial_y^2) \\ &= \Delta_{\mathbb{R}^2}. \end{aligned}$$

General Dirac operators share many of the same properties of D_{CR}.

III. The A(P)S theorem

FLASHBACK:

Gauss-Bonnet: index version If M is an oriented, compact, 2-dim., Riem. manifold without boundary, then

$$D_{GB} : C^\infty(M, \Lambda^{ev}) \to C^\infty(M, \Lambda^{odd})$$

is Fredholm, and

$$\begin{array}{ccc} \operatorname{ind} D_{\mathrm{GB}} & = & \frac{1}{2\pi}\int_M K \\ \uparrow & & \uparrow \\ \text{analytical} & & \text{geometrical} \end{array}$$

The Index formula interpretation of Gauss-Bonnet.

III. The A(P)S theorem

Michael Atiyah (1929–) and Isadore Singer (1924–)

 

Atiyah-Singer index theorem (1963):

$D : C^\infty(M, E) \to C^\infty(M, F)$ is Fredholm, and

$$\operatorname{ind} D = \int_M K_{AS}$$

$$\uparrow \qquad\qquad \uparrow$$

analytical geometrical

where $K_{AS} = \widehat{A}(M)\mathrm{Ch}((E \oplus F)/\mathrm{Sp})$, an *explicitly* defined polynomial in the curvatures of M, E, and F.

III. The A(P)S theorem

Question: How does the AS formula

$$\operatorname{ind} D = \int_M K_{AS}$$

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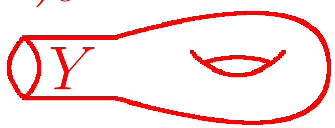
$$M \cong [0,1)_s \times Y$$



M

- Functional Ana. Data: Let

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be a Dirac operator.

III. The A(P)S theorem

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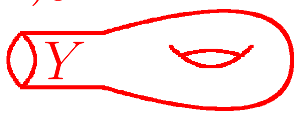

M

- Assumptions on $D : C^\infty(M,E) \to C^\infty(M,F)$:

Over the collar $[0,1)_s \times Y$, assume

$$g = ds^2 + h$$

$$E \cong E|_{s=0} \ , \ F \cong F|_{s=0}$$

$$D = \Gamma(\partial_s + D_Y),$$

where

$$\Gamma : E|_{s=0} \to F|_{s=0} \quad , \quad \Gamma^*\Gamma = \mathrm{Id},$$

and $D_Y : C^\infty(Y, E|_{s=0}) \to C^\infty(Y, E|_{s=0})$

is a self-adjoint Dirac operator on Y.

III. The A(P)S theorem

Question: Is $D : C^\infty(M,E) \to C^\infty(M,F)$ Fredholm, and if so, what is $\mathrm{ind}\, D =$?

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Ex: Consider $M = [0, 1]_s \times \mathbb{S}^1_y$ and $D_{CR} = \partial_s + i\partial_y$.

$\ker D_{CR} =$ hol. functions on $[0, 1] \times \mathbb{R}$, of period 2π in y.

(Note: $e^{kz} = e^{kx}e^{iky} \in \ker D_{CR}$ for all $k \in \mathbb{Z}$.

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At least two ways to fix this problem.

1) Put boundary conditions.

2) Attach an infinite cylinder. $\leftarrowtail \leftarrowtail \leftarrowtail$

Ex con't: $M = [0,1]_s \times \mathbb{S}^1_y$ and $D_{CR} = \partial_s + i\partial_y$.

$\widehat{M} = (-\infty, \infty)_s$

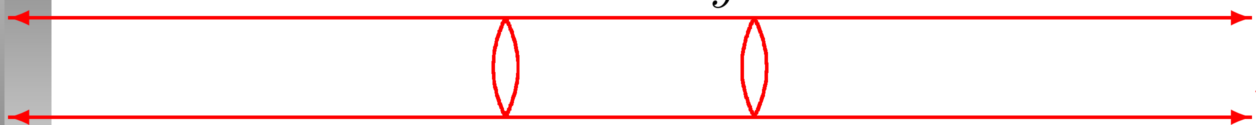
Let $\widehat{C}^\infty(\widehat{M}, \mathbb{C})$ = smooth functions on $(-\infty, \infty) \times \mathbb{S}^1$ that $\to 0$ exp. as $|s| \to \infty$.

Consider

$$\widehat{D}_{CR} = \partial_s + i\partial_y : \widehat{C}^\infty(\widehat{M}, \mathbb{C}) \to \widehat{C}^\infty(\widehat{M}, \mathbb{C}).$$

III. The A(P)S theorem

Ex con't: $M = [0,1]_s \times \mathbb{S}^1_y$ and $D_{CR} = \partial_s + i\partial_y$.



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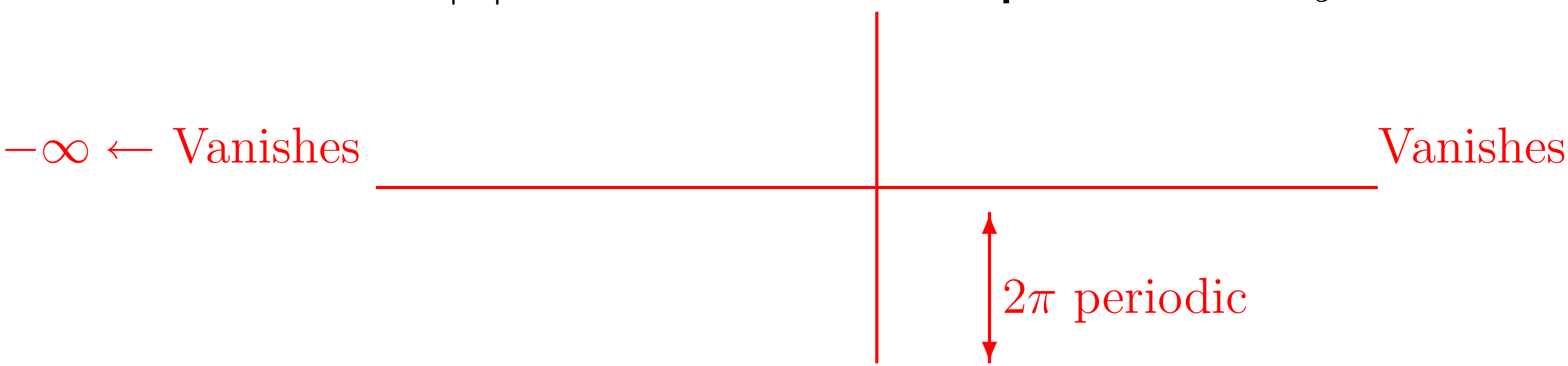
$\therefore \ker \widehat{D}_{CR} =$ holomorphic functions on $\mathbb{C}$ that $\to 0$ exp. as $|s| \to \infty$ and are 2π-periodic in y.



