Math. Z. (2008) 259:65–95
DOI 10.1007/s00209-007-0212-6

Mathematische Zeitschrift

Resolvents of cone pseudodifferential operators, asymptotic expansions and applications

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Accepted: 28 November 2006 / Published online: 30 June 2007

Abstract We study the structure and asymptotic behavior of the resolvent of elliptic cone pseudodifferential operators acting on weighted Sobolev spaces over a compact manifold with boundary. We obtain an asymptotic expansion of the resolvent as the spectral parameter tends to infinity, and use it to derive corresponding heat trace and zeta function expansions as well as an analytic index formula.

Keywords Pseudodifferential operators · Manifolds with conical singularities · Resolvents · Heat kernels · Zeta functions · Analytic index formulas

Mathematics Subject Classification (2000) Primary 58J35; Secondary 58J40 · 58J37 · 58J20

1 Introduction

In this paper we study the structure and asymptotic behavior of the resolvent of elliptic cone *pseudodifferential* operators acting on weighted Sobolev spaces over a compact manifold with boundary. Our results complete (and contain) the existing descriptions of the resolvent of a cone differential operator (on Sobolev spaces), and provide a first account on the structure of resolvents, heat kernels, and complex powers of pseudodifferential operators on manifolds with conic singularities.

Resolvent and heat kernel asymptotics on conic manifolds have been studied by many authors since the seminal papers by Cheeger [7,8]. For certain classes of first and second order symmetric operators there are contributions by Callias [5], Callias and Uhlmann [6], Brüning and Seeley [2,4], and Mooers [33], to mention just a few. Lesch [22] generalized the

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techniques of Brüning and Seeley and obtained more general results for selfadjoint differential operators of arbitrary order.

Following Schulze's approach for the study of operators on manifolds with edges, see e.g. [36], the first author developed a parameter-dependent calculus (cf. [14]) that describes the resolvent of an elliptic cone differential operator that is not necessarily selfadjoint. In particular, he introduced the appropriate notion of parameter-dependent ellipticity that guarantees the existence of the resolvent and provides good norm estimates. In Sect. 4 we will show that this ellipticity condition is not only sufficient but also necessary. Later in [24–26], following Melrose's approach [29], the second author studied the resolvent of elliptic cone differential operators from a more geometric viewpoint. To this end, he developed a parameter-dependent calculus that gives a precise description of the Schwarz kernel of the resolvent, providing a more convenient framework to analyze heat kernels, zeta functions, and other geometric invariants, see e.g. [15].

In the setting of resolvents of close extensions of a cone differential operator, there are recent results by Schrohe and Seiler [35], Falomir et al. [11], and Falomir et al. [10]. More recently, Gil et al. [18] proved the existence of the resolvent and sectors of minimal growth for the closed extensions of a general cone differential operator. To the best of our knowledge, resolvents of elliptic cone pseudodifferential operators have not been studied before in any setting.

In this work we consider a cone pseudodifferential operator $A \in x^{-\mu}\Psi_b^{\mu}(M)$, where M is a compact manifold with boundary, x is a boundary defining function for ∂M, μ is a positive real number, and $\Psi_b^{\mu}(M)$ is the class of b-pseudodifferential operators of order μ, as introduced by Melrose. Our main goal is to give a precise description of the resolvent $(A-\lambda)^{-1}$ when A satisfies the aforementioned parameter-dependent ellipticity on a sector $\Lambda \subset \mathbb{C}$. We obtain an asymptotic expansion in λ as $|\lambda| \to \infty$, and use it to derive heat trace asymptotics and zeta function expansions. For this purpose, we extend the existing pseudodifferential calculi introduced in [24,25] and define two new classes of operators arising in the parametrix construction used to analyze the resolvent.

As in the case of a differential operator, the construction of a good parameter-dependent parametrix of $A-\lambda$ is crucial to describe the fine structure of the resolvent and its asymptotic behavior in λ. However, when the given operator is not differential but rather a genuine pseudodifferential operator, for instance $\sqrt{\Delta}$, the parametrix construction requires a more delicate analytic treatment. The general idea is to design a parameter-dependent pseudodifferential calculus tailoring the new features of the operators into the geometry of their Schwartz kernels.

To illustrate the main technical difficulty in the parametrix construction for the operator family $A-\lambda$, let us discuss the related (but much simpler) situation of an operator in the b-calculus. Given a parameter-elliptic b-differential operator A, one can construct a parametrix $B(\lambda)$ of $A-\lambda$ such that

$$(A-\lambda)B(\lambda) = 1 + R(\lambda), \tag{1.1}$$

where $R(\lambda)$ is in the calculus with bounds, of order $-\infty$, vanishing to infinite order as $|\lambda| \to \infty$ in Λ. For a b-pseudodifferential operator, the error term $R(\lambda)$ in (1.1) can only be made to vanish to order -1 in the calculus with bounds. Nonetheless, this decay already implies that $R(\lambda) \to 0$ as $|\lambda| \to \infty$, thus $1 + R(\lambda)$ can be inverted for large λ, and consequently, the resolvent exists and belongs to the calculus.

However, when A is a cone pseudodifferential operator, the additional weight factor $x^{-\mu}$ in A makes the situation more complicated: One can obtain an expression similar to (1.1), but the boundary defining function x, the spectral parameter λ, and the bounds, are all coupled in a way that the operator $1 + R(\lambda)$ is unfortunately *not* invertible even for large λ. A novelty of this paper is the development of two new parameter-dependent calculi with bounds which incorporate the coupling of the boundary defining function, the spectral parameter, and the bounds. We introduce these operator classes and show the corresponding composition theorems. This will allow us to further modify $R(\lambda)$ and get a true residual term that decays as $|\lambda| \to \infty$, so that $1 + R(\lambda)$ can be inverted within the calculus.

Once the resolvent of an elliptic cone pseudodifferential operator is understood, we use its structure to study the corresponding heat kernels and complex powers. In particular, the short-time asymptotic expansion of the heat trace obtained in this paper is used to get part of an analytic index formula consisting of two terms; a term coming from the heat trace asymptotics of an associated operator with no boundary spectrum, and a second term that resembles the eta invariant. This formula relies on an index formula by Piazza [34] and on a factorization theorem proposed by Schulze and proved by Witt [42].

We now outline the content of this paper. We begin in Sect. 2 by reviewing various conormal spaces of functions on manifolds with corners as introduced in Melrose's seminal paper [28]. With this background, in Sect. 3 we define and discuss the new parameter-dependent pseudodifferential calculi that are needed in Sect. 4 to construct a good parametrix for a parameter-elliptic cone pseudodifferential operator. In Sect. 5 we use the structure of these calculi to obtain resolvent, heat kernel, and zeta function expansions. Finally, in Sect. 6, we discuss the index of the closure of an elliptic cone operator. The material of this section is a version of [17].

2 Manifolds with corners, asymptotics, and *b*-operators

An n-dimensional manifold with corners Z is a topological space with C^∞ structure given by local charts of the form $[0, 1)^k \times (-1, 1)^{n-k}$, where k can run between 0 and n depending on where the chart is located in the manifold. Each boundary hypersurface H is embedded and has a globally defined boundary defining function; a nonnegative function in $C^\infty(Z)$ that vanishes only on H where it has a nonzero differential.

Asymptotic expansions Let $\mathcal{U} = [0, 1)_x^k \times (-1, 1)_y^{n-k}$. Then for $a \in \mathbb{R}^k$ the space of symbols $\Sigma^a(\mathcal{U})$ consists of those smooth functions $u \in C^\infty(\mathring{\mathcal{U}})$ of the form

$$u(x, y) = x_1^{a_1} \cdots x_k^{a_k}\, v(x, y),$$

where for each α and β, $(x\partial_x)^\alpha \partial_y^\beta v(x, y)$ is a bounded function.

Let $\mathbb{N}$ be the set of positive integers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An *index set* E is a discrete subset of $\mathbb{C} \times \mathbb{N}_0$ such that

- $(z, k) \in E \Rightarrow (z, \ell) \in E$ for all $0 \le \ell \le k$, and
- given any $N \in \mathbb{R}$, the set $\{(z, k) \in E \mid \Re z \le N\}$ is finite.

If in addition, $(z, k) \in E \Rightarrow (z + \ell, k) \in E$ for all $\ell \in \mathbb{N}_0$, then E is called a C^∞ index set. For simplicity, we will use the words "index set" instead of "C^∞ index set" unless stated otherwise. A discrete subset $D \subset \mathbb{C}$ will be referred to as an index set by means of the identification $D \cong \{(z, 0) \mid z \in D\}$.

Given an index set E, a function $u \in \Sigma^a(\mathcal{U})$ is said to have an *asymptotic expansion* at $x_1 = 0$ with index set E if, for each $N > 0$,

$$u(x, y) = \sum_{(z,k)\in E,\, \Re z \le N} x_1^z (\log x_1)^k\, u_{(z,k)}(x', y) \;+\; x_1^N\, u_N(x, y) \tag{2.1}$$

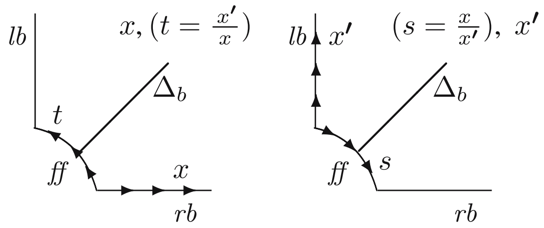

Fig. 1 Each of these coordinates together with coordinates on Y^2 define projective coordinates on M_b^2 near *ff*

with $u_N(x, y) \in \Sigma^a(\mathcal{U})$ and $u_{(z,k)}(x', y) \in \Sigma^{a'}(\mathcal{U}')$, where $a = (a_1, a')$, $x = (x_1, x')$, and $\mathcal{U}' = [0, 1)_{x'}^{k-1} \times (-1, 1)_y^{n-k}$. Furthermore, given $\kappa > 0$, the function u is said to have a *partial expansion* at $x_1 = 0$ with index set E of order κ if u admits the expansion (2.1) for all $N \le \kappa$. In fact, it is sufficient to check that (2.1) holds for $N = \kappa$. Observe that a function has an asymptotic expansion at $x_1 = 0$ with index set E if and only if it has a partial expansion at $x_1 = 0$ with index set E of any order $\kappa > 0$. Note also that if $E = \varnothing$, then the expansion property (2.1) holds for $N = \kappa$ if and only if u vanishes to order κ at $x_1 = 0$. Asymptotic and partial asymptotic expansions at any other boundary $x_i = 0$ are defined similarly.

