The Atiyah-Singer Index Theorem II

Paul Loya

Where we ended last time

Let M be an oriented, compact, even-dim. Riemannian manifold without boundary.

- Let E and F be Hermitian vector bundles on M and let

$$L : C^\infty(M, E) \to C^\infty(M, F)$$

be a "Gauss-Bonnet type operator" (technically called a **Dirac operator**).

- (**Atiyah-Singer Index Theorem, 1963**) L is Fredholm and the following index formula holds:

$$\boxed{\begin{array}{ccc} \operatorname{ind} L & = & \int_M K_{AS}, \\ \textbf{analytical} & & \textbf{geometrical} \end{array}}$$

where K_{AS} is an **explicitly** defined polynomial in the curvatures of M, E, and F.

Questions

- Last lecture we talked about a “poor man’s” Gauss-Bonnet operator, but a . . .

“poor man’s ____” is a cheaper, simpler version of ____.

So, what is the true Gauss-Bonnet operator?

Questions

- Last lecture we talked about a "poor man's" Gauss-Bonnet operator, but a . . .

"poor man's ____" is a cheaper, simpler version of ____.

So, what is the true Gauss-Bonnet operator?

- What is a Dirac operator?

Questions

- Last lecture we talked about a “poor man’s” Gauss-Bonnet operator, but a . . .

  “poor man’s ____” is a cheaper, simpler version of ____.

  So, what is the true Gauss-Bonnet operator?

- What is a Dirac operator?

- What is K_{AS}?

Outline of talk

I. Review of differential operators

Outline of talk

I. Review of differential operators

II. Review of the principal symbol and ellipticity

Outline of talk

I. Review of differential operators

II. Review of the principal symbol and ellipticity

III. Dirac operators

Outline of talk

I. Review of differential operators

II. Review of the principal symbol and ellipticity

III. Dirac operators

IV. The true Gauss-Bonnet operator

Outline of talk

I. Review of differential operators

II. Review of the principal symbol and ellipticity

III. Dirac operators

IV. The true Gauss-Bonnet operator

V. The term K_{AS}

I. Review of differential operators

Preview of Part I

- A **(linear) differential operator** is a linear map given by taking linear combinations of partial derivatives and multiplying by smooth functions.

- The poor man's Gauss Bonnet operator is a first order differential operator.

I. Review of differential operators

Ex 1. $L : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ is the **Laplacian** or **Laplace operator**:

$$L = \Delta = -\partial_x^2 - \partial_y^2;$$

$$\Delta f = -\partial_x^2 f - \partial_y^2 f.$$

Δ is an example of a **second order operator**.

I. Review of differential operators

Ex 1. $L : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ is the **Laplacian** or **Laplace operator**:

$$L = \Delta = -\partial_x^2 - \partial_y^2;$$

$$\Delta f = -\partial_x^2 f - \partial_y^2 f.$$

Δ is an example of a **second order operator**.

Ex 2. $L : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$

$$L = -\partial_x^2 - \partial_y^2 + 5\partial_x - x^2\partial_y + 10e^{-x-y}.$$

Another second order operator.

I. Review of differential operators

Ex 3. The **Cauchy-Riemann operator** is the operator $L : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ defined by

$$D_{CR} = \partial_x + i\partial_y.$$

D_{CR} is an example of a **first order operator**. This operator is the fundamental operator of complex analysis!

I. Review of differential operators

Ex 3. The **Cauchy-Riemann operator** is the operator $L : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ defined by

$$D_{CR} = \partial_x + i\partial_y.$$

D_{CR} is an example of a **first order operator**. This operator is the fundamental operator of complex analysis!

Ex 4. Another first order operator is $L : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ defined by

$$L = \partial_x + i\partial_y + 2\sin(x^2 + y^2).$$

I. Review of differential operators

Ex 5. Recall that the **poor man's Gauss-Bonnet operator** is the operator

$$L_{GB} : C^\infty(M, TM) \to C^\infty(M, \mathbb{R}^2)$$

defined by

$$L_{GB}(v) = (-\mathrm{curl}\, v \ , \ \mathrm{div}\, v)$$

where v is a vector field on M.

I. Review of differential operators

Ex 5. Recall that the **poor man's Gauss-Bonnet operator** is the operator

$$L_{GB} : C^\infty(M, TM) \to C^\infty(M, \mathbb{R}^2)$$

defined by

$$L_{GB}(v) = (-\mathrm{curl}\, v\ ,\ \mathrm{div}\, v)$$

where v is a vector field on M.