III. The A(P)S theorem

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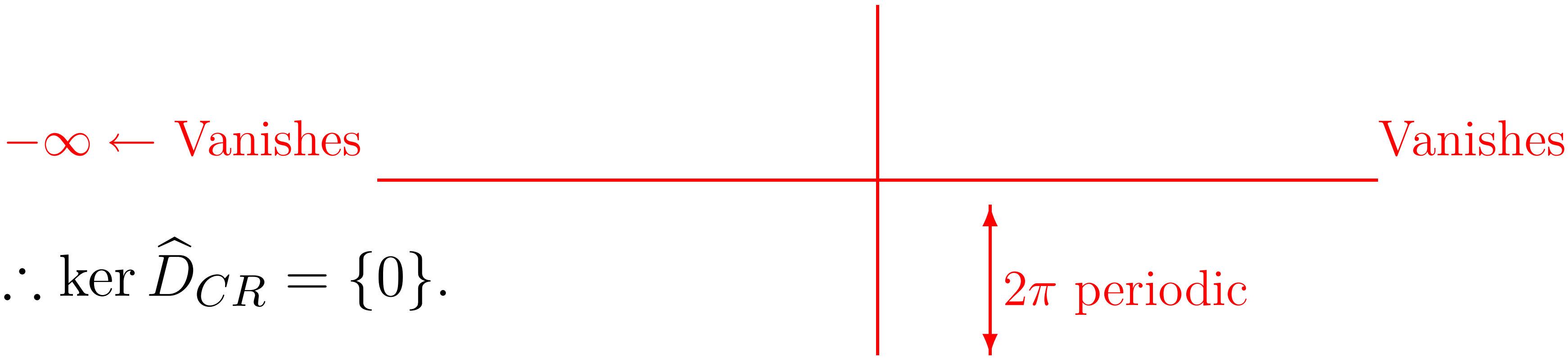

$\therefore \ker \widehat{D}_{CR} = \{0\}$.

III. The A(P)S theorem

Recall: E, F are Hermitian vector bundles over an even-dimensional, compact, oriented, Riemannian manifold M with smooth boundary and

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

is a Dirac operator. Over the collar $[0,1)_s \times Y$,

$$g = ds^2 + h$$

$$E \cong E|_{s=0}\ ,\ \ F \cong F|_{s=0}$$

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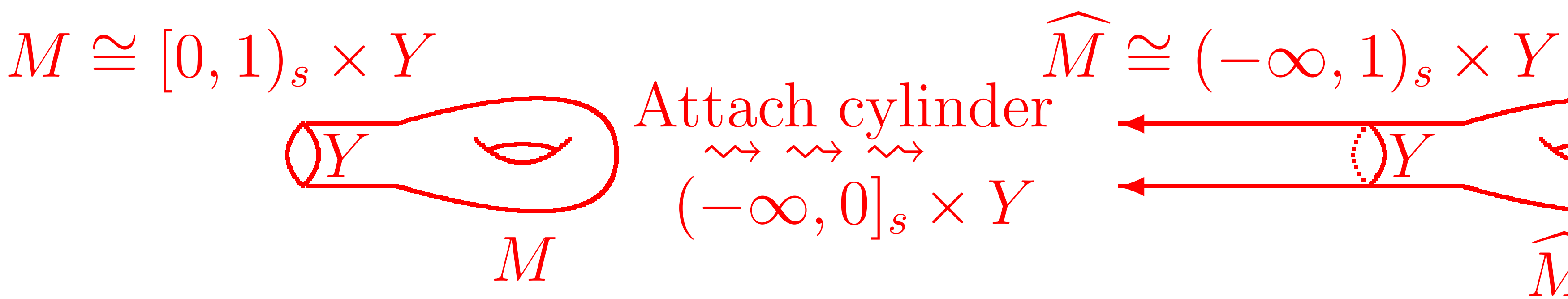
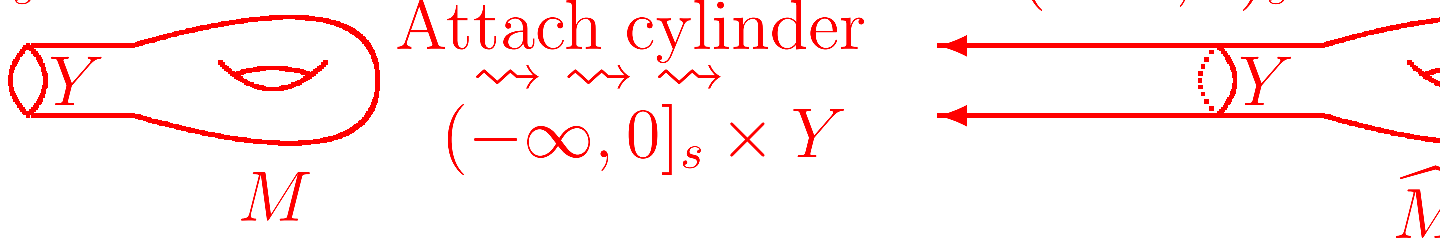
Attach infinite cylinder:



III. The A(P)S theorem

$M \cong [0,1)_s \times Y$ $\qquad$ $\widehat{M} \cong (-\infty,1)_s \times Y$



$g = ds^2 + h$ extends to a metric on $\widehat{M}$

E extends to a v.b. $\widehat{E}$ on $\widehat{M}$

F extends to a v.b. $\widehat{F}$ on $\widehat{M}$

D extends to an operator $\widehat{D}$ on $\widehat{M}$

Let

$\widehat{C}^\infty(\widehat{M},\widehat{E}) =$ smooth sections that $\to 0$ exp. as $s \to -\infty$.

Question: Is $\widehat{D} : \widehat{C}^\infty(\widehat{M},\widehat{E}) \to \widehat{C}^\infty(\widehat{M},\widehat{F})$ Fredholm?

III. The A(P)S theorem

Atiyah (1929–), Patodi (1945–1976), and Singer (1924–)

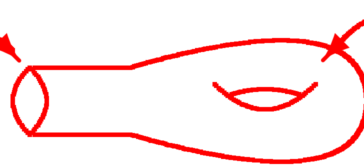
  

Answer: **Atiyah-Patodi-Singer index theorem (1975):**
$\widehat{D} : \widehat{C}^\infty(\widehat{M}, \widehat{E}) \to \widehat{C}^\infty(\widehat{M}, \widehat{F})$ is Fredholm, and

$$\boxed{\operatorname{ind} \widehat{D} = \int_M K_{AS} - \frac{1}{2}\Big(\eta(D_Y) + \dim\ker D_Y\Big).}$$

where K_{AS} is the Atiyah-Singer polynomial and $\eta(D_Y)$ is the eta invariant of D_Y.