On a manifold with corners Z one can define asymptotic expansions at a hypersurface H with index set E by reference to local coordinates. First of all, a function $u \in C^\infty(\mathring{Z})$ is said to be in $\Sigma^0(Z)$, if for any patch $\mathcal{U}$ on Z and for any $\varphi \in C_c^\infty(\mathcal{U})$, the function φu is an element of $\Sigma^0(\mathcal{U})$. Let $H_1, \dots, H_m$ be the hypersurfaces of Z with corresponding boundary defining functions $\rho_1, \dots, \rho_m$. For $a \in \mathbb{R}^m$ we define

$$\Sigma^a(Z) = \{\rho_1^{a_1} \cdots \rho_m^{a_m}\, v \mid v \in \Sigma^0(Z)\}.$$

A function $u \in \Sigma^a(Z)$ has a partial expansion at H with index set E of order κ, if for any patch $\mathcal{U} = [0, 1)_{x_1} \times \mathcal{U}'$ on Z with $H \cap \mathcal{U} = \{x_1 = 0\}$, and for any $\varphi \in C_c^\infty(\mathcal{U})$, the function φu has a partial expansion at $x_1 = 0$ with index set E of order κ in the sense described above.

If $\mathcal{E}$ is a collection of index sets $\mathcal{E} = \{E_{H_1}, \dots, E_{H_\ell}\}$ corresponding to some family of hypersurfaces $H_1, \dots, H_\ell$ of Z, then we denote by $\mathcal{A}_\kappa^{\mathcal{E}}(Z)$ the space of functions $u \in \Sigma^a(Z)$ for some $a \in \mathbb{R}^m$ such that for each H, u has a partial expansion at H with index set E_H of order κ. Finally, we define

$$A^{\mathcal{E}}(Z) = \bigcap_{\kappa > 0} \mathcal{A}_\kappa^{\mathcal{E}}(Z).$$

Blow-up and pseudodifferential operators Let M be a smooth manifold of dimension n with boundary $Y = \partial M$. Then the product $M^2 = M \times M$ is a manifold with corners in the above sense. The blow-up $M_b^2 = [M^2; Y^2]$ of M^2 along Y^2 (cf. [29]) is then a new manifold with corners that has an atlas consisting of the usual coordinate patches on $M^2 \setminus Y^2$ together with polar coordinate patches over Y^2 in M^2. For instance, if $M^2 = [0, \infty)_x \times [0, \infty)_{x'}$, then M_b^2 is the set $[0, \infty)_r \times (\mathbb{S}^1 \cap M^2)_\theta$ with $(r, \theta) = (\|(x, x')\|, \tan^{-1}(x'/x))$. In this paper we will work with the more convenient projective coordinates $(x, x') \mapsto (x, t)$ with $t = x'/x$, or $(x, x') \mapsto (s, x')$ with $s = x/x'$. The boundary hypersurfaces *lb*, *rb*, and *ff* (for "left boundary", "right boundary", and "front face", respectively) of M_b^2 together with the projective coordinates are shown in Fig. 1.

Henceforth we fix a b-measure $\mathfrak{m}$ on M and we denote by $\mathfrak{m}'$ the lift of $\mathfrak{m}$ to M^2 under the right projection $M^2 \ni (x, x') \mapsto x' \in M$.

Definition 2.2 For $\mu \in \mathbb{R}$, the space $\Psi_b^\mu(M)$ of b-pseudodifferential operators consists of operators A on $C^\infty(M)$ that have a Schwartz kernel K_A satisfying the following two conditions:

- Given $\varphi \in C_c^\infty(M_b^2 \setminus \Delta_b)$, the kernel φK_A is of the form $k \cdot \mathfrak{m}'$, where k is a smooth function on M_b^2 that vanishes to infinite order at the boundaries *lb* and *rb*.
- Given a coordinate patch of M_b^2 overlapping Δ_b of the form $\mathcal{U}_y \times \mathbb{R}_z^n$ such that $\Delta_b \cong \mathcal{U} \times \{0\}$, and given $\varphi \in C_c^\infty(\mathcal{U} \times \mathbb{R}^n)$, we have

$$(\varphi K_A) = \int e^{iz\cdot\xi}\, a(y,\xi)\, đ\xi \cdot \mathfrak{m}',$$

  where $y \mapsto a(y, \xi)$ is smooth with values in $S_{c\ell}^\mu(\mathbb{R}^n)$; the space of classical symbols of order μ.

The space $\mathrm{Diff}_b^m(M)$ of totally characteristic differential operators of order m is clearly contained in $\Psi_b^m(M)$.

3 Parametric pseudodifferential calculus

The spaces of parametric symbols and pseudodifferential operators discussed in this section are intended to describe operator families of the form $B(A-\lambda)^{-1}$, where A and B are both cone pseudodifferential operators on an compact manifold M, and λ is a spectral parameter living on a sector $\Lambda \subset \mathbb{C}$. Our symbol calculus is somewhat related to the weakly parametric calculus from Grubb and Seeley [20], see also [41].

Symbol spaces For μ, $p \in \mathbb{R}$ and $d > 0$ we define $S^{\mu,p,d}(\mathbb{R}^n; \Lambda)$ as the space of functions $a \in C^\infty(\mathbb{R}^n \times \Lambda)$ such that

$$|\partial_\xi^\alpha \partial_\lambda^\beta a(\xi,\lambda)| \le C_{\alpha\beta}(1+|\xi|)^{\mu-p-|\alpha|}(1+|\xi|+|\lambda|^{1/d})^{p-d|\beta|}.$$

The space $S_r^{\mu,p,d}(\mathbb{R}^n; \Lambda)$, $p/d \in \mathbb{Z}$, consists of elements $a \in S^{\mu,p,d}(\mathbb{R}^n; \Lambda)$ such that if we set

$$\tilde{a}(\xi, z) := z^{p/d} a(\xi, 1/z),$$

then $\tilde{a}(\xi, z)$ is smooth at $z = 0$, and

$$|\partial_\xi^\alpha \partial_z^\beta \tilde{a}(\xi,z)| \le C_{\alpha\beta}(1+|\xi|)^{\mu-p-|\alpha|+d|\beta|}(1+|z||\xi|^d)^{p/d-|\beta|} \tag{3.1}$$

uniformly for $|z| \le 1$. Further let $S_{r,c\ell}^{\mu,p,d}(\mathbb{R}^n; \Lambda)$ be the space of elements $a \in S_r^{\mu,p,d}(\mathbb{R}^n; \Lambda)$ that, for every $N \in \mathbb{N}$, admit a decomposition

$$a(\xi,\lambda) = \sum_{j=0}^{N-1} \chi(\xi) a_{\mu-j}(\xi,\lambda) + r_N(\xi,\lambda), \tag{3.2}$$

where $r_N \in S_r^{\mu-N,p,d}(\mathbb{R}^n; \Lambda)$, $\chi \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) = 0$ for $|\xi| \le \frac{1}{2}$ and $\chi(\xi) = 1$ for $|\xi| \ge 1$, and where each $a_{\mu-j}(\xi, \lambda)$ has the following properties:

- $a_{\mu-j}(\delta\xi, \delta^d\lambda) = \delta^{\mu-j} a_{\mu-j}(\xi, \lambda)$ for every $\delta > 0$,
- $z^{p/d} a_{\mu-j}(\xi, 1/z)$ is smooth at $z = 0$.

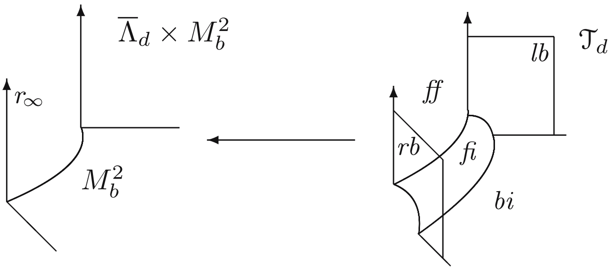

Fig. 2 The manifold $\mathcal{T}_d$ near infinity. Here, $r_\infty = 0$ defines the boundary at $|\lambda| = \infty$

Example 3.3 Let $a(\xi)$ be a homogeneous function in $\xi \in \mathbb{R}^n$ of degree $\mu \in \mathbb{R}$ that never takes values in a sector Λ for $\xi \neq 0$, and let $b(\xi)$ be a homogeneous function of degree $\mu' \in \mathbb{R}$. Given $\ell \in \mathbb{N}_0$, set

$$q(\xi, \lambda) = b(\xi)(a(\xi) - \lambda)^{-\ell}.$$

Then, $\chi(\xi)\, q(\lambda, \xi) \in S^{\mu'-\ell\mu, -\ell\mu, \mu}_{r,c\ell}(\mathbb{R}^n; \Lambda)$. Here, the cut-off function $\chi(\xi)$ is needed because $a(\xi)$ and $b(\xi)$ are, in general, not smooth at $\xi = 0$.

Parameter-dependent operators We first review some spaces of parameter-dependent pseudodifferential operators used to capture resolvents of cone differential operators (see [24,25]). Henceforth we shall fix a boundary defining function ϱ for *ff*. Recall that $\mathfrak{m}'$ denotes the fixed b-measure $\mathfrak{m}$ lifted to M^2 under the right projection $M^2 \ni (x, x') \mapsto x' \in M$.

Definition 3.4 Given $\mu, p, d \in \mathbb{R}$ with $p/d \in \mathbb{Z}$ and $d > 0$, the space $\Psi^{\mu,p,d}_c(M; \Lambda)$ consists of parameter-dependent operators $A(\lambda)$ that have a Schwartz kernel $K_{A(\lambda)}$ satisfying the following two conditions:

- Given $\varphi \in C^\infty_c(M^2_b \setminus \Delta_b)$, the kernel $\varphi K_{A(\lambda)}$ is of the form $k(\varrho^d\lambda, q) \cdot \mathfrak{m}'$, where $k(\lambda, q)$ is a smooth function of $(\lambda, q) \in \Lambda \times M^2_b$ that vanishes to infinite order in q at the sets *lb* and *rb*, and is such that if we define $\widetilde{k}(z, q) = z^{p/d} k(1/z, q)$, then $\widetilde{k}(z, q)$ is smooth at $z = 0$.
- Given a coordinate patch of M^2_b overlapping Δ_b of the form $\mathcal{U}_y \times \mathbb{R}^n_\zeta$ such that $\Delta_b \cong \mathcal{U} \times \{0\}$, and given $\varphi \in C^\infty_c(\mathcal{U} \times \mathbb{R}^n)$, we have

$$\varphi K_{A(\lambda)} = \int e^{i\zeta\cdot\xi}\, a(y, \xi, \varrho^d\lambda)\, đ\xi \cdot \mathfrak{m}',$$

where $y \mapsto a(y, \xi, \lambda)$ is smooth with values in $S^{\mu,p,d}_{r,c\ell}(\mathbb{R}^n; \Lambda)$.