Let $M = \mathbb{R}^2$. Given a vector field $v = f\vec{\imath} + g\vec{\jmath}$ on $\mathbb{R}^2$,

$$\mathrm{curl}\, v = (\partial_x g - \partial_y f)\vec{k}$$

$$\mathrm{div}\, v = \partial_x f + \partial_y g.$$

I. Review of differential operators

Therefore,

$$L_{GB}(f\vec{\imath} + g\vec{\jmath}) = (\partial_y f - \partial_x g,\ \partial_x f + \partial_y g).$$

We can also write L_{GB} as a matrix:

$$\boxed{L_{GB}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \partial_y & -\partial_x \\ \partial_x & \partial_y \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}.}$$

Therefore, the poor man's Gauss-Bonnet operator is a first order differential operator.

In general, a differential operator L is of **m-th order** if each term of L involves at most m differentiations.

I. Review of differential operators

Summary of Part I

- A **differential operator** is a linear map given by taking linear combinations of partial derivatives and multiplying by smooth functions.

- The Laplacian is a second order differential operator

- The Cauchy-Riemann operator and the poor man's Gauss Bonnet operator are first order differential operators.

II. Principal symbol and ellipticity

Preview of Part II

- Principal = first or of highest importance, rank, worth.

- The principal symbol of a differential operator is a (matrix of) *polynomials* determined by the “most important” part of the operator.

- A differential operator is elliptic if its principal symbol is invertible.

II. Principal symbol and ellipticity

- Principal symbol.

Ex 2 con't: Consider the operator:

$$L = -\partial_x^2 - \partial_y^2 + 5\partial_x - x^2\partial_y + 10e^{-x-y}.$$

II. Principal symbol and ellipticity

- Principal symbol.

Ex 2 con't: Consider the operator:

$$L = -\partial_x^2 - \partial_y^2 + 5\partial_x - x^2\partial_y + 10e^{-x-y}.$$

The principal symbol of L is

$$\begin{aligned}\sigma(L)(\xi_1, \xi_2) &= -(i\xi_1)^2 - (i\xi_2)^2 \\ &= \xi_1^2 + \xi_2^2 \\ &= |\xi|^2 \ \left(\text{where } \xi = (\xi_1, \xi_2)\right) \\ &= \text{squared length of } \xi.\end{aligned}$$

II. Principal symbol and ellipticity

- Principal symbol.

Ex 1 con't: Consider the Laplacian:

$$\Delta = -\partial_x^2 - \partial_y^2.$$

II. Principal symbol and ellipticity

- Principal symbol.

Ex 1 con't: Consider the Laplacian:

$$\Delta = -\partial_x^2 - \partial_y^2.$$

The principal symbol of Δ is

$$\begin{aligned}\sigma(\Delta)(\xi_1, \xi_2) &= -(i\xi_1)^2 - (i\xi_2)^2 \\ &= \xi_1^2 + \xi_2^2 \\ &= |\xi|^2 \ \text{(where } \xi = (\xi_1, \xi_2)) \\ &= \text{squared length of } \xi.\end{aligned}$$

II. Principal symbol and ellipticity

- Principal symbol.

Ex 4 con't: Consider the operator

$$L = \partial_x + i\partial_y + 2\sin(x^2 + y^2).$$

II. Principal symbol and ellipticity

- Principal symbol.

Ex 4 con't: Consider the operator

$$L = \partial_x + i\partial_y + 2\sin(x^2 + y^2).$$

The principal symbol of L is

$$\begin{aligned} \sigma(L)(\xi_1, \xi_2) &= i\xi_1 + i(i\xi_2) \\ &= i\xi_1 - \xi_2. \end{aligned}$$

II. Principal symbol and ellipticity

- Principal symbol.

Ex 4 con't: Consider the operator

$$L = \partial_x + i\partial_y + 2\sin(x^2 + y^2).$$

The principal symbol of L is

$$\begin{aligned}\sigma(L)(\xi_1, \xi_2) &= i\xi_1 + i(i\xi_2) \\ &= i\xi_1 - \xi_2.\end{aligned}$$

Ex 3 con't: For the Cauchy-Riemann operator $D_{CR} = \partial_x + i\partial_y$, we have

$$\sigma(D_{CR})(\xi_1, \xi_2) = i\xi_1 - \xi_2.$$

II. Principal symbol and ellipticity

- Principal symbol.

Ex 5 con't: Consider the poor man's Gauss-Bonnet operator (written as a matrix)

$$L_{GB} = \begin{pmatrix} \partial_y & -\partial_x \\ \partial_x & \partial_y \end{pmatrix}.$$

II. Principal symbol and ellipticity

- Principal symbol.