$\eta(D_Y) + \dim\ker D_Y$
"boundary term"



$\int_M K_{AS}$
"interior term"

III. The A(P)S theorem

$$\boxed{\operatorname{ind}\widehat{D} = \int_M K_{AS} - \frac{1}{2}\Big(\eta(D_Y) + \dim\ker D_Y\Big).}$$

Notes:

1) Without a boundary, we have $\operatorname{ind} D = \int_M K_{AS}$. Thus, adding a boundary adds a boundary term.

$$\boxed{\operatorname{ind}\widehat{D} = \int_M K_{AS} - \frac{1}{2}\Big(\eta(D_Y) + \dim\ker D_Y\Big).}$$

Notes:

1) Without a boundary, we have $\operatorname{ind} D = \int_M K_{AS}$. Thus, adding a boundary adds a boundary term.

2) $\eta(D_Y)$ is a "*spectral invariant*": Recall that if $\{\lambda_j\}$ are the eigenvalues of D_Y,

$$\eta(D_Y) \text{ “}{=}\text{” } \sum_{\lambda_j \neq 0} \mathrm{sign}(\lambda_j)$$

$$= \sum_{\lambda_j > 0} 1 - \sum_{\lambda_j < 0} 1$$

$$= \#\text{ of pos. e.v.} \quad - \quad \#\text{ of neg. e.v.}$$

Notes:

3) Another expression for $\eta(D_Y)$ is

$$\eta(D_Y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \mathrm{Tr}\left(D_Y e^{-tD_Y^2} \right) dt.$$

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"Proof:" We have

$$\begin{aligned}
\frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \mathrm{Tr}\left(D_Y e^{-tD_Y^2} \right) dt &= \sum_j \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \lambda_j e^{-t\lambda_j^2}\, dt \\
&= \sum_{\lambda_j \neq 0} \frac{1}{\sqrt{\pi}} \frac{|\lambda_j|}{\lambda_j} \int_0^\infty t^{-1/2} e^{-t}\, dt \\
&= \sum_{\lambda_j \neq 0} \mathrm{sign}(\lambda_j) = \eta(D_Y).
\end{aligned}$$

Summary of Part III

- Dirac operators on compact manifolds without boundary are *always* Fredholm. The Atiyah-Singer theorem computes the index in terms of geo./top. data.

- On compact manifolds with boundary, Dirac operators are *never* Fredholm. However, the operator $\widehat{D}$ on the noncompact manifold $\widehat{M}$ is always Fredholm.

- $$\boxed{\begin{array}{ccccc} \operatorname{ind}\widehat{D} & = & \int_M K_{AS} & -\frac{1}{2}\Big(\eta(D_Y) & +\dim\ker D_Y\Big) \\ \uparrow & & \uparrow & \uparrow & \\ \text{analytical} & & \text{geometrical} & \text{spectral} & \end{array}}$$

Preview of Part IV

- **Question:** How does the APS formula
$$\operatorname{ind} \widehat{D} = \int_M K_{AS} - \frac{1}{2}\Big(\eta(D_Y) + \dim\ker D_Y\Big).$$
change when M has corners?

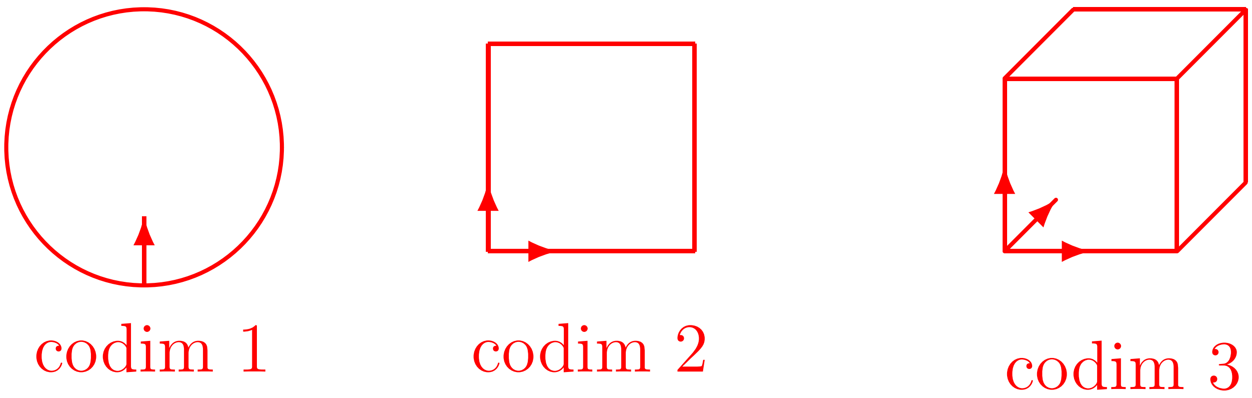
- **Answer:** We pick up additional terms from the corners.

- Like the Gauss-Bonnet formula, these extra terms are “exterior angles”.

IV. Index formulas on mwcs

- Top/Geo Data: Let E, F be Hermitian vector bundles over an even-dimensional, compact, oriented, Riemannian manifold M with a corner of codim. 2.

Ex:



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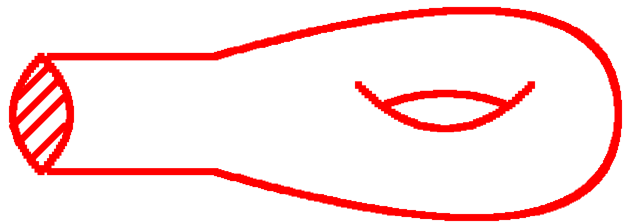


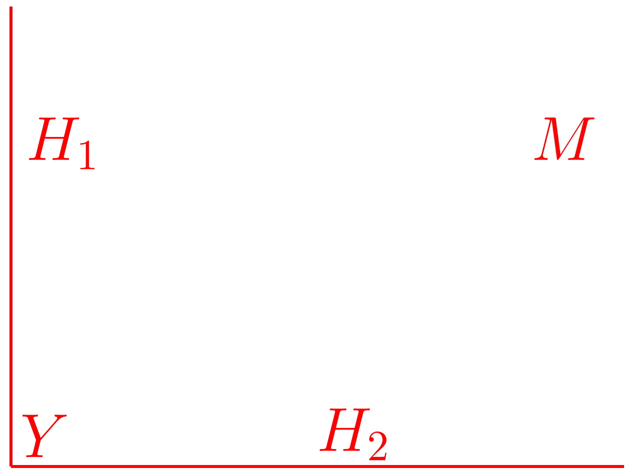
- Functional Ana. Data: Let

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

be a Dirac operator.