Let $[\Lambda; \{0\}]$ be the sector Λ blown-up at at the origin; that is, Λ with polar coordinates taken at $\lambda = 0$, let $\overline{\Lambda}$ denote the stereographic compactification of $[\Lambda; \{0\}]$ in the Riemann sphere. Coordinates on $\overline{\Lambda}$ near the blown-up origin are $\rho_0 = |\lambda|$ and $\theta = \lambda/|\lambda|$; near $\lambda = \infty$ the coordinates are $\rho_\infty = |\lambda|^{-1}$ and $\theta = \lambda/|\lambda|$. Let $d > 0$ and let $\overline{\Lambda}_d = \{\lambda^{1/d} \mid \lambda \in \overline{\Lambda}\}$ so that the radial coordinates on $\overline{\Lambda}_d$ are $r_0 = |\lambda|^{1/d}$ near the origin and $r_\infty = |\lambda|^{-1/d}$ near infinity.

We consider (see Fig. 2)

$$\mathcal{T}_d := [\overline{\Lambda}_d \times M^2_b; \{r_\infty = 0\} \times ff_b], \tag{3.5}$$

the blow-up of $\overline{\Lambda}_d \times M^2_b$ along $\{r_\infty = 0\} \times ff_b$, where ff_b is the front face of M^2_b.

The blown-up manifold $\mathcal{T}_d$ has eight boundary hypersurfaces, five of which are illustrated in Fig. 2, namely, *fi* (face at infinity), *bi* (boundary at infinity), and the three hypersurfaces *lb*, *rb*, and *ff*, induced by the corresponding boundaries of the manifold M_b^2. The other three hypersurfaces are $\{r_0 = 0\}$ and the endpoints of the angular variable θ. Because we are interested in asymptotics for $|\lambda|$ near infinity, these three hypersurfaces will play only minor roles.

Assumption 3.6 From now on, all functions depending on λ, either implicitly (as functions on $\mathcal{T}_d$, for instance) or explicity (as functions on $\overline{\Lambda}$), are assumed to be smooth in $\theta = \lambda/|\lambda|$ and symbols of order zero at $\{r_0 = 0\}$.

We now use the notation from Sect. 2 to describe the various residual classes of pseudodifferential operators with asymptotics that appear in the parametrix construction of parameter-dependent elliptic operators.

Definition 3.7 Let $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff}, E_{fi}, \varnothing)$ be an index family for $\mathcal{T}_d$ associated to the faces (lb, rb, ff, fi, bi). We denote by $\Psi_c^{-\infty,d,\mathcal{E}}(M;\Lambda)$ the space of those parameter-dependent operators $A(\lambda)$ that have a Schwartz kernel of the form $K_{A(\lambda)} = k \cdot \mathfrak{m}'$ where $k \in \mathcal{A}^{\mathcal{E}}(\mathcal{T}_d)$. Thus k defines a function on $\mathcal{T}_d$ that vanishes to infinite order at *bi* and have asymptotic expansions at the hypersurfaces *lb*, *rb*, *ff*, and *fi*, determined by the index sets E_{lb}, E_{rb}, E_{ff}, and E_{fi}, respectively.

Two new parameter-dependent residual classes In order to capture the resolvents of elliptic *pseudodifferential* operators we need to introduce two new classes of smoothing operators satisfying only conormal bounds. We begin by recalling the calculus with bounds (cf. [29, Sect. 5.16]).

Definition 3.8 Let $\alpha > 0$ and let $\mathcal{E} = (\varnothing, \varnothing, \mathbb{N}_0)$ be the index family on M_b^2 associating the empty sets to its left and right boundaries, and $\mathbb{N}_0$ to its front face. The space $\Psi_b^{-\infty,\alpha}(M)$ denotes the class of operators A having a Schwartz kernel of the form $K_A = k \cdot \mathfrak{m}'$, with $k \in \mathcal{A}^{\mathcal{E}}_{\alpha+\varepsilon}(M_b^2)$ for some $\varepsilon > 0$. More precisely, if ρ_l and ρ_r are boundary defining functions for the left and right boundaries of M_b^2, then the function $\rho_l^{-\alpha-\varepsilon}\rho_r^{-\alpha-\varepsilon} k$ is a symbol in $\Sigma^0(M_b^2)$ having a partial expansion at the front face of M_b^2 with index set $\mathbb{N}_0$ of order $\alpha + \varepsilon$.

Definition 3.9 Let $N \in \mathbb{N}$ and $d > 0$. For $m \in \mathbb{N}$ we define $\Psi_{m,N}^{-\infty,d}(M;\Lambda)$ as the space of those parameter-dependent operators $A(\lambda)$ whose Schwartz kernel $K_{A(\lambda)}$ is of the form $k(\varrho^d\lambda, q)\cdot\mathfrak{m}'$ with $k(\lambda, q)$ satisfying the following properties:

(a) For some $\varepsilon > 0$, $\rho_l^{-Nd-\varepsilon}\rho_r^{-Nd-\varepsilon}k$ is a symbol in $\Sigma^0(\overline{\Lambda} \times M_b^2)$ having a partial expansion at the face $\overline{\Lambda} \times ff$ with index set $\mathbb{N}_0$ of order $Nd + \varepsilon$. Again, ρ_l and ρ_r are boundary defining functions for *lb* and *rb* in M_b^2,

(b) For each $N' \leq N$,

$$k(\lambda, q) = \sum_{j=m}^{N'-1} \lambda^{-j} f_j(q) + \lambda^{-N'} k_{N'}(\lambda, q),$$

where $f_j \in \mathcal{A}^{\mathcal{E}}_{2Nd-jd}(M_b^2)$ with $\mathcal{E} = (\varnothing, \varnothing, \mathbb{N}_0)$, and $k_{N'}$ satisfies (a) with Nd replaced by $2Nd - N'd$. If $m \geq N$, then we disregard the summation and require instead $k(\lambda, q) = \lambda^{-N} k_N(\lambda, q)$, where k_N satisfies (a).

The next lemma relates the two parameter-dependent spaces introduced in Definitions 3.4 and 3.9; the proof follows immediately from the definitions.

Lemma 3.10 *If $p/d \in -\mathbb{N}$, then for any $N \in \mathbb{N}$,*

$$\Psi_c^{-\infty,p,d}(M;\Lambda) \subset \Psi_{m,N}^{-\infty,d}(M;\Lambda), \quad m = -p/d.$$

Our next space of operators is a calculus with bounds version of the space $\Psi_c^{-\infty,d,\mathcal{E}}(M;\Lambda)$ introduced in Definition 3.7.

Definition 3.11 Let $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff}, E_{fi})$ be an index family for $\mathcal{T}_d$ associated to the faces (lb, rb, ff, fi). Then we define $\Psi_N^{-\infty,d,\mathcal{E}}(M;\Lambda)$ as those parameter-dependent operators $A(\lambda)$ that have a Schwartz kernel of the form $K_{A(\lambda)} = k \cdot \mathfrak{m}'$, where k is a symbol on $\mathcal{T}_d$, of order Nd at bi, that satisfies:

- Given $\varphi \in C^\infty(\mathcal{T}_d)$ supported near *fi*, φk is in $\mathcal{A}^{\mathcal{E}}_{Nd+\varepsilon}(\mathcal{T}_d)$ for some $\varepsilon > 0$.
- Given $\psi \in C^\infty(\mathcal{T}_d)$ supported away from *fi*, ψk is the kernel of a parameter-dependent operator in $\Psi_{N,N}^{-\infty,d}(M;\Lambda)$.

Observe that

$$\Psi_c^{-\infty,d,\mathcal{E}}(M;\Lambda) = \bigcap_{N\in\mathbb{N}} \Psi_N^{-\infty,d,\mathcal{E}}(M;\Lambda).$$

Lemma 3.12 *We have*

$$\Psi_{N,N}^{-\infty,d}(M;\Lambda) \subset \Psi_N^{-\infty,d,\mathcal{E}}(M;\Lambda),$$

where $\mathcal{E}$ is the index family on $\mathcal{T}_d$ given by $\mathcal{E} = (\varnothing, \varnothing, \mathbb{N}_0, \mathbb{N}_0)$.

Proof Let $A(\lambda) \in \Psi_{N,N}^{-\infty,d}(M;\Lambda)$ and let $K_{A(\lambda)} = k(\varrho^d\lambda, q)\,\mathfrak{m}'$ be its kernel with all the properties described in Definition 3.9. In particular, the operator $\widetilde{A}(\lambda)$ with kernel $k(\lambda, q)\,\mathfrak{m}'$ is such that $\lambda^N \widetilde{A}(\lambda)$ belongs to $\Sigma^0(\overline{\Lambda}, \Psi_b^{-\infty,Nd}(M))$. By definition, we only need to analyze $k(\varrho^d\lambda, q)$ locally in coordinates near the face *fi*. By symmetry it suffices to consider the kernel only away from one of the lateral boundaries of M_b^2; for instance, away from the left boundary *lb*. Since our kernels are smooth in $\theta = \lambda/|\lambda|$ and in the variables on the boundary, we shall omit these variables in what follows. Thus consider the coordinates $q = (x,t) \in \mathcal{U} \subset M_b^2$, with x defining *ff* and $t = x'/x$ defining *rb*, see Fig. 1. If $\rho = |\lambda|$, then for some $\varepsilon > 0$ we can write

$$k(\lambda, q) = t^{Nd+\varepsilon} g(\rho, x, t),$$

where g is a symbol in $\Sigma^0(\mathbb{R}_+ \times \mathcal{U})$ that can be expanded in x at $x = 0$ with index set $\mathbb{N}_0$ of order $Nd + \varepsilon$. In particular, k has a partial asymptotic expansion at $t = 0$ with index set $E_{rb} = \varnothing$ of order $Nd + \varepsilon$.