Ex 5 con't: Consider the poor man's Gauss-Bonnet operator (written as a matrix)

$$L_{GB} = \begin{pmatrix} \partial_y & -\partial_x \\ \partial_x & \partial_y \end{pmatrix}.$$

The principal symbol of L is

$$\sigma(L_{GB})(\xi_1, \xi_2) = \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix}.$$

II. Principal symbol and ellipticity

- Principal symbol.

Let L be an m-th differential operator and let $x_1, x_2, \dots, x_n$ be the variables it differentiates with respect to.

In the terms of L containing m partial derivatives, replace

$$\partial_{x_1} \text{ by } i\xi_1 \ , \qquad \partial_{x_2} \text{ by } i\xi_2 \ \ , \qquad \dots \ \ , \qquad \partial_{x_n} \text{ by } i\xi_n.$$

The resulting function of the real variables $\xi_1, \dots, \xi_n$ is called the **principal symbol** of L:

$$\sigma(L)(\xi_1, \dots, \xi_n) \quad \text{or} \quad \sigma(L)(\xi),$$

where $\xi = (\xi_1, \dots, \xi_n)$.

II. Principal symbol and ellipticity

- Ellipticity.

Recall that

$$\sigma(\Delta)(\xi) = |\xi|^2.$$

For $\xi \neq 0$; that is, $\xi = (\xi_1, \xi_2) \neq (0,0)$,

$$\sigma(\Delta)(\xi)^{-1}$$

is defined.

II. Principal symbol and ellipticity

- Ellipticity.

Recall that

$$\sigma(\Delta)(\xi) = |\xi|^2.$$

For $\xi \neq 0$; that is, $\xi = (\xi_1, \xi_2) \neq (0, 0)$,

$$\sigma(\Delta)(\xi)^{-1}$$

is defined.

Similarly, for $\xi \neq 0$

$$\sigma(D_{CR})(\xi) = i\xi_1 - \xi_2$$

is invertible.

II. Principal symbol and ellipticity

- Ellipticity.

The poor man's Gauss-Bonnet operator,

$$\sigma(L_{GB})(\xi) = \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix},$$

also has the same property:

For $\xi \neq 0$, $\sigma(L_{GB})(\xi)$ is an invertible matrix.

(Notice that $\det \sigma(L_{GB})(\xi) = -\xi_2^2 - \xi_1^2 = -|\xi|^2$.)

II. Principal symbol and ellipticity

- Ellipticity.

A differential operator L is **elliptic** if for $\xi \neq 0$, the principal symbol $\sigma(L)(\xi)$ is invertible.

Thus, Δ, D_{CR}, and L_{GB} are elliptic.

II. Principal symbol and ellipticity

- Ellipticity.

A differential operator L is **elliptic** if for $\xi \neq 0$, the principal symbol $\sigma(L)(\xi)$ is invertible.

Thus, Δ, D_{CR}, and L_{GB} are elliptic.

Most operators are not elliptic! E.g.

$$L = \partial_x^2 - \partial_y + 10.$$

We have $\sigma(L)(\xi_1, \xi_2) = -(i\xi_1)^2 = \xi_1^2$. Then $\xi = (0, 1) \neq 0$, but

$$\sigma(L)(\xi) = 0 \text{ is not invertible.}$$

II. Principal symbol and ellipticity

Summary of Part II

- Examples: $\sigma(\Delta)(\xi) = |\xi|^2$, $\sigma(D_{CR})(\xi) = i\xi_1 - \xi_2$, and

$$\sigma(L_{GB})(\xi) = \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix}$$

- Significance: Laplacians involve geometry. What is the significance of the last two examples?

- The three operators above are elliptic.

- Can also define differential operators, principal symbols, and ellipticity when manifolds and vector bundles are involved.

III. Dirac operators

Preview of Part III

- Recall the Laplacian is a second order operator such that

$$\sigma(\Delta)(\xi) = |\xi|^2.$$

Thus, Δ captures geometry.

- A Dirac operator is a first order operator whose principal symbol “squared” is $|\xi|^2$.

III. Dirac operators

Ex. For $D_{CR} = \partial_x + i\partial_y$, we have

$$\sigma(D_{CR})(\xi) = i\xi_1 - \xi_2,$$

so

$$\begin{aligned}\overline{\sigma(D_{CR})(\xi)}\,\sigma(D_{CR})(\xi) &= \overline{(i\xi_1 - \xi_2)}(i\xi_1 - \xi_2) \\ &= (-i\xi_1 - \xi_2)(i\xi_1 - \xi_2) \\ &= \xi_1^2 + \xi_2^2 \\ &= |\xi|^2.\end{aligned}$$

Hence we can obtain lengths (geometry) by conjugating and then multiplying!