IV. Index formulas on mwcs

- Assume $D : C^\infty(M, E) \to C^\infty(M, F)$ is of product-type near ∂M.



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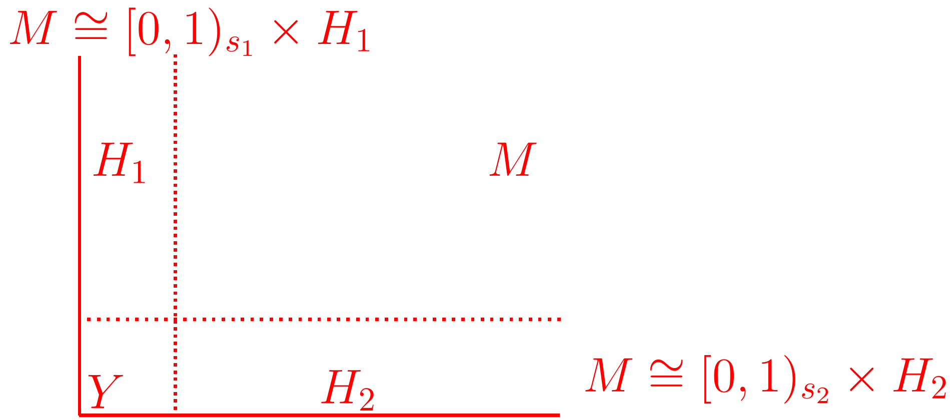

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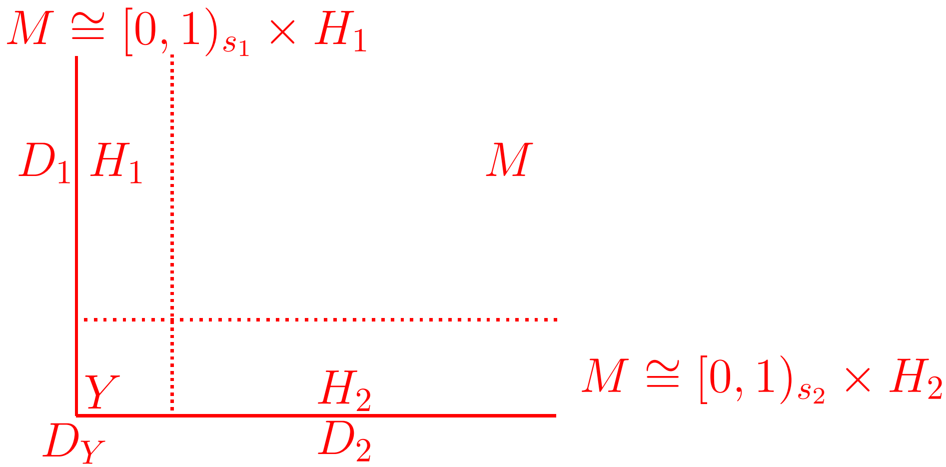

E.g. over the collar $[0,1)_{s_1} \times H_1$, assume

$$g = ds_1^2 + h_1$$

$$E \cong E|_{s_1=0}\ ,\ \ F \cong F|_{s_1=0}$$

$$D = \Gamma_1(\partial_{s_1} + D_1),$$

where D_1 is a (formally) s.a. Dirac operator on H_1 and $\Gamma_1 : E|_{s_1=0} \to F|_{s_1=0}$ satisfies $\Gamma_1^*\Gamma_1 = \mathrm{Id}$.

IV. Index formulas on mwcs

Question: Is $D : C^\infty(M, E) \to C^\infty(M, F)$ Fredholm, and if so, what is $\operatorname{ind} D =$?

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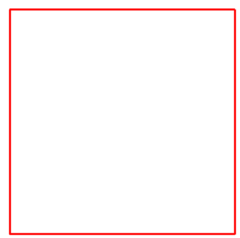

M

$\ker D_{CR} =$ hol. functions on the rectangle.

$\Longrightarrow \quad \dim\ker D_{CR} = \infty$.

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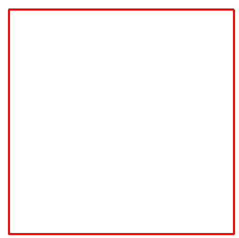

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$\Longrightarrow \quad \dim \ker D_{CR} = \infty$.

What do we do? **Copy mwb case:** Attach infinite cylinders.

IV. Index formulas on mwcs

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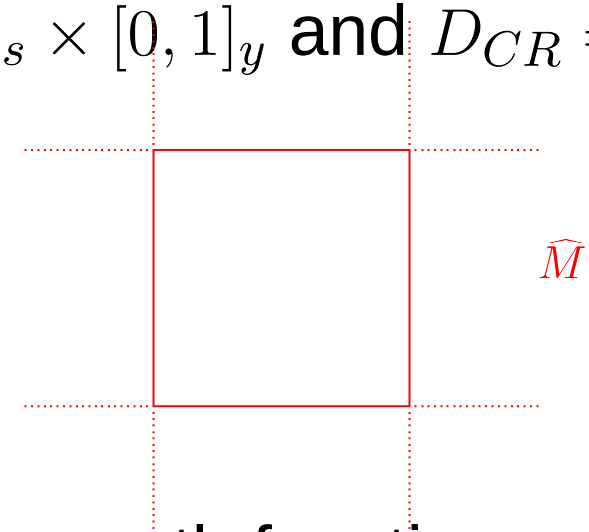

Let

$$\widehat{C}^\infty(\widehat{M}, \mathbb{C}) = \text{smooth functions on } \widehat{M} \text{ that} \to 0$$
$$\text{exp. as } |s| \to \infty \text{ and } |y| \to \infty.$$

Consider

$$\widehat{D}_{CR} = \partial_s + i\partial_y : \widehat{C}^\infty(\widehat{M}, \mathbb{C}) \to \widehat{C}^\infty(\widehat{M}, \mathbb{C}).$$

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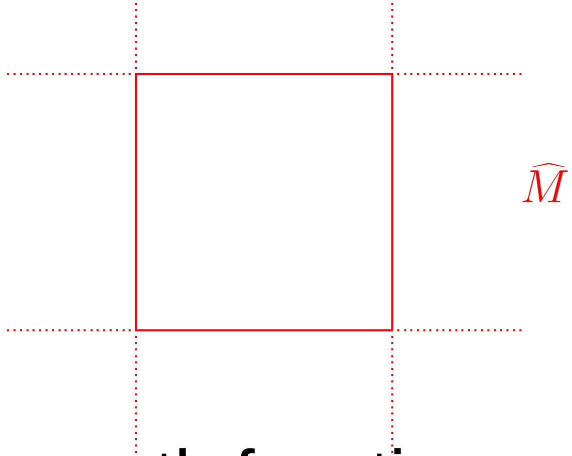