We now lift $k(x^d\lambda, q)$ to $\mathcal{T}_d$. Near *ff* and *fi*, the variable $r = \rho^{-1/d}$ defines *fi* and $v = x/r$ defines *ff*, and in these coordinates,

$$k(x^d\lambda, q) = t^{Nd+\varepsilon} g(v^d, rv, t).$$

The asymptotic properties of g imply that $g(v^d, rv, t)$ can be expanded in r and v with index set $\mathbb{N}_0$ of order $Nd + \varepsilon$. On the other hand, near *fi* and bi, x defines *fi* and $w = v^{-1}$ defines bi, and in these coordinates,

$$k(x^d\lambda, q) = t^{Nd+\varepsilon} g(w^{-d}, x, t).$$

Now, since $\lambda^N \widetilde{A}(\lambda) \in \Sigma^0(\overline{\Lambda}, \Psi_b^{-\infty,Nd}(M))$, the function g can actually be written as $g(\rho, x, t) = \rho^{-N} h(\rho, x, t)$, where h is a symbol in $\Sigma^0(\mathbb{R}_+ \times \mathcal{U})$. Therefore,

$$k(x^d\lambda, q) = t^{Nd+\varepsilon} w^{Nd} h(w^{-d}, x, t).$$

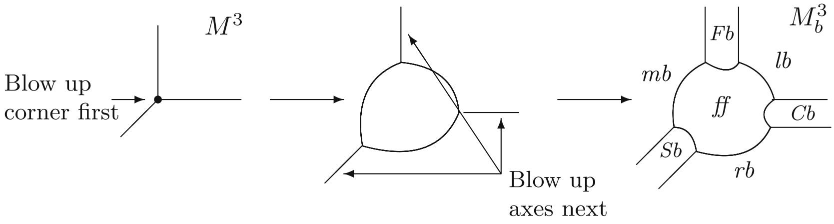

Fig. 3 The blown-up manifold M_b^3 and its boundary hypersurfaces

The asymptotic properties of g imply that $h(w^{-d}, x, t)$ can be expanded in x at $x = 0$ with index set $\mathbb{N}_0$ of order $Nd + \varepsilon$. In conclusion, we have proven that k defines a function on $\mathcal{T}_d$ that vanishes to order Nd at bi and has partial expansions of order $Nd + \varepsilon$ with index sets $E_{rb} = \varnothing$, $E_{ff} = \mathbb{N}_0$ and $E_{fi} = \mathbb{N}_0$. The same is true with rb replaced by lb, thus $A(\lambda) \in \Psi_N^{-\infty,d,\mathcal{E}}(M; \Lambda)$ with $\mathcal{E} = (\varnothing, \varnothing, \mathbb{N}_0, \mathbb{N}_0)$. □

Composition of pseudodifferential operators In order to prove essential composition properties of our new parameter-dependent spaces, we need to review how b-pseudodifferential operators are composed. Let A and B be operators on $C^\infty(M)$ with Schwartz kernels K_A and K_B, respectively, that are smooth on M^2 and vanish to infinite order at the boundary $Y^2 = \partial M^2$. Then we know that AB is also a smoothing operator, and

$$K_{AB}(u, w) = \int_{v \in M} K_A(u, v) K_B(v, w). \tag{3.13}$$

We can write this purely in terms of pullbacks and pushforwards of distributions as follows. Let $\pi_F, \pi_S, \pi_C : M^3 \to M^2$ be the maps

$$\pi_F(u, v, w) = (u, v), \quad \pi_S(u, v, w) = (v, w), \quad \pi_C(u, v, w) = (u, w)$$

(F, S, and C stand for 'first', 'second', and 'composite'). Writing $K_A = k_A\, \mathfrak{m}'$ and $K_B = k_B\, \mathfrak{m}'$, where k_A and k_B are smooth functions on M^2 vanishing to infinite order at the boundary Y^2, we have

$$(\pi_C^* \mathfrak{m}\, \pi_F^* K_A\, \pi_S^* K_B)(u, v, w) = k_A(u, v)\, k_B(v, w)\, \mathfrak{m}(u)\mathfrak{m}(v)\mathfrak{m}(w),$$

where on the left-hand side, $\mathfrak{m}$ represents the fixed b-measure on M lifted to M^2 under the left projection, that is, $\mathfrak{m}(u, w) = \mathfrak{m}(u)$ for all $(u, w) \in M^2$. In particular, $\pi_C^* \mathfrak{m}\, \pi_F^* K_A\, \pi_S^* K_B$ is a density on M^3 and so its pushforward to M^2 under π_C is well-defined. By (3.13) and the definition of the pushforward $(\pi_C)_*$ we get

$$\mathfrak{m}\, K_{AB} = (\pi_C)_* (\pi_C^* \mathfrak{m}\, \pi_F^* K_A\, \pi_S^* K_B). \tag{3.14}$$

This identity shows that we can determine the Schwartz kernel of AB by analyzing pullbacks, products, and pushforwards of the kernels K_A and K_B. Now, since our operators are actually in $\Psi_b^*(M)$, in order to get a similar identity for the Schwartz kernel of AB, we first introduce the blown-up manifold M_b^3.

The manifold M_b^3 is defined by blowing-up (that is, introducing polar coordinates around) the manifold Y^3 in M^3 and then blowing-up the submanifolds coming from the codimension two corners of M^3. The manifold M_b^3 along with its various faces are shown in Fig. 3. Let

$\pi_{F,b}, \pi_{S,b}, \pi_{C,b} : M_b^3 \to M_b^2$ be the maps π_F, π_S, π_C expressed in the polar coordinates of M_b^3 and M_b^2. Then we can express the composition (3.14) in terms of these new maps:

$$\mathfrak{m}\, K_{AB} = (\pi_{C,b})_*(\pi_{C,b}^* \mathfrak{m}\, \pi_{F,b}^* K_A\, \pi_{S,b}^* K_B). \tag{3.15}$$

Written in this way, $\mathfrak{m}$, K_A, K_B, and K_{AB} are understood to be lifted to M_b^2. The formula (3.15) is the key to proving composition properties of our parameter-dependent operators.

Composition theorems for parameter-dependent operators We begin by stating a composition result whose proof is almost exactly the same as the proof of [24, Theorem 4.4], so we leave out the details.

Theorem 3.16 *We have*

$$x^\nu \Psi_c^{\mu,p,d}(M;\Lambda) \circ x^{\nu'} \Psi_c^{\mu',p',d}(M;\Lambda) \subset x^{\nu+\nu'} \Psi_c^{\mu+\mu',p+p',d}(M;\Lambda).$$

The following theorem is proved in [25, Proposition 4.1].

Theorem 3.17 *We have*

$$x^\nu \Psi_b^{\mu}(M) \circ x^{\nu'} \Psi_c^{\mu',p,d}(M;\Lambda) \subset x^{\nu+\nu'} \Psi_c^{\mu+\mu',p,d}(M;\Lambda);$$

and

$$x^{\nu'} \Psi_c^{\mu',p,d}(M;\Lambda) \circ x^\nu \Psi_b^{\mu}(M) \subset x^{\nu+\nu'} \Psi_c^{\mu+\mu',p,d}(M;\Lambda).$$

This theorem states that the parameter-dependent spaces $\Psi_c^{*,p,d}(M;\Lambda)$ are closed under composition with cone pseudodifferential operators. We next consider how these spaces behave under composition with the calculus with bounds and their parameter-dependent versions. The next theorem is established by following the proof of [25, Proposition 4.2], taking into account the bounds. To avoid reproducing the proof of loc. cit., we shall omit the details.

Theorem 3.18 *Let $\nu, \mu \in \mathbb{R}$ and $d, m, N \in \mathbb{N}$. Let $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff}, E_{fi})$.*

(i) $\Psi_{m,N}^{-\infty,d}(M;\Lambda)$ *is closed under compositions with* $x^\nu \Psi_b^\mu(M)$.

(ii) *If $B \in x^\nu \Psi_b^\mu(M)$ and $R \in \Psi_N^{-\infty,d,\mathcal{E}}(M;\Lambda)$, then*

$$BR \in x^\nu \Psi_N^{-\infty,d,\mathcal{E}}(M;\Lambda) \quad \text{and} \quad x^{-\nu} R x^\nu \in \Psi_N^{-\infty,d,\mathcal{E}_\nu}(M;\Lambda),$$

where $\mathcal{E}_\nu = (E_{lb} - \nu, E_{rb} + \nu, E_{ff}, E_{fi})$.

(iii) *Let $p/d \in -\mathbb{N}$. For any $N > 0$, we have*

$$\Psi_b^{-\infty,\,2Nd}(M) \circ \Psi_c^{\mu,p,d}(M;\Lambda) \subset \Psi_{m,N}^{-\infty,d}(M;\Lambda),$$
$$\Psi_c^{\mu,p,d}(M;\Lambda) \circ \Psi_b^{-\infty,\,2Nd}(M) \subset \Psi_{m,N}^{-\infty,d}(M;\Lambda),$$

where $m = -p/d$;

$$\Psi_{m',N}^{-\infty,d}(M;\Lambda) \circ \Psi_c^{\mu,p,d}(M;\Lambda) \subset \Psi_{m,N}^{-\infty,d}(M;\Lambda),$$
$$\Psi_c^{\mu,p,d}(M;\Lambda) \circ \Psi_{m',N}^{-\infty,d}(M;\Lambda) \subset \Psi_{m,N}^{-\infty,d}(M;\Lambda),$$

where $m = \min\{m', -p/d\}$.

(iv) $\Psi_N^{-\infty,d,\mathcal{E}}(M;\Lambda)$ *is closed under composition with* $\Psi_c^{\mu,p,d}(M;\Lambda)$*, for instance,*

$$\Psi_c^{\mu,p,d}(M;\Lambda) \circ \Psi_N^{-\infty,d,\mathcal{E}}(M;\Lambda) \subset \Psi_N^{-\infty,d,\mathcal{E}}(M;\Lambda).$$

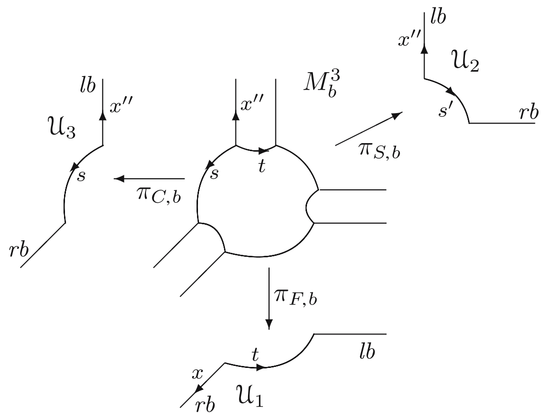

Fig. 4 Projective coordinates and projections on $\mathcal{U}$

We next consider composition in our first new parameter-dependent calculus.

Theorem 3.19 *We have*

$$\Psi^{-\infty,d}_{m,N_1}(M;\Lambda)\circ\Psi^{-\infty,d}_{m',N_2}(M;\Lambda)\subset\Psi^{-\infty,d}_{m+m',N}(M;\Lambda),$$

where $N=\min\{N_1,N_2\}$.

Proof Since $\Psi^{-\infty,d}_{m,N_1}(M;\Lambda)\subset\Psi^{-\infty,d}_{m,N}(M;\Lambda)$ and $\Psi^{-\infty,d}_{m',N_2}(M;\Lambda)\subset\Psi^{-\infty,d}_{m',N}(M;\Lambda)$, which follows from the definition of these spaces, we may assume that $N=N_1=N_2$. Thus given $A\in\Psi^{-\infty,d}_{m,N}(M;\Lambda)$ and $B\in\Psi^{-\infty,d}_{m',N}(M;\Lambda)$, we need to show that $AB\in\Psi^{-\infty,d}_{m+m',N}(M;\Lambda)$. To simplify the notation, we assume that $M=[0,1)_x$ and $\Lambda=\mathbb{R}_+$. The argument in the general case is basically the same, the main difference being the appearance of the tangential variables on Y that make the proof notationally more complicated. We will use projective coordinates (see Fig. 1).