III. Dirac operators

Ex. For the poor man's Gauss-Bonnet operator, we have

$$\sigma(L_{GB})(\xi) = \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix}.$$

III. Dirac operators

Ex. For the poor man's Gauss-Bonnet operator, we have

$$\sigma(L_{GB})(\xi) = \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \sigma(L_{GB})(\xi)^* \sigma(L_{GB})(\xi) &= \begin{pmatrix} -i\xi_2 & -i\xi_1 \\ i\xi_1 & -i\xi_2 \end{pmatrix} \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix} \\ &= \begin{pmatrix} \xi_1^2 + \xi_2^2 & 0 \\ 0 & \xi_1^2 + \xi_2^2 \end{pmatrix} \\ &= |\xi|^2. \end{aligned}$$

III. Dirac operators

Definition: A first order differential operator L is called a **Dirac(-type) operator** if L is elliptic and

$$\sigma(L)(\xi)^*\sigma(L)(\xi) = |\xi|^2.$$

Therefore, D_{CR} and L_{GB} are Dirac operators.

III. Dirac operators

Definition: A first order differential operator L is called a **Dirac(-type) operator** if L is elliptic and

$$\sigma(L)(\xi)^* \sigma(L)(\xi) = |\xi|^2.$$

Therefore, D_{CR} and L_{GB} are Dirac operators.

- Recall

$$\sigma(\Delta)(\xi) = |\xi|^2.$$

Thus, we can think of a Dirac operator as an operator such that when you square it (really, the principal symbol), you get the (principal symbol of the) Laplacian.

Hence, a Dirac operator is a type of “square root” of a Laplacian.

III. Dirac operators

- Dirac operators can be defined when Riemannian manifolds and Hermitian vector bundles are involved: as a first order differential operator L that is elliptic and

$$\sigma(L)(\xi)^*\sigma(L)(\xi) = |\xi|^2.$$

III. Dirac operators

- Dirac operators can be defined when Riemannian manifolds and Hermitian vector bundles are involved: as a first order differential operator L that is elliptic and

$$\sigma(L)(\xi)^*\sigma(L)(\xi) = |\xi|^2.$$

- *Now we understand the hypothesis of Atiyah-Singer!*

"Let E and F be Hermitian vector bundles on M and let

$$L : C^\infty(M, E) \to C^\infty(M, F)$$

be a "Gauss-Bonnet type operator" (technically called a **Dirac operator**)."

III. Dirac operators

Summary of Part III

- A Dirac operator is a first order differential operator whose principal symbol “squared” is the symbol of the Laplacian.
- Like Laplacians, Dirac operators capture the geometry of the manifold.
- Advantage of Dirac operators: They are first order instead of second order. (Hence are simpler “in principle.”)

IV. The true Gauss-Bonnet operator

Preview of Part IV

- Differential forms are objects you integrate (in line and area integrals).

- The exterior derivative d is just the gradient and curl "all-in-one".

- The Gauss-Bonnet operator is $D_{GB} = d + d^*$.

IV. The true Gauss-Bonnet operator

- Differential forms. **(Focus on $\mathbb{R}^2$.)**

$$C^\infty(\mathbb{R}^2,\Lambda^0) = C^\infty(\mathbb{R}^2) = \text{0-forms}$$

$$C^\infty(\mathbb{R}^2,\Lambda^1) = \text{1-forms} \quad f\,dx + g\,dy$$

$$C^\infty(\mathbb{R}^2,\Lambda^2) = \text{2-forms} \quad f\,dx\wedge dy$$

There are no 3-forms on $\mathbb{R}^2$.

IV. The true Gauss-Bonnet operator

- Differential forms. **(Focus on $\mathbb{R}^2$.)**

$$C^\infty(\mathbb{R}^2, \Lambda^0) = C^\infty(\mathbb{R}^2) = \text{0-forms}$$

$$C^\infty(\mathbb{R}^2, \Lambda^1) = \text{1-forms} \quad f\,dx + g\,dy$$

$$C^\infty(\mathbb{R}^2, \Lambda^2) = \text{2-forms} \quad f\,dx \wedge dy$$

There are no 3-forms on $\mathbb{R}^2$.

Think of

$$dx \longleftrightarrow \vec{\imath}\ , \quad dy \longleftrightarrow \vec{\jmath}\ , \quad dx \wedge dy \longleftrightarrow \vec{k}.$$

Remark: 1-forms are objects usually found in line integrals and 2-forms are found in area integrals.