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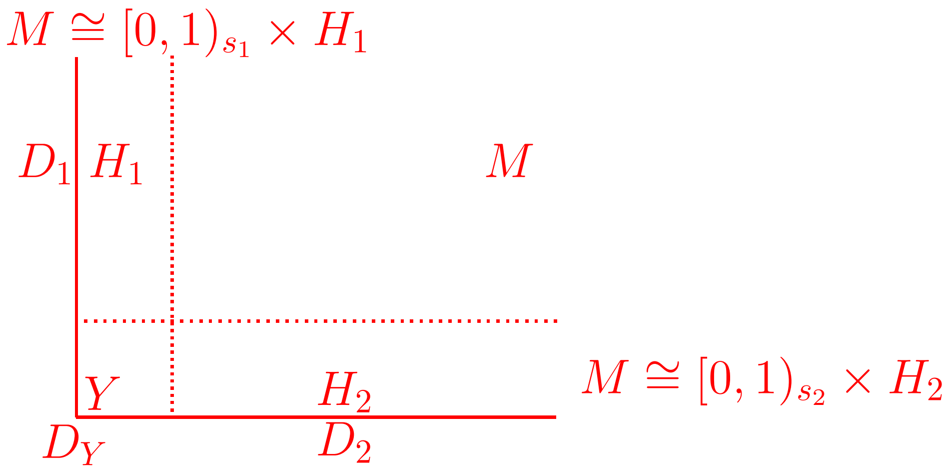
$$\Longrightarrow \ker \widehat{D}_{CR} = \text{holomorphic functions on } \mathbb{C} \text{ that} \to 0$$
$$\text{exp. as } |s| \to \infty \text{ and } |y| \to \infty$$
$$= \{0\}.$$

IV. Index formulas on mwcs

Recall: E, F are Hermitian vector bundles over an even-dimensional, compact, oriented, Riemannian manifold M with a corner of codim. 2 and

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

is a Dirac operator of product-type near ∂M:



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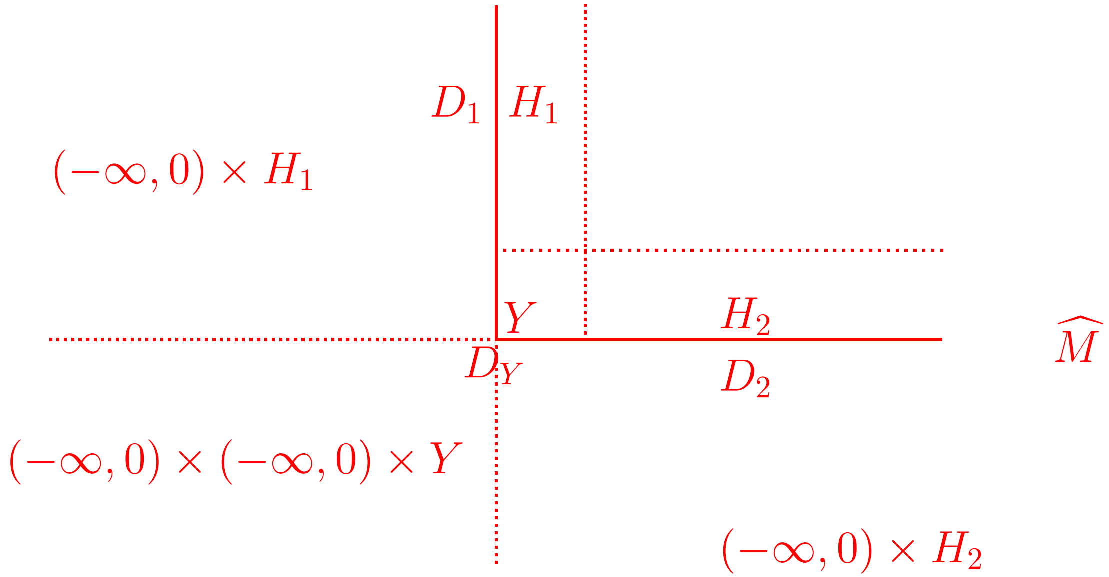

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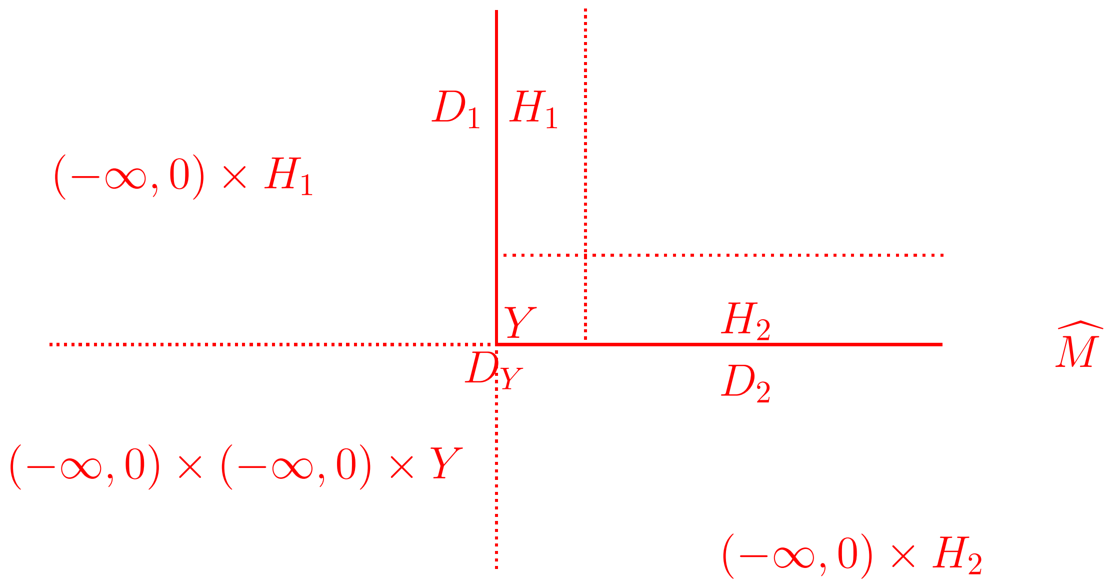
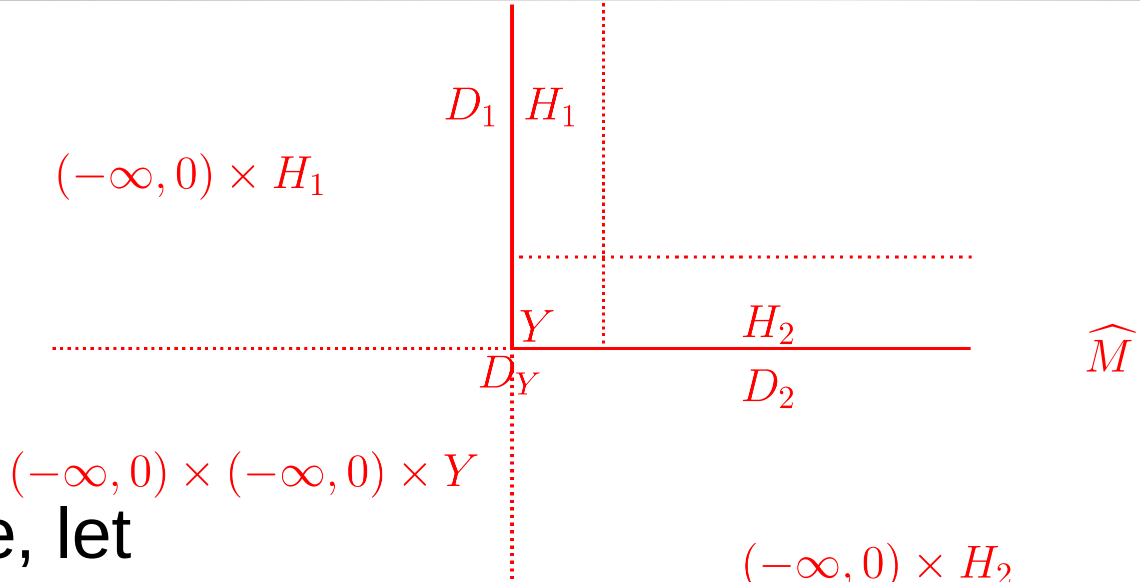