In the following, we denote by x,x',x'' the coordinates on the left, middle, and right factors of $M^3=[0,1)^3$ and we assume that $\mathfrak{m}=|dx/x|$. To show that $AB\in\Psi^{-\infty,d}_{m+m',N}(M;\Lambda)$, we use the formula (3.15) above. To do so, we will assume that the lifted kernel $\pi^*_{C,b}\mathfrak{m}\,\pi^*_{F,b}K_A\,\pi^*_{S,b}K_B$ is supported in a neighborhood $\mathcal{U}\subset M^3_b$ of the intersection of mb, ff, and Fb (see Fig. 3). On $\mathcal{U}$ we introduce projective coordinates as follows. First, we blow-up the origin in M^3 and define, away from the hypersurface $\{x''=0\}$, the coordinates (s,s',x'') with $s=x/x''$ and $s'=x'/x''$. Next, we blow-up the x''-axis to get M^3_b, and define on $\mathcal{U}$ the coordinates (s,t,x'') with $t=s'/s=x'/x$. In conclusion, we get the projective coordinates

$$(s,t,x'')\in\mathcal{U}\quad\text{with}\quad s=\frac{x}{x''}\quad\text{and}\quad t=\frac{x'}{x}. \tag{3.20}$$

By definition, $\pi_{F,b}(s,t,x'')$ is the image of $\pi_F(x,x',x'')=(x,x')$ written in coordinates $(x,x'/x)$ on M^2_b. Similarly, $\pi_{S,b}(s,t,x'')$ is the image of $\pi_S(x,x',x'')=(x',x'')$ written in coordinates $(x'/x'',x'')$. The appropriate choice of projective coordinates on M^2_b for the images of π_F and π_S is illustrated in Fig. 4. By means of (3.20) we finally get

$$\pi_{F,b}(s,t,x'')=(sx'',t),\quad \pi_{S,b}(s,t,x'')=(st,x''). \tag{3.21}$$

Let $\mathcal{U}_1=\pi_{F,b}(\mathcal{U})$. In the coordinates $(x,t)\in\mathcal{U}_1$ (see Fig. 4), the kernel of A is of the form $K_A=k_1(x^d\lambda,x,t)\,|dx'/x'|$, where for some $\varepsilon>0$, $t^{-Nd-\varepsilon}k_1(\lambda,x,t)$ is a symbol in

$\Sigma^0(\mathbb{R}_+ \times \mathcal{U}_1)$ that can be expanded at $x = 0$ with index set $\mathbb{N}_0$ of order $Nd + \varepsilon$. Moreover, for each $N' \leq N$,

$$k_1(\lambda, x, t) = \sum_{j=m}^{N'-1} \lambda^{-j} f_j(x, t) + \lambda^{-N'} k_{1,N'}(\lambda, x, t), \tag{3.22}$$

where all the coefficients satisfy the properties listed in Definition 3.9.

Let $\mathcal{U}_2 = \pi_{S,b}(\mathcal{U})$. In the coordinates $(s', x'') \in \mathcal{U}_2$ (Fig. 4), the kernel of B is of the form $K_B = k_2((x'')^d\lambda, s', x'')\, |dx''/x''|$, where for some $\varepsilon > 0$, $(s')^{-Nd-\varepsilon} k_2(\lambda, s', x'')$ is a symbol in $\Sigma^0(\mathbb{R}_+ \times \mathcal{U}_2)$ that can be expanded at $x'' = 0$ with index set $\mathbb{N}_0$ of order $Nd + \varepsilon$. The function $k_2(\lambda, s', x'')$ also admits an expansion similar to (3.22) with the obvious change of variables. Using the formulas for $\pi_{S,b}$ and $\pi_{F,b}$ in (3.21), it follows that on $\mathcal{U}$,

$$\pi_{C,b}^* \mathfrak{m}\, \pi_{F,b}^* K_A\, \pi_{S,b}^* K_B = k_1((sx'')^d\lambda, sx'', t)\, k_2((x'')^d\lambda, st, x'')\left|\frac{dsdtdx''}{stx''}\right|.$$

Furthermore, $\pi_{C,b}(s, t, x'')$ is the image of $\pi_C(x, x', x'') = (x, x'')$ written in coordinates $(x/x'', x'')$ on M_b^2, thus

$$\pi_{C,b}(s, t, x'') = (s, x'').$$

By the definition of pushforward,

$$(\pi_{C,b})_*(\pi_{C,b}^* \mathfrak{m}\, \pi_{F,b}^* K_A\, \pi_{S,b}^* K_B) = k_3((x'')^d\lambda, s, x'')\left|\frac{dsdx''}{sx''}\right|,$$

where

$$k_3(\lambda, s, x'') = \int k_1(s^d\lambda, sx'', t)\, k_2(\lambda, st, x'')\frac{dt}{t}$$

From the properties of A and B it follows easily that $s^{-Nd-\varepsilon} k_3(\lambda, s, x'')$ is a symbol in $\Sigma^0(\mathbb{R}_+ \times \mathcal{U}_3)$ having a partial expansion at $x'' = 0$ with index set $\mathbb{N}_0$ of order $Nd + \varepsilon$. Moreover, the formula (3.22) corresponding to k_2 (denoting the coefficients by g_j instead of f_j) implies that given $N' \leq N$,

$$k_3(\lambda, s, x'') = \sum_{j=m'}^{N'-1} \lambda^{-j} \int k_1(s^d\lambda, sx'', t)\, g_j(st, x'')\, \frac{dt}{t} + \lambda^{-N'} \int k_1(s^d\lambda, sx'', t)\, k_{2,N'}(\lambda, st, x'')\, \frac{dt}{t}.$$

Now for each j, expanding $k_1(\lambda, x, t)$ in λ up to order $N' - j$, we find that

$$k_3(\lambda, s, x'') = \sum_{j=m+m'}^{N'-1} \lambda^{-j} h_j(s, x'') + \lambda^{-N'} k_{3,N'}(\lambda, s, x''), \tag{3.23}$$

where

$$h_j(s, x'') = \sum_{\ell=m'}^{j-m} s^{-(j-\ell)d} \int f_{j-\ell}(sx'', t)\, g_\ell(st, x'')\frac{dt}{t}$$

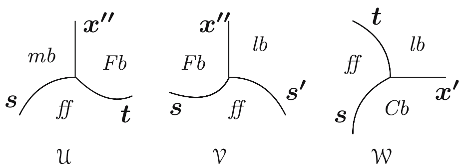

Fig. 5 Three representative coordinate patches on M_b^3

and

$$k_{3,N'}(\lambda, s, x'') = \sum_{j=m'}^{N'-1} s^{-(N'-j)d} \int k_{1,N'-j}(s^d\lambda, s, x'')\, g_\ell(st, x'') \frac{dt}{t} + \int k_1(s^d\lambda, sx'', t)\, k_{2,N'}(\lambda, st, x'') \frac{dt}{t}.$$

It remains to verify that the coefficients in the expansion (3.23) have the properties required in part (b) of Definition 3.9. For the h_j's this follows from the corresponding properties of the functions $f_{j-\ell}$ and g_ℓ. In particular, expanding these functions at $x'' = 0$ according to (2.1) we get the required expansion for h_j. Notice that these expansions are partial expansions with index set $\mathbb{N}_0$ of order $2Nd - jd + \ell d$ for $f_{j-\ell}$ and $2Nd - \ell d$ for g_ℓ, which are both of order greater than $2Nd - jd$. Therefore, the resulting asymptotic expansion for $h_j(s, x'')$ at $x'' = 0$ is of the same type. The properties of $k_{3,N'}(\lambda, s, x'')$ follow in a similar manner. □

Remark 3.24 In the previous proof, we restricted ourselves to a neighborhood $\mathcal{U} \subset M_b^3 = [0, 1)_b^3$ of the intersection of the faces *mb*, *ff*, and Fb (see Fig. 3). But in fact, in this model case, we need six coordinate patches to cover the entire manifold. However, by symmetry, there are only three patches that require slightly different treatments. For instance, we could choose in addition to $\mathcal{U}$ a neighborhood $\mathcal{V}$ of the intersection of Fb, *ff*, and *lb*, and a neighborhood $\mathcal{W}$ of the intersection of *ff*, Cb, and *lb*, to complete a set of representative local coordinates, see Fig. 5. Since the calculations on $\mathcal{V}$ and $\mathcal{W}$ are similar in nature, and in order to avoid an overloading of technical computations, we decided to skip them. Nonetheless, to demonstrate these different treatments without repeating our arguments, in the proofs of Theorems 3.27 and 3.29 we will work on $\mathcal{V}$ and $\mathcal{W}$, respectively.

For index sets E and F, we define the extended union of these sets by

$$E \,\overline{\cup}\, F = E \cup F \cup \{(z, k + \ell + 1) \mid (z, k) \in E,\ (z, \ell) \in F\}. \tag{3.25}$$

Given $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff}, E_{fi})$ and $\mathcal{F} = (F_{lb}, F_{rb}, F_{ff}, F_{fi})$, we define the index family $\mathcal{E}\hat{\circ}\mathcal{F} = (G_{lb}, G_{rb}, G_{ff}, G_{fi})$ as follows:

$$\begin{aligned} G_{lb} &= E_{lb} \,\overline{\cup}\, (E_{ff} + F_{lb}), \quad G_{rb} = (E_{rb} + F_{ff}) \,\overline{\cup}\, F_{rb}, \\ G_{ff} &= (E_{ff} + F_{ff}) \,\overline{\cup}\, (E_{lb} + F_{rb}), \quad \text{and} \quad G_{fi} = E_{fi} + F_{fi}. \end{aligned} \tag{3.26}$$

Our second new parameter-dependent space has the following properties.

Theorem 3.27 *Provided that* $E_{rb} + F_{lb} > 0$, *we have*

$$\Psi_N^{-\infty,d,\mathcal{E}}(M; \Lambda) \circ \Psi_N^{-\infty,d,\mathcal{F}}(M; \Lambda) \subset \Psi_N^{-\infty,d,\mathcal{E}\hat{\circ}\mathcal{F}}(M; \Lambda).$$

Proof Let $A \in \Psi_N^{-\infty,d,\mathcal{E}}(M; \Lambda)$ and $B \in \Psi_N^{-\infty,d,\mathcal{F}}(M; \Lambda)$. We will use the formula (3.15) to show that $AB \in \Psi_N^{-\infty,d,\mathcal{E}\hat{\circ}\mathcal{F}}(M; \Lambda)$. As in the previous proof, we assume that $M = [0, 1)_x$ and $\Lambda = \mathbb{R}_+$. We also use projective coordinates near *lb* on the product $[0, 1)_b^2$ (see Fig. 1).