IV. The true Gauss-Bonnet operator

- The wedge.

The "wedge" $\wedge$ has the defining "cross product" property

$$\alpha \wedge \beta = -\beta \wedge \alpha$$

for any 1-forms α and β. $\Big($ cf. $v \times w = -w \times v$. $\Big)$

IV. The true Gauss-Bonnet operator

- The wedge.

The "wedge" $\wedge$ has the defining "cross product" property

$$\alpha \wedge \beta = -\beta \wedge \alpha$$

for any 1-forms α and β. $\Big($ cf. $v \times w = -w \times v$. $\Big)$

Ex. $dx \wedge dy = -dy \wedge dx$. $\Big($cf. $\vec{\imath} \times \vec{\jmath} = -\vec{\jmath} \times \vec{\imath}\Big)$.

IV. The true Gauss-Bonnet operator

- The wedge.

The "wedge" $\wedge$ has the defining "cross product" property

$$\alpha \wedge \beta = -\beta \wedge \alpha$$

for any 1-forms α and β. $\Big($ cf. $v \times w = -w \times v$. $\Big)$

Ex. $$dx \wedge dy = -dy \wedge dx. \qquad \left(\text{cf. } \vec{\imath} \times \vec{\jmath} = -\vec{\jmath} \times \vec{\imath}\right).$$

Ex.

$$\alpha \wedge \alpha = -\alpha \wedge \alpha.$$

Therefore, $\alpha \wedge \alpha = 0$. In particular,

$$dx \wedge dx = 0 \quad \text{and} \quad dy \wedge dy = 0. \qquad \left(\text{cf. } \vec{\imath} \times \vec{\imath} = 0\right).$$

IV. The true Gauss-Bonnet operator

The **exterior derivative**

$$d : C^\infty(\mathbb{R}^2, \Lambda^k) \to C^\infty(\mathbb{R}^2, \Lambda^{k+1})$$

is the differential operator

$$d = \partial_x \, dx + \partial_y \, dy$$

acting componentwise. $\left(\text{cf. } \nabla = \partial_x \, \vec{\imath} + \partial_y \, \vec{\jmath}.\right)$

IV. The true Gauss-Bonnet operator

The **exterior derivative**

$$d : C^\infty(\mathbb{R}^2, \Lambda^k) \longrightarrow C^\infty(\mathbb{R}^2, \Lambda^{k+1})$$

is the differential operator

$$d = \partial_x \, dx + \partial_y \, dy$$

acting componentwise. $\left(\text{cf. } \nabla = \partial_x \, \vec{\imath} + \partial_y \, \vec{\jmath}.\right)$

Note: d really consists of three maps

$$C^\infty(\mathbb{R}^2, \Lambda^0) \xrightarrow{d} C^\infty(\mathbb{R}^2, \Lambda^1) \xrightarrow{d} C^\infty(\mathbb{R}^2, \Lambda^2) \xrightarrow{d} 0.$$

($d = 0$ on 2-forms since there are no 3-forms.)

IV. The true Gauss-Bonnet operator

0-forms:

$$d : C^\infty(\mathbb{R}^2, \Lambda^0) \to C^\infty(\mathbb{R}^2, \Lambda^1).$$

For $f \in C^\infty(\mathbb{R}^2, \Lambda^0) = C^\infty(\mathbb{R}^2)$,

$$df = \partial_x f \, dx + \partial_y f \, dy.$$

$\big($cf. $\nabla f = \partial_x f \, \vec{\imath} + \partial_y f \, \vec{\jmath}.\big)$ Thus,

$$d = \text{gradient on 0-forms.}$$

IV. The true Gauss-Bonnet operator

1-forms:

$$d(f\,dx + g\,dy) = df \wedge dx + dg \wedge dy$$

IV. The true Gauss-Bonnet operator

1-forms:

$$
\begin{aligned}
d(f\,dx + g\,dy) &= df \wedge dx + dg \wedge dy \\
&= (\partial_x f\,dx + \partial_y f\,dy) \wedge dx + (\partial_x g\,dx + \partial_y g\,dy) \wedge dy \\
&= \partial_y f\,dy \wedge dx + \partial_x g\,dx \wedge dy \\
&= -\partial_y f\,dx \wedge dy + \partial_x g\,dx \wedge dy \\
&= (\partial_x g - \partial_y f)\,dx \wedge dy.
\end{aligned}
$$