The Riemannian metric, E, F, and D extend to $\widehat{M}$; denote the extended objects by “hats”.

IV. Index formulas on mwcs

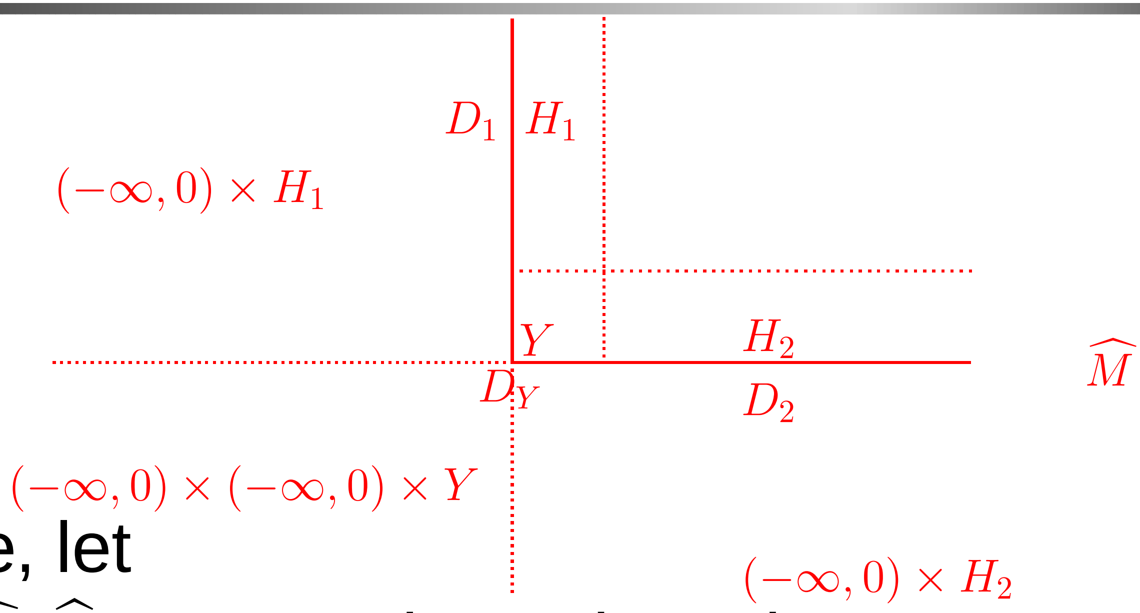


As before, let

$\widehat{C}^\infty(\widehat{M},\widehat{E})$ = smooth sections that $\to 0$ exp. at ∞.

Question: Is $\widehat{D}:\widehat{C}^\infty(\widehat{M},\widehat{E})\to\widehat{C}^\infty(\widehat{M},\widehat{F})$ Fredholm?

IV. Index formulas on mwcs



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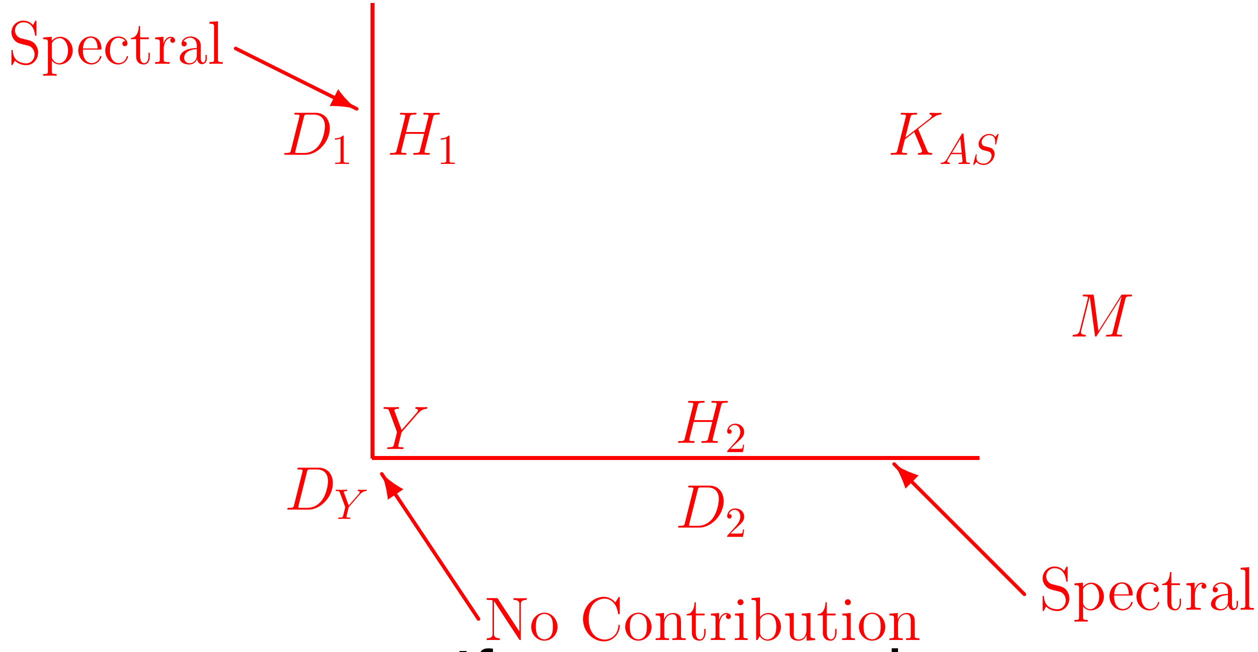
Question: Is $\widehat{D}:\widehat{C}^\infty(\widehat{M},\widehat{E})\to\widehat{C}^\infty(\widehat{M},\widehat{F})$ Fredholm?