Let x, x', x'' be the coordinates on the left, middle, and right factors of $[0, 1)^3$ and assume that $\mathfrak{m} = |dx/x|$. In this proof we now assume the lifted kernel $\pi_{C,b}^* \mathfrak{m}\, \pi_{F,b}^* K_A\, \pi_{S,b}^* K_B$ to be supported in a neighborhood $\mathcal{V} \subset M_b^3$ of the intersection of Fb, ff, and lb (see Fig. 3). On $\mathcal{V}$ we may use the coordinates

$$(s, s', x'') \quad \text{with } s = \frac{x}{x'} \quad \text{and} \quad s' = \frac{x'}{x''} \quad \text{(see Fig. 5).}$$

The projections $\pi_{F,b}$, $\pi_{S,b}$, and $\pi_{C,b}$ all map $\mathcal{V}$ onto a neighborhood of lb in M_b^2, and we have

$$\begin{aligned} &\pi_{F,b}(s, s', x'') = (s, s'x''), \quad \pi_{S,b}(s, s', x'') = (s', x''), \\ &\pi_{C,b}(s, s', x'') = (ss', x''). \end{aligned} \tag{3.28}$$

If $r = \lambda^{-1/d}$ and $v' = x'/r$, then near lb in $\mathcal{T}_d$, we can write $K_A = k_1(r, s, v')|dx'/x'|$, where for some $\varepsilon > 0$, $k_1(r, s, v')$ has expansions at $r = 0$, $v' = 0$, and $s = 0$, with index sets E_{fi}, E_{ff}, and E_{lb} of order $Nd+\varepsilon$, respectively, and for $v' \geq 1$, $k_1(r, s, v') = (v')^{-Nd-\varepsilon} \widetilde{k_1}(r, s, v')$ where $\widetilde{k_1}(r, s, v')$ is a symbol of order 0 in v' and has expansions at $r = 0$ and $s = 0$ with index sets E_{fi} and E_{lb} of order $Nd + \varepsilon$, respectively. Similarly, $K_B = k_2(r, s, v')|dx'/x'|$ where $k_2(r, s, v')$ has analogous properties as $k_1(r, s, v')$ but with index sets given by $\mathcal{F}$. Using the formulas in (3.28), it follows that

$$\pi_{C,b}^* \mathfrak{m}\, \pi_{F,b}^* K_A\, \pi_{S,b}^* K_B = k_1(r, s, s'x''/r) k_2(r, s', x''/r) \left| \frac{ds\,ds'\,dx''}{ss'x''} \right|.$$

Hence, as $\pi_{C,b}(s, s', x'') = (ss', x'')$, by the definition of pushforward, we obtain

$$(\pi_{C,b})_* (\pi_{C,b}^* \mathfrak{m}\, \pi_{F,b}^* K_A\, \pi_{S,b}^* K_B) = k_3(r, s, v') \left| \frac{ds\,dx'}{sx'} \right|,$$

where

$$k_3(r, s, v') = \int k_1(r, s/s', s'v') k_2(r, s', v') \frac{ds'}{s'}.$$

Now the asymptotic properties of k_1 and k_2 together with Melrose's pushforward theorem (see [24, Appendix]) imply that $k_3(r, s, v')$ has expansions at $r = 0$, $s = 0$, and $v' = 0$, with index sets $E_{\mathit{fi}} + F_{\mathit{fi}}$, $E_{\mathit{ff}} + F_{\mathit{ff}}$, and $E_{lb} \overline{\cup} (E_{\mathit{ff}} + F_{lb})$ of order $Nd+\varepsilon$, respectively. Moreover, for $v' \geq 1$, $k_3(r, s, v') = (v')^{-Nd-\varepsilon} \widetilde{k_3}(r, s, v')$ where $\widetilde{k_3}(r, s, v')$ is a symbol of order 0 in v' and has expansions at $r = 0$ and $s = 0$ with index sets $E_{\mathit{fi}} + F_{\mathit{fi}}$ and $E_{lb} \overline{\cup} (E_{\mathit{ff}} + F_{lb})$ of order $Nd + \varepsilon$, respectively. □

Finally, we consider the composition of our two new parameter-dependent spaces.

Theorem 3.29 *We have*

$$\begin{aligned} &\Psi_{m,N}^{-\infty,d}(M; \Lambda) \circ \Psi_N^{-\infty,d,\mathcal{E}}(M; \Lambda) \subset \Psi_N^{-\infty,d,\mathcal{E}}(M; \Lambda), \\ &\Psi_N^{-\infty,d,\mathcal{E}}(M; \Lambda) \circ \Psi_{m,N}^{-\infty,d}(M; \Lambda) \subset \Psi_N^{-\infty,d,\mathcal{E}}(M; \Lambda). \end{aligned}$$

Proof Let $A \in \Psi_{m,N}^{-\infty,d}(M; \Lambda)$ and $B \in \Psi_N^{-\infty,d,\mathcal{E}}(M; \Lambda)$. As in our previous proofs, we will use the composition formula (3.15) to show that $AB \in \Psi_N^{-\infty,d,\mathcal{E}}(M; \Lambda)$. Again, we consider $M = [0, 1)_x$, $\Lambda = \mathbb{R}_+$, and introduce the following coordinates on $[0, 1)_b^2$ (see Fig. 1):

$$(s, x') \text{ near } lb, \quad \text{and} \quad (x, t) \text{ near } rb, \quad \text{where } s = \frac{x}{x'} \text{ and } t = \frac{x'}{x}. \tag{3.30}$$

Let x, x', x'' be the coordinates on the left, middle, and right factors of $[0, 1)^3$ and assume that $\mathfrak{m} = |dx/x|$. According to Remark 3.24, in the present proof we will assume that the lifted kernel $\pi^*_{C,b}\mathfrak{m}\,\pi^*_{F,b}K_A\,\pi^*_{S,b}K_B$ is supported in a neighborhood $\mathcal{W}$ of the intersection of *ff*, *Cb*, and *lb* in M^3_b (see Fig. 3). Here, we may use the coordinates $(s, x', t) \in \mathcal{W}$, where $s = x/x''$ and $t = x''/x'$ (see Fig. 5). The projections $\pi_{C,b}$ and $\pi_{F,b}$ map $\mathcal{W}$ onto a neighborhood of *lb* in M^2_b, and $\pi_{S,b}$ maps $\mathcal{W}$ onto a neighborhood of *rb*. Moreover, in the coordinates (3.30) on M^2_b near *lb*, we have

$$\pi_{F,b}(s, x', t) = (st, x'), \quad \pi_{C,b}(s, x', t) = (s, x't), \tag{3.31}$$

and near *rb*, we have

$$\pi_{S,b}(s, x', t) = (x', t). \tag{3.32}$$

Near *lb* in M^2_b, we can write $K_A = k_1((x')^d\lambda, s, x')|dx'/x'|$, where for some $\varepsilon > 0$, $s^{-Nd-\varepsilon}k_1(\lambda, s, x')$ is a symbol of order 0 in all variables that can be expanded at $x' = 0$ with index set $\mathbb{N}_0$ of order $Nd + \varepsilon$, and for each $N' \leq N$, k_1 can be written in the form

$$k_1(\lambda, s, x') = \sum_{j=m}^{N'-1} \lambda^{-j} f_j(s, x') + \lambda^{-N'} f_{N'}(\lambda, s, x'), \tag{3.33}$$

where $f_{N'}(\lambda, s, x')$ satisfies the same conditions as $k_1(\lambda, s, x')$ but with N replaced by $2Nd - N'd$, and for each $m \leq j \leq N' - 1$, $f_j(s, x')$ satisfies the same conditions as $k_1(\lambda, s, x')$ but with Nd replaced by $2Nd - jd$; of course, without the condition (3.33). If $v = x/r$ where $r = \lambda^{-1/d}$, then near *rb* in $\mathcal{T}_d$, we can write $K_B = k_2(r, v, t)|dx'/x'|$, where for some $\varepsilon > 0$, $k_2(r, v, t)$ has expansions at $r = 0$, $v = 0$, and $t = 0$, with index sets E_{fi}, E_{ff}, and E_{rb} of order $Nd + \varepsilon$, respectively, and for $v \geq 1$, $k_2(r, v, t) = v^{-Nd-\varepsilon}\widetilde{k_2}(r, v, t)$ where $\widetilde{k_2}(r, v, t)$ is a symbol of order 0 in v and has expansions at $r = 0$ and $t = 0$ with index sets E_{fi} and E_{rb} of order $Nd + \varepsilon$, respectively. Using the formulas for $\pi_{F,b}$ and $\pi_{S,b}$ in (3.31) and (3.32), it follows that on $\mathcal{W} \subset M^3_b$,

$$\begin{aligned}\pi^*_{C,b}\mathfrak{m}\,\pi^*_{F,b}K_A\,\pi^*_{S,b}K_B &= k_1((x')^d\lambda, st, x')k_2(r, x'/r, t)\left|\frac{ds\,dt\,dx'}{stx'}\right| \\ &= k_1((x'/r)^d, st, x')k_2(r, x'/r, t)\left|\frac{ds\,dt\,dx'}{stx'}\right|,\end{aligned}$$

since $r = \lambda^{-1/d}$. Hence, as $\pi_{C,b}(s, x', t) = (s, x't)$, working out the definition of pushforward we obtain

$$(\pi_{C,b})_*(\pi^*_{C,b}\mathfrak{m}\,\pi^*_{F,b}K_A\,\pi^*_{S,b}K_B) = k_3(r, v', s)\left|\frac{ds\,dx'}{sx'}\right|,$$

where

$$k_3(r, v', s) = \int k_1((v'/t)^d, st, rv')k_2(r, v'/t, t)\frac{dt}{t}.$$

Using the asymptotic properties of $k_1(\lambda, s, x')$ given in (3.33) and the asymptotic properties of $k_2(r, v, t)$, one can show that $k_3(r, v', s)$ has expansions at $r = 0$, $v' = 0$, and $s = 0$, with index sets F_{fi}, F_{ff}, and $\varnothing$ of order $Nd + \varepsilon$, respectively, and for $v' \geq 1$, $k_3(r, v', s) = (v')^{-Nd-\varepsilon}\widetilde{k_3}(r, v', s)$ where $\widetilde{k_3}(r, v', s)$ is a symbol of order 0 in v' and has expansions at $r = 0$ and $s = 0$ with index sets F_{fi} and $\varnothing$ of order $Nd + \varepsilon$, respectively. □

4 Resolvents and parametrix construction

In this section we let $\mu > 0$ and consider a cone pseudodifferential operator $A \in x^{-\mu}\Psi_b^{\mu}(M)$, where x is a boundary defining function for ∂M and $\Psi_b^{\mu}(M)$ is the class of b-operators, cf. Definition 2.2. It is well-known (see e.g. [29] or [37]) that A can be extended as a bounded operator

$$A : x^{\alpha} H_b^s(M) \to x^{\alpha-\mu} H_b^{s-\mu}(M), \tag{4.1}$$

where the space $H_b^s(M)$ is defined as follows. We fix a b-measure $\mathfrak{m}$ and let $L_b^2(M)$ be the Hilbert space of square integrable functions with respect to $\mathfrak{m}$. For $s \in \mathbb{N}$, the space $H_b^s(M)$ consists of all $u \in L_b^2(M)$ such that $Pu \in L_b^2(M)$ for every differential operator $P \in \mathrm{Diff}_b^s(M)$. For an arbitrary $s \in \mathbb{R}$, the space $H_b^s(M)$ can be defined by duality and interpolation.