IV. The true Gauss-Bonnet operator

1-forms:

$$\begin{aligned} d(f\,dx + g\,dy) &= df \wedge dx + dg \wedge dy \\ &= (\partial_x f\,dx + \partial_y f\,dy) \wedge dx + (\partial_x g\,dx + \partial_y g\,dy) \wedge dy \\ &= \partial_y f\,dy \wedge dx + \partial_x g\,dx \wedge dy \\ &= -\partial_y f\,dx \wedge dy + \partial_x g\,dx \wedge dy \\ &= (\partial_x g - \partial_y f)\,dx \wedge dy. \end{aligned}$$

(cf. $\text{curl}(f\,\vec{\imath} + g\,\vec{\jmath}) = (\partial_x g - \partial_y f)\,\vec{k}$.) Thus,

$$d = \text{curl on 1-forms.}$$

IV. The true Gauss-Bonnet operator

- Adjoint: If L is an $m \times n$ matrix, we have

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

The adjoint (conjugate transpose) L^* is an $n \times m$ matrix, so

$$L^* : \mathbb{R}^m \longrightarrow \mathbb{R}^n.$$

Taking the adjoint switches the domain and codomain.

IV. The true Gauss-Bonnet operator

- Adjoint: If L is an $m \times n$ matrix, we have

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

The adjoint (conjugate transpose) L^* is an $n \times m$ matrix, so

$$L^* : \mathbb{R}^m \longrightarrow \mathbb{R}^n.$$

Taking the adjoint switches the domain and codomain.

Recall

$$C^\infty(\mathbb{R}^2, \Lambda^0) \xrightarrow{d} C^\infty(\mathbb{R}^2, \Lambda^1) \xrightarrow{d} C^\infty(\mathbb{R}^2, \Lambda^2).$$

There is an adjoint

$$C^\infty(\mathbb{R}^2, \Lambda^2) \xrightarrow{d^*} C^\infty(\mathbb{R}^2, \Lambda^1) \xrightarrow{d^*} C^\infty(\mathbb{R}^2, \Lambda^0).$$

IV. The true Gauss-Bonnet operator

- What is d^*?

IV. The true Gauss-Bonnet operator

- What is d^*?

2-forms: $d^* : C^\infty(\mathbb{R}^2, \Lambda^2) \xrightarrow{d^*} C^\infty(\mathbb{R}^2, \Lambda^1)$:

$$d^*(f\,dx \wedge dy) = \partial_y f\,dx - \partial_x f\,dy.$$

IV. The true Gauss-Bonnet operator

- What is d^*?

2-forms: $d^* : C^\infty(\mathbb{R}^2, \Lambda^2) \xrightarrow{d^*} C^\infty(\mathbb{R}^2, \Lambda^1)$:

$$d^*(f\, dx \wedge dy) = \partial_y f\, dx - \partial_x f\, dy.$$

Note:

$$\operatorname{curl}(f\, \vec{k}) = \partial_y f\, \vec{\imath} - \partial_x f\, \vec{\jmath}.$$

Therefore,

$$d^* = \operatorname{curl} \text{ on 2-forms.}$$

IV. The true Gauss-Bonnet operator

- What is d^*?

1-forms: $d^* : C^\infty(\mathbb{R}^2, \Lambda^1) \xrightarrow{d^*} C^\infty(\mathbb{R}^2, \Lambda^0)$:

$$d^*(f\,dx + g\,dy) = -(\partial_x f + \partial_y g).$$

IV. The true Gauss-Bonnet operator

- What is d^*?

1-forms: $d^* : C^\infty(\mathbb{R}^2, \Lambda^1) \xrightarrow{d^*} C^\infty(\mathbb{R}^2, \Lambda^0)$:

$$d^*(f\,dx + g\,dy) = -(\partial_x f + \partial_y g).$$

Note:

$$\mathrm{div}\,(f\,\vec{\imath} + g\,\vec{\jmath}) = \partial_x f + \partial_y g.$$

Therefore,

$$d^* = -\mathrm{div} \text{ on 1-forms.}$$

IV. The true Gauss-Bonnet operator

- The true Gauss-Bonnet operator. Recall

$$d : C^\infty(\mathbb{R}^2, \Lambda^k) \to C^\infty(\mathbb{R}^2, \Lambda^{k+1}).$$

$$d^* : C^\infty(\mathbb{R}^2, \Lambda^{k+1}) \to C^\infty(\mathbb{R}^2, \Lambda^k).$$

IV. The true Gauss-Bonnet operator

- The true Gauss-Bonnet operator. Recall

$$d : C^\infty(\mathbb{R}^2, \Lambda^k) \to C^\infty(\mathbb{R}^2, \Lambda^{k+1}).$$

$$d^* : C^\infty(\mathbb{R}^2, \Lambda^{k+1}) \to C^\infty(\mathbb{R}^2, \Lambda^k).$$

- $C^\infty(\mathbb{R}^2, \Lambda^{ev}) =$ linear combination of 0 and 2 forms
- $C^\infty(\mathbb{R}^2, \Lambda^{odd}) = C^\infty(\mathbb{R}^2, \Lambda^1) =$ 1-forms. Then,

$$D_{GB} = d + d^* : C^\infty(\mathbb{R}^2, \Lambda^{ev}) \to C^\infty(\mathbb{R}^2, \Lambda^{odd})$$

is called the **THE Gauss-Bonnet operator**.