Theorem: $\widehat{D}:\widehat{C}^\infty(\widehat{M},\widehat{E})\to\widehat{C}^\infty(\widehat{M},\widehat{F})$ is Fredholm if and only if $\ker D_Y=0$.

Corner Principle: $A:\widehat{C}^\infty(\widehat{M},\widehat{E})\to\widehat{C}^\infty(\widehat{M},\widehat{F})$ is Fredholm if and only if $\ker A_Y=0$.

IV. Index formulas on mwcs



Theorem (Müller, 1996): If $\ker D_Y = 0$, then

$$\boxed{\begin{array}{ccc} \operatorname{ind}\widehat{D} = & \int_M K_{AS} - \frac{1}{2}\sum_{i=1}^{2}\Big(\eta(D_i) + \dim\ker D_i\Big) & \\ \uparrow & \uparrow & \uparrow \\ \text{analytical} & \text{geometrical} & \text{spectral} \end{array}}$$

What happens if $\ker D_Y \neq 0$?

IV. Index formulas on mwcs

Theorem (Melrose-Nistor): $\ker D_Y$ is a symplectic vector space and given any Lagrangian subspace $\Lambda \subseteq \ker D_Y$, there is an operator $\mathcal{R}$ such that

$$\widehat{D} + \mathcal{R} : \widehat{C}^\infty(\widehat{M}, \widehat{E}) \to \widehat{C}^\infty(\widehat{M}, \widehat{F})$$

is Fredholm.

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is Fredholm.

Idea: Write $\ker D_Y = \Lambda \oplus \Lambda^\perp$ and define

$$r : \ker D_Y \to \ker D_Y \quad \text{by} \quad r = \begin{cases} +1 & \text{on } \Lambda \\ -1 & \text{on } \Lambda^\perp. \end{cases}$$

Choose $\mathcal{R}$ such that its restriction $\mathcal{R}_Y$ to Y is r. Then $\ker(D_Y + \mathcal{R}_Y) = 0$, so $\widehat{D} + \mathcal{R}$ is Fredholm.

Question: What is $\operatorname{ind}(\widehat{D} + \mathcal{R}) =$?

FLASHBACK: For an oriented, compact, 2-dimensional, Riemannian manifold M with corners,

$$\chi(M) \quad = \quad \frac{1}{2\pi}\int_M K \quad + \quad \frac{1}{2\pi}(\text{curv of } \partial M)$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

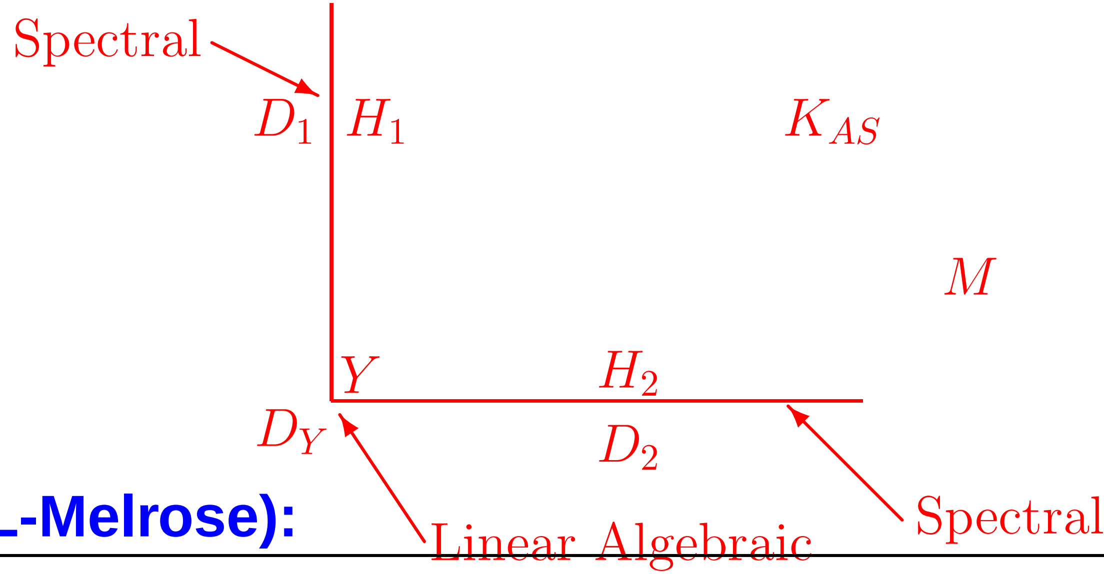
topological — geometrical — boundary correction

$$+\tfrac{1}{2\pi}(\text{sum of exterior angles})$$

$$\uparrow$$

linear algebra, correction from corners

IV. Index formulas on mwcs



Theorem (L-Melrose):

$$\boxed{\begin{aligned}\operatorname{ind}(\widehat{D} + \mathcal{R}) = \ & \int_M K_{AS} - \frac{1}{2}\sum_{i=1}^{2}\Big(\eta(D_i) + \dim\ker D_i\Big) \\ & -\frac{1}{2}\dim(\Lambda \cap \Lambda_{sc}) + \frac{1}{2\pi}\text{ext. } \angle(\Lambda, \Lambda_{sc}).\end{aligned}}$$

Remarks • Recall $\Lambda \subseteq \ker D_Y$.

- $\Lambda_{sc} \subseteq \ker D_Y$ is the “scattering Lagrangian.”
- First explain ext. $\angle$, then Λ_{sc}.