Remark 4.2 For $s \geq s'$ and $\alpha \geq \alpha'$ the embedding $x^{\alpha} H_b^s(M) \hookrightarrow x^{\alpha'} H_b^{s'}(M)$ is continuous. If $\alpha > \alpha'$, then it is compact if $s > s'$, Hilbert–Schmidt if $s > s' + \frac{n}{2}$, and trace class if $s > s' + n$, where $n = \dim M$.

For $A \in x^{-\mu}\Psi_b^{\mu}(M)$ we let ${}^b\sigma_{\mu}(A)$ be the totally characteristic principal symbol of $x^{\mu}A$ in $\Psi_b^{\mu}(M)$. The operator A is said to be b-elliptic if ${}^b\sigma_{\mu}(A)$ is invertible on ${}^bT^*M \setminus 0$, where ${}^bT^*M$ denotes the b-cotangent bundle, cf. [29]. The Fredholm property of (4.1) is determined by the indicial family (or conormal symbol) $\widehat{A}(z)$ associated with A. It is defined as the operator family

$$\widehat{A}(z) : C^{\infty}(Y) \to C^{\infty}(Y) : u \mapsto x^{\mu-z} A(x^z \tilde{u})|_{x=0},$$

where $Y = \partial M$ and $\tilde{u}$ is some extension of u. The set

$$\mathrm{spec}_b(A) = \{z \in \mathbb{C} \mid \widehat{A}(z) : H^{\mu}(Y) \to L^2(Y) \text{ is not invertible}\}$$

is called the *boundary spectrum of* A. If A is b-elliptic, then its boundary spectrum is discrete and we have the following result, cf. [21,29,31,37].

Theorem 4.3 *If $A \in x^{-\mu}\Psi_b^{\mu}(M)$ is b-elliptic, then for every $\alpha \in \mathbb{R}$ such that $\mathrm{spec}_b(A) \cap \{z \in \mathbb{C} \mid \Im z = -\alpha\} = \varnothing$, the operator (4.1) is Fredholm for every $s \in \mathbb{R}$.*

In order to ensure the existence of the resolvent and be able to describe it within our calculus, we need a natural notion of parameter-dependent ellipticity that resembles Agmon's condition at the symbol level and takes into account the singular global behavior of the operator near the boundary. Following [14] we will define the parameter-ellipticity with help of a model operator $A_{\wedge}$ living on the model cone $Y^{\wedge} := \overline{\mathbb{R}}_+ \times Y$. More precisely, with A we associate the operator

$$A_{\wedge} : C_c^{\infty}(Y^{\wedge}) \to C^{\infty}(Y^{\wedge}) : u \mapsto \lim_{\varrho \to 0} \varrho^{\mu} \kappa_{\varrho} \varphi A(\psi \kappa_{\varrho}^{-1} u), \tag{4.4}$$

where κ_{ϱ} is defined by $(\kappa_{\varrho} u)(x, y) := u(\varrho x, y)$, $\varrho > 0$, and where φ and ψ are smooth functions supported in a collar neighborhood of Y $(= \partial M = \partial Y^{\wedge})$ so that $\psi \kappa_{\varrho}^{-1} u$ and $\varphi A(\psi \kappa_{\varrho}^{-1} u)$ can be regarded as functions on both manifolds M and $Y^{\wedge}$.

On $Y^{\wedge}$ it is convenient to introduce Schulze's (cone) Sobolev spaces $\mathcal{K}^{s,\alpha}(Y^{\wedge})$ for $s, \alpha \in \mathbb{R}$, defined as follows. Let $\omega \in C_c^{\infty}(\overline{\mathbb{R}}_+)$ with $\omega(r) = 1$ near $r = 0$. Then the space $\mathcal{K}^{s,\alpha}(Y^{\wedge})$ consists of distributions u such that $\omega u \in r^{\alpha} H_b^s(Y^{\wedge})$, and such that given any coordinate patch $\mathcal{U}$ on Y diffeomorphic to an open subset of $\mathbb{S}^{n-1}$ and function $\varphi \in C_c^{\infty}(\mathcal{U})$, we have

$(1-\omega)\varphi\, u \in H^s(\mathbb{R}^n)$ where $\mathbb{R}_+ \times \mathbb{S}^{n-1}$ is identified with $\mathbb{R}^n \setminus \{0\}$ via polar coordinates. By definition, we have

$$\mathcal{K}^{0,0}(Y^\wedge) = H_b^0(Y^\wedge) = L_b^2(Y^\wedge).$$

These spaces have been systematically considered by Schulze in his edge calculus, see e.g. [36,37]. For $A \in x^{-\mu}\Psi_b^\mu(M)$, the associated model operator $A_\wedge$ extends as a bounded operator $A_\wedge : \mathcal{K}^{s,\alpha}(Y^\wedge) \to \mathcal{K}^{s-\mu,\alpha-\mu}(Y^\wedge)$ for every $s, \alpha \in \mathbb{R}$.

Definition 4.5 Let $A \in x^{-\mu}\Psi_b^\mu(M)$ and let Λ be a sector in $\mathbb{C}$ containing the origin. The operator family $A - \lambda$ is said to be *parameter-elliptic* on Λ with respect to $\alpha \in \mathbb{R}$, if and only if

(a) ${}^b\sigma_\mu(A)(\xi) - \lambda$ is invertible for all $\xi \neq 0$ and $\lambda \in \Lambda$,
(b) $A_\wedge - \lambda : \mathcal{K}^{s,\alpha}(Y^\wedge) \to \mathcal{K}^{s-\mu,\alpha-\mu}(Y^\wedge)$ is invertible for every $\lambda \in \Lambda$ sufficiently large, and for some $s \in \mathbb{R}$.

These conditions imply that $\operatorname{spec}_b(A) \cap \{z \in \mathbb{C} \mid \Im z = -\alpha\} = \varnothing$.

It is worth mentioning that in Definition 4.5(b) the cone Sobolev space $\mathcal{K}^{s,\alpha}(Y^\wedge)$ cannot be replaced by the weighted space $x^\alpha H_b^s(Y^\wedge)$. This is a consequence of Proposition 4.18 and the following elementary observation.

Lemma 4.6 *Let $A_\wedge$ be as in (4.4) and let $\mu > 0$ be such that $\mathcal{K}^{\mu,\mu}(Y^\wedge)$ is a proper subspace of $x^\mu H_b^\mu(Y^\wedge)$. If $A_\wedge : x^\mu H_b^\mu(Y^\wedge) \to L_b^2(Y^\wedge)$ is invertible, then $A_\wedge : \mathcal{K}^{\mu,\mu}(Y^\wedge) \to L_b^2(Y^\wedge)$ is not surjective.*

Proof Let $u \in x^\mu H_b^\mu(Y^\wedge) \setminus \mathcal{K}^{\mu,\mu}(Y^\wedge)$ and assume that $A_\wedge$ is surjective. Then there exists a function $v \in \mathcal{K}^{\mu,\mu}$ such that $A_\wedge v = A_\wedge u \in L_b^2(Y^\wedge)$. But $v \in x^\mu H_b^\mu(Y^\wedge)$ and $A_\wedge$ is injective, so $u = v \in \mathcal{K}^{\mu,\mu}$ which contradicts the assumption on u. □

If $\Lambda \subset \mathbb{C}$ is a sector not containing the positive real axis, then every operator $A \in x^{-\mu}\Psi_b^\mu(M)$ such that $A : x^\alpha H_b^s(M) \to x^{\alpha-\mu}H_b^{s-\mu}(M)$ is positive and selfadjoint, is parameter-elliptic on Λ with respect to α. This follows from Proposition 4.18.

Example 4.7 (cf. Example 3.3 in [15]) Let M be a compact n-manifold with boundary and let g be a Riemannian metric on M which, near the boundary, coincides with the cone metric $dx^2 + x^2 g_Y$, where g_Y is a metric on $Y = \partial M$. The corresponding measure is of the form $x^n\mathfrak{m}$ for a b-measure $\mathfrak{m}$. Let Δ_g be the Laplace-Beltrami operator associated to the metric g. This operator is symmetric on $L^2(M, x^n\mathfrak{m}) = x^{-n/2}L_b^2(M)$ and therefore, the operator

$$A = -x^{n/2-1}\Delta_g\, x^{-n/2+1} + x^{-2}a^2 \in x^{-2}\Psi_b^2(M) \tag{4.8}$$

is symmetric on $x^{-1}L_b^2(M)$ for every real number a. For $u \in C_c^\infty(M)$ supported near the boundary, we have

$$Au = x^{-2}\left((xD_x)^2 - \Delta_Y + \tfrac{(n-2)^2}{4} + a^2\right)u,$$

where Δ_Y is the Laplacian corresponding to g_Y. For $a > 1$ the boundary spectrum of $x^2 A$ does not intersect the strip $\{\sigma \in \mathbb{C} \mid |\Im\sigma| < 1\}$ so that A with domain $xH_b^2(M)$ is positive and selfadjoint in $x^{-1}L_b^2(M)$. In particular, $A - \lambda$ is parameter-elliptic with respect to $\alpha = 1$ on any sector $\Lambda \subset \mathbb{C}$ contained in the resolvent set of A.

Example 4.9 Let A be the operator (4.8). If $T \in x^{-1}\Psi_b^1(M)$ is symmetric on $x^{-1}L_b^2(M)$, then the operator $A+T$ with domain $xH_b^2(M) \hookrightarrow H_b^1(M)$ is also positive and selfadjoint, and therefore parameter-elliptic with respect to $\alpha = 1$. Observe that $T : H_b^1(M) \to x^{-1}L_b^2(M)$ is bounded.

We are now ready to prove that our parameter-dependent operators capture the resolvent of a cone pseudodifferential operators. We begin by defining certain index sets that appear in Theorem 4.11. We define

$$\widehat{E}^{\pm}(\alpha) = \big\{(z+r,k) \,|\, r \in \mathbb{N}_0,\ \tau = \mp iz \in \operatorname{spec}_b(A) + i\mu,\\ 1 \le k+1 \le \sum_{\ell=0}^{r} \operatorname{ord}(\tau - i\mu \mp i\ell),\ \text{and } \Re z > \pm(\alpha-\mu)\big\},$$

where the order of a pole $\tau \in \operatorname{spec}_b(A)$ of the inverse of the conormal symbol $\widehat{A}(\tau)$ is denoted by $\operatorname{ord}(\tau)$. Setting $\check{E}^{\pm}(\alpha) = \widehat{E}^{\pm}(\alpha) \,\overline{\cup}\, \widehat{E}^{\pm}(\alpha)$ and $E(\alpha) = \mathbb{N}\,\overline{\cup}(\widehat{E}^{+}(\alpha) + \widehat{E}^{-}(\alpha))$, we define

$$\mathcal{E}(\alpha) = (\check{E}^{+}(\alpha), \check{E}^{-}(\alpha), E(\alpha), \mathbb{N}_0), \quad \text{where } \check{E}^{\pm}(\alpha) = \widehat{E}^{\pm}(\alpha)\overline{\cup}\widehat{E}^{\pm}(\alpha). \tag{4.10}$$

Theorem 4.11 *Let $A \in x^{-\mu}\Psi_b^{\mu}(M)$, $\mu > 0$, be such that $A - \lambda$ is parameter-elliptic on Λ with respect to some $\alpha \in \mathbb{R}$. Then for $\lambda \in \Lambda$ sufficiently large,*

$$A - \lambda : x^{\alpha} H_b^s(M) \to x^{\alpha-\mu} H_b^{s-\mu}(M)$$

is invertible for any $s \in \mathbb{R}$, and

$$(A-\lambda)^{-1} \in x^{\mu}\Psi_c^{-\mu,-\mu,\mu}(M;\Lambda) + x^{\mu}\Psi_c^{-\infty,\mu,\mathcal{E}(\alpha)}(M;\Lambda),$$

where $\mathcal{E}(\alpha)$ is the index family defined in (4.10). Moreover, for $\alpha = \mu = s$ we have that

$$(A-\lambda)^{-1} : L_b^2(M) \to x^{\mu} H_b^{\mu}(M) \tag{4.12}$$

is uniformly bounded in λ.