IV. The true Gauss-Bonnet operator

- Exercises:

1) Check that D_{GB} is a Dirac operator.

2) How is the poor man's Gauss-Bonnet operator related to d and d^*?

IV. The true Gauss-Bonnet operator

- Exercises:

1) Check that D_{GB} is a Dirac operator.
2) How is the poor man's Gauss-Bonnet operator related to d and d^*?

- Given any Riemannian manifold M, we can define differential forms, d, and d^*. Let

$$C^\infty(M, \Lambda^{ev}) = \text{ even forms } , \; C^\infty(M, \Lambda^{odd}) = \text{odd forms.}$$

Then,

$$D_{GB} = d + d^* : C^\infty(M, \Lambda^{ev}) \to C^\infty(M, \Lambda^{odd})$$

is called **THE Gauss-Bonnet operator**. D_{GB} is a Dirac operator.

IV. The true Gauss-Bonnet operator

Notes:

- The operator

$$\Delta := (d + d^*)^2$$

is called the **Laplacian** or **Laplace operator**.

- By definition, $d + d^*$ is a square root of the Laplacian.

- $d + d^*$ and Δ are important in "Hodge theory," a subject which relates the kernels and cokernels of these operators to the topology of the manifold. In particular,

$$\text{ind}\, D_{GB} = \chi(M).$$

IV. The true Gauss-Bonnet operator

Summary of Part IV

- Differential forms are objects you integrate.

- The exterior derivative d is the gradient and curl "all-in-one".

- The adjoint d^* is the curl and divergence "all-in-one".

- The Gauss-Bonnet operator is the Dirac operator

$$D_{GB} = d + d^* : C^\infty(M, \Lambda^{ev}) \to C^\infty(M, \Lambda^{odd}).$$

V. The integrand K_{AS}

Preview of Part V

- (One of the) most beautiful formulas in the world:

$$\boxed{\operatorname{ind} L = \frac{1}{(4\pi i)^m} \int_M \sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)} \operatorname{STr}\left(e^{K_E + K_F + \frac{1}{4}K}\right).}$$

V. The integrand K_{AS}

Data:

- Let M be an oriented, compact, even-dim. Riemannian manifold and let E and F be Hermitian vector bundles on M.
- Let K = curvature of M, K_E = curvature of E, K_F = curvature of F.
- Let $L : C^\infty(M, E) \to C^\infty(M, F)$ be a Dirac operator.

V. The integrand K_{AS}

Data:

- Let M be an oriented, compact, even-dim. Riemannian manifold and let E and F be Hermitian vector bundles on M.
- Let K = curvature of M, K_E = curvature of E, K_F = curvature of F.
- Let $L : C^\infty(M, E) \to C^\infty(M, F)$ be a Dirac operator.

Atiyah-Singer: L is Fredholm and

$$\operatorname{ind} L = \int_M K_{AS},$$

where K_{AS} is an **explicitly** defined polynomial in K, K_E, K_F.

V. The integrand K_{AS}

- $\hat{A}$-genus of M:

$$\sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)}.$$

You can actually make sense of this.

V. The integrand K_{AS}

- $\hat{A}$-genus of M:

$$\sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)}.$$

You can actually make sense of this.

- Twisted Chern character

$$\mathrm{STr}\left(e^{K_E+K_F+\frac{1}{4}K}\right).$$

where “STr” is called a “super trace”.

V. The integrand K_{AS}

- $\hat{A}$-genus of M:

$$\sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)}.$$

You can actually make sense of this.

- Twisted Chern character

$$\mathrm{STr}\left(e^{K_E+K_F+\frac{1}{4}K}\right).$$

where “STr” is called a “super trace”.

- $K_{AS} = \frac{1}{(4\pi i)^m}\sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)}\mathrm{STr}\left(e^{K_E+K_F+\frac{1}{4}K}\right)$,

where $\dim M = 2m$.