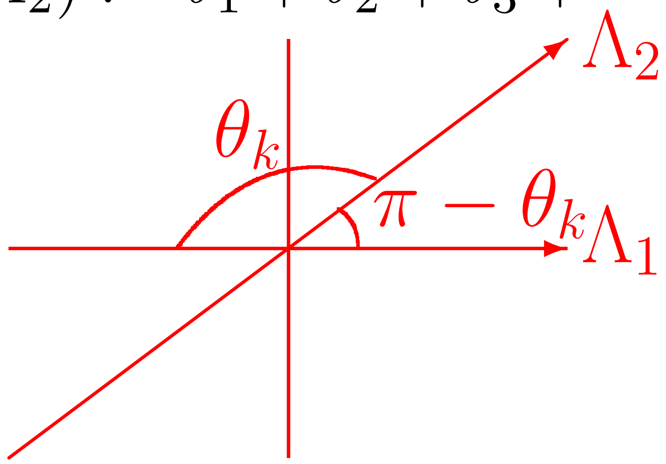
- Let V be a Hermitian symplectic vector space (E.g. $V = \ker D_Y$.)
- If Λ_1 and Λ_2 are two Lagrangian subspaces of V, then one can show that

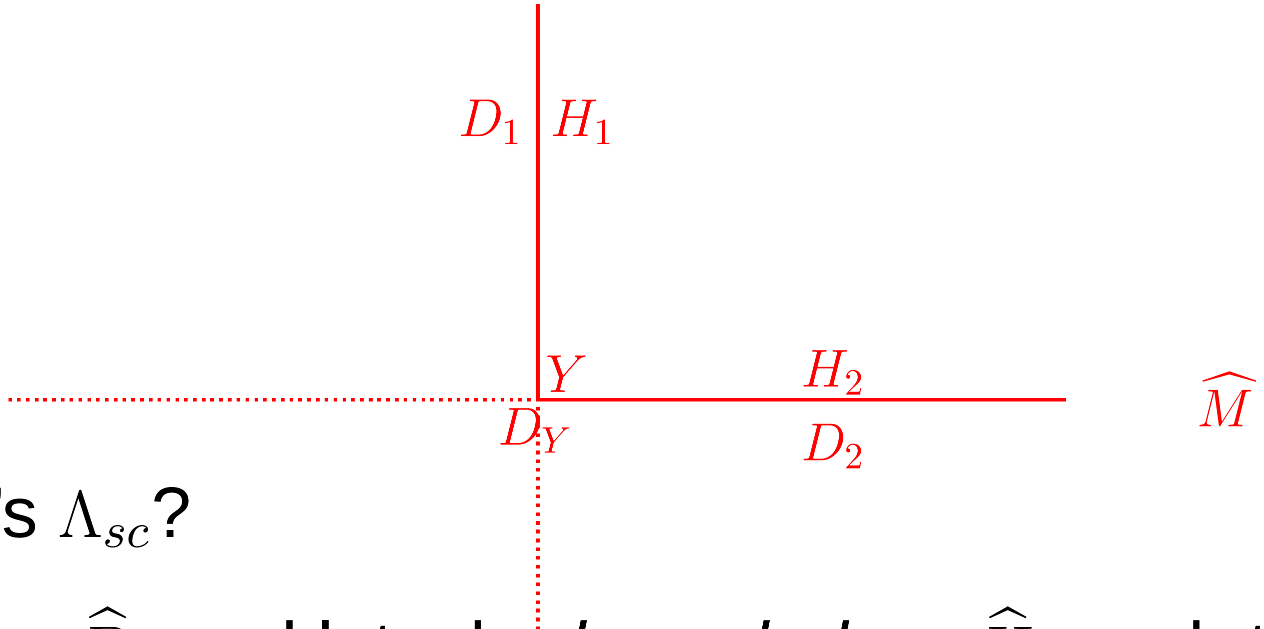
$$V \cong \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \cdots \oplus \mathbb{C}^2 \quad \text{s.t.}$$

$$\Lambda_1 = \mathrm{span}\,(1,1) \text{ in } k\text{-th copy of } \mathbb{C} \times \mathbb{C}$$

$$\Lambda_2 = \mathrm{span}\,(e^{i(\pi-\theta_k)},1) \text{ in } k\text{-th copy of } \mathbb{C} \times \mathbb{C}.$$

Then ext. $\angle(\Lambda_1, \Lambda_2) := \theta_1 + \theta_2 + \theta_3 + \cdots$.





- What's Λ_{sc}?

Consider $\widehat{D}_2$ and let u be *bounded* on $\widehat{H}_2$ such that $\widehat{D}_2 u = 0$. A-P-S showed that

$$\lim_{s_2 \to -\infty} u(s_2, y) \ \text{ exists}$$

and this limit lies in $\ker D_Y$.

$$\Lambda_{sc,2} := \{\text{limiting values}\} \subseteq \ker D_Y .$$

Similarly, can define $\Lambda_{sc,1}$.

Summary of Part V

- On compact mwcs of codim. 2, Dirac operators are never Fredholm. Unfortunately, there is no theory of BVP's for Dirac operators on mwcs.

- So, we try to get a Fredholm problem by considering the operator $\widehat{D}$ on the noncompact manifold $\widehat{M}$.

- The operator $\widehat{D}$ is Fredholm if and only if $\ker(D_Y) = 0$.

- (Müller) In this case, we have the index formula:

$$\operatorname{ind} \widehat{D} = \int_M K_{AS} - \frac{1}{2} \sum_{i=1}^{2} \big(\eta(D_i) + \dim \ker D_i \big)$$

Summary of Part V

- If $\ker D_Y \neq 0$, then to any $\Lambda \subseteq \ker D_Y$ there is an operator $\mathcal{R}$ such that $\widehat{D} + \mathcal{R}$ is Fredholm.

- We have

$$\operatorname{ind}(\widehat{D} + \mathcal{R}) = \int_M K_{AS} - \frac{1}{2}\sum_{i=1}^{2}\Big(\eta(D_i) + \dim\ker D_i\Big) - \frac{1}{2}\dim(\Lambda \cap \Lambda_{sc}) + \frac{1}{2\pi}\text{ext. } \angle(\Lambda, \Lambda_{sc})$$

$\uparrow$ analytical — $\uparrow$ geometrical — $\uparrow$ spectral — $\uparrow$ symplectic geometry

Summary of Talk

- The Gauss-Bonnet formula for a mwc has contributions from the interior of the manifold and the faces of its boundary.

- The Atiyah-Singer theorem (and its extensions to mwb and mwc) are "higher" Gauss-Bonnet formulas.

- **Advertisement:** Main ingredient to do index theory: A space of Ψdos tailored to the geometric situation at hand.

Next lecture we'll discuss the b-calculus, the "right" space of Ψdos for manifolds with cylindrical ends. Then we'll use it to prove the A-P-S index theorem.