V. The integrand K_{AS}

The **Atiyah-Singer theorem** in all its glory: Given

- An oriented, compact, even dimensional (say $2m$) Riemannian manifold M
- Hermitian vector bundles E and F over M
- Dirac operator $L : C^\infty(M, E) \to C^\infty(M, F)$.

Then,

$$\boxed{\operatorname{ind} L = \frac{1}{(4\pi i)^m} \int_M \sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)} \operatorname{STr}\left(e^{K_E + K_F + \frac{1}{4}K}\right).}$$

V. The integrand K_{AS}

Ex. Consider the Gauss-Bonnet operator:

$$D_{GB} : C^\infty(M, \Lambda^{ev}) \to C^\infty(M, \Lambda^{odd}).$$

Recall that (via "Hodge theory") ind $D_{GB} = \chi(M)$.

One can work out that

$$K_{AS} = \frac{1}{(2\pi)^m} \frac{(-K)^m}{m!} = \frac{1}{(2\pi)^m} \mathrm{Pf}(-K),$$

where $\mathrm{Pf}(-K)$ is called the Pfaffian of M.

$\therefore$ **Gauss-Bonnet-Chern theorem:**

$$\boxed{\chi(M) = \frac{1}{(2\pi)^m} \int_M \mathrm{Pf}(-K).}$$

Summary of Talk

Question: What is the true Gauss-Bonnet operator?

Answer: The operator $d + d^*$ acting on even forms. d is the exterior derivative (= gradient and curl "all-in-one") and d^* is the adjoint of d.

Question: What is a Dirac operator?

Answer: Basically a "square root" of a Laplacian.

Question: What is K_{AS}?

Answer: So beautiful, the AS thm. has to be repeated:

$$\boxed{\operatorname{ind} L = \frac{1}{(4\pi i)^m} \int_M \sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)} \operatorname{STr}\left(e^{K_E + K_F + \frac{1}{4}K}\right).}$$

WAIT!

Isn't this supposed to be a conference on SINGULAR analysis?

A-S on singular manifolds

We know that for a *smooth* manifold M without boundary,

$$\operatorname{ind} L = \int_M K_{AS}.$$

What about SINGULAR manifolds like

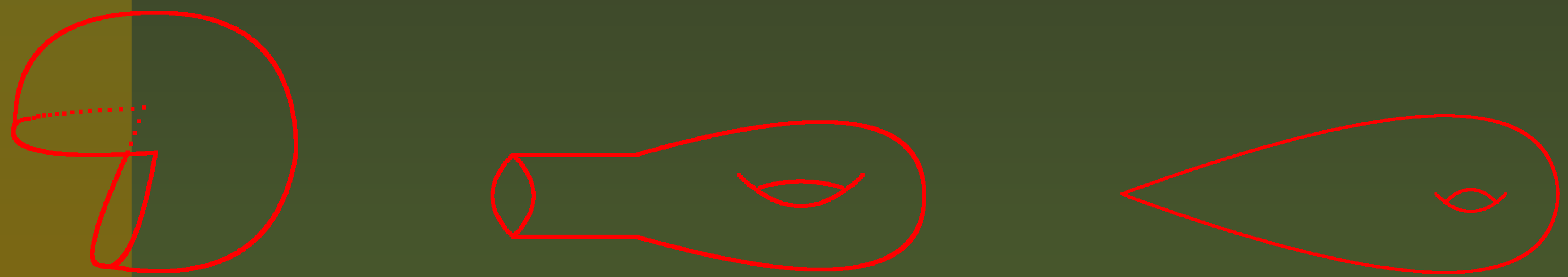
What is the A-S theorem for such manifolds?

A-S on singular manifolds

We know that for a *smooth* manifold M without boundary,

$$\operatorname{ind} L = \int_M K_{AS}.$$

What about SINGULAR manifolds like



What is the A-S theorem for such manifolds?

Answer: Next week!

An advertisement

We'll talk about

1) Index theorems on singular manifolds.

2) The proof of the A-S theorem: pseudodifferential operators and the heat kernel.

3) The proof of the A-S theorem for a singular manifold: "Exotic" pseudodifferential operators and the heat kernel.

An advertisement

We'll talk about

1) Index theorems on singular manifolds.

2) The proof of the A-S theorem: pseudodifferential operators and the heat kernel.

3) The proof of the A-S theorem for a singular manifold: "Exotic" pseudodifferential operators and the heat kernel.

$$\boxed{\operatorname{ind} L = \frac{1}{(4\pi i)^m} \int_M \sqrt{\det\left(\frac{K/2}{\sinh(K/2)}\right)} \operatorname{STr}\left(e^{K_E + K_F + \frac{1}{4}K}\right).}$$