Proof Let $(x, y) \in \mathcal{U} = [0, c)_x \times \mathbb{R}^{n-1}$ be local coordinates near the boundary of M and let $a_\mu(x, y, \xi)$ denote the totally characteristic principal symbol of A. Given $\varepsilon > 0$, let $\chi \in C^\infty(\mathbb{R}^n)$ with $\chi(\xi) = 0$ for $|\xi| < \varepsilon$ and $\chi(\xi) = 1$ for $|\xi| > 2\varepsilon$. Let

$$b_{-\mu}(x, y, \xi, \lambda) = \chi(\xi)(a_\mu(x, y, \xi) - x^{\mu}\lambda)^{-1} \tag{4.13}$$

Observe that (x', y', z), where $z = (\log(x/x'), y - y')$, are coordinates on M_b^2 near Δ_b. Given $\varphi \in C_c^\infty(\mathcal{U})$ and $\psi(z) \in C_c^\infty(\mathbb{R}^n)$ where $\psi(z) = 1$ on a neighborhood of $z = 0$, define the Schwartz kernel of $B(\lambda)$ by

$$K_{B(\lambda)} = \varphi(x', y')\,\psi(z) \int e^{iz\cdot\xi}\, b_{-\mu}(x', y', \xi, \lambda)\, đ\xi \cdot \mathfrak{m}',$$

where $\mathfrak{m}' = |(dx'/x')dy'|$. Then, by definition, $B(\lambda) \in \Psi_c^{-\mu,-\mu,\mu}(M;\Lambda)$ (cf. Example 3.3). Since the principal b-symbol of Ax^{μ} is also a_μ, and

$$(a_\mu(x, y, \xi) - x^{\mu}\lambda)\, b_{-\mu}(x, y, \xi, \lambda)\chi(\xi) = 1 + (\chi(\xi) - 1),$$

the composition properties of the b-calculus show that

$$(A - \lambda)x^{\mu}B(\lambda) = (Ax^{\mu} - x^{\mu}\lambda)B(\lambda) = \varphi - S(\lambda) + T, \tag{4.14}$$

where $S(\lambda) \in \Psi_c^{-1,-\mu,\mu}(M;\Lambda)$, and the Schwartz kernel of T is given by

$$K_T = \varphi(x', y')\,\psi(z) \int e^{iz\cdot\xi}\, (\chi(\xi) - 1)\, đ\xi \cdot \mathfrak{m}'.$$

Since $\chi(\xi) = 1$ for $|\xi| > 2\varepsilon$, $\chi(\xi) - 1 = 0$ for $|\xi| > 2\varepsilon$, which implies that T is a b-pseudodifferential operator of order $-\infty$ with a symbol supported in $|\xi| < 2\varepsilon$ and whose Schwartz kernel $K_T \to 0$ in the C^∞ topology as a smooth function on M_b^2. In particular, the mapping properties of b-pseudodifferential operators [29] imply that the L_b^2 norm of T tends to 0 as $\varepsilon \to 0$. If $\mathcal{U}$ is a coordinate patch on the interior of M, a similar argument shows that given $\varphi \in C_c^\infty(\mathcal{U})$, there is a $B(\lambda) \in \Psi_c^{-\mu,-\mu,\mu}(M;\Lambda)$ such that (4.14) holds.

Let $\{\mathcal{U}_i\}_{i=1}^N$ be coordinate patches covering M such that, as in (4.14), there exists a $B_i(\lambda) \in \Psi_c^{-\mu,-\mu,\mu}(M;\Lambda)$ satisfying $(A-\lambda)x^\mu B_i(\lambda) = \varphi_i - S_i(\lambda) + T_i$, where $S_i(\lambda) \in \Psi_c^{-1,-\mu,\mu}(M;\Lambda)$, and where φ_i is a smooth function supported in $\mathcal{U}_i$. Setting $B_0(\lambda) = \sum_{i=1}^N B_i(\lambda) \in \Psi_c^{-\mu,-\mu,\mu}(M;\Lambda)$ and assuming that the φ_i form a partition of unity of M, we obtain

$$(A-\lambda)x^\mu B_0(\lambda) = I - S_0(\lambda) + T, \tag{4.15}$$

where $T \in \Psi_b^{-\infty}(M)$ and $S_0(\lambda) \in \Psi_c^{-1,-\mu,\mu}(M;\Lambda)$. Theorem 3.16 shows that $S_0(\lambda)^j \in \Psi_c^{-j,-j\mu,\mu}(M;\Lambda)$ for each j. Thus we can choose $S_0'(\lambda) \in \Psi_c^{-1,-\mu,\mu}(M;\Lambda)$ such that $S_0'(\lambda) \sim \sum_{j=1}^\infty S_0(\lambda)^j$, where the right-hand side is an asymptotic sum. This implies that

$$(I - S_0(\lambda))(I + S_0'(\lambda)) = I - R_1(\lambda), \quad R_1(\lambda) \in \Psi_c^{-\infty,-\infty,\mu}(M;\Lambda).$$

Multiplying both sides of (4.15) by $I + S_0'(\lambda)$, we obtain

$$(A-\lambda)x^\mu B_1(\lambda) = I - S_1(\lambda) + T,$$

where $B_1(\lambda) = B_0(\lambda) + B_0(\lambda)S_0'(\lambda) \in \Psi_c^{-\mu,-\mu,\mu}(M;\Lambda)$ by Theorem 3.16, and $S_1(\lambda) = R_1(\lambda) - TS_0'(\lambda) \in \Psi_c^{-\infty,-\mu,\mu}(M;\Lambda)$ by Theorem 3.17.

By choosing $\varepsilon > 0$ sufficiently small, we may assume that $I+T$ is invertible. The inverse of $I+T$ is of the form $I+T'$ where $T' \in \Psi_b^{-\infty,\beta}(M)$ for some $\beta > 0$ that depends on the width of the strip on which the conormal symbol of T is invertible. Moreover, since $\|T\|_{L_b^2} \to 0$ as $\varepsilon \to 0$, the arguments found in [29, Chap. 5] imply that $\beta > 0$ can be choosing arbitrarily large by choosing $\varepsilon > 0$ smaller. Choose any $N >> 0$ and let $\varepsilon > 0$ be chosen so that $T' \in \Psi_b^{-\infty,2N\mu}(M)$. Then multiplying both sides of the previous displayed equation by $I + T'$, we obtain

$$(A-\lambda)x^\mu B_2(\lambda) = I - S_2(\lambda), \tag{4.16}$$

where $B_2(\lambda) = B_1(\lambda) + B_1(\lambda)T' \in \Psi_c^{-\mu,-\mu,\mu}(M;\Lambda) + \Psi_{1,N}^{-\infty,\mu}(M;\Lambda)$ by Theorem 3.18, and $S_2(\lambda) = S_1(\lambda) + S_1(\lambda)T' \in \Psi_{1,N}^{-\infty,\mu}(M;\Lambda)$ by Lemma 3.10 and Theorem 3.18. By Theorem 3.19, $S_2(\lambda)^j \in \Psi_{j,N}^{-\infty,\mu}(M;\Lambda)$, which implies that $S_2'(\lambda) = \sum_{j=1}^{N-1} S_2(\lambda)^j \in \Psi_{1,N}^{-\infty,\mu}(M;\Lambda)$ satisfies

$$(I - S_2(\lambda))(I + S_2'(\lambda)) = I - S_3(\lambda), \quad S_3(\lambda) = S_2(\lambda)^N \in \Psi_{N,N}^{-\infty,\mu}(M;\Lambda).$$

Multiplying both sides of (4.16) by $I + S_2'(\lambda)$, we obtain

$$(A-\lambda)x^\mu B_3(\lambda) = I - S_3(\lambda),$$

where $B_3(\lambda) = B_2(\lambda) + B_2(\lambda)S_2'(\lambda) \in \Psi_c^{-\mu,-\mu,\mu}(M;\Lambda) + \Psi_{1,N}^{-\infty,\mu}(M;\Lambda)$ by Theorems 3.18 and 3.19, and by Lemma 3.12 we have $S_3(\lambda) \in \Psi_{N,N}^{-\infty,\mu}(M;\Lambda) \subset \Psi_N^{-\infty,\mu,\mathcal{E}}(M;\Lambda)$ where $\mathcal{E}$ is the index family on $\mathcal{T}_d$ given by $\mathcal{E} = (\varnothing, \varnothing, \mathbb{N}_0, \mathbb{N}_0)$.

Finally, using the localized inverse $(A_\wedge - \lambda)^{-1}$ (which exists by condition (b) in Definition 4.5) one can modify the parametrix $x^\mu B_3(\lambda)$ to get a remainder term that decays as

$1/|\lambda|$. Then, by means of a standard Neumann series argument, this new parametrix can be further refined to obtain the exact resolvent. The difficulty is to understand the pseudodifferential structure of the resolvent which requires understanding the structure of $(A_\wedge - \lambda)^{-1}$. This analysis is rather long but can be done following the same arguments as in the proof of Theorem 6.3 in [24]. The conclusion is that

$$(A-\lambda)^{-1} \in x^\mu \Psi_c^{-\mu,-\mu,\mu}(M;\Lambda) + x^\mu \Psi_{1,N}^{-\infty,\mu}(M;\Lambda) + x^\mu \Psi_N^{-\infty,\mu,\mathcal{F}}(M;\Lambda),$$

where $\mathcal{F} = (F_{lb}, F_{rb}, F_{ff}, \mathbb{N}_0)$, with $F_{lb} > \alpha - \mu$, $F_{rb} > -(\alpha-\mu)$, and $F_{ff} > 0$. Now, since $\Psi_{1,\infty}^{-\infty,\mu}(M;\Lambda) \subset \Psi_c^{-\infty,-\mu,\mu}(M;\Lambda)$ and $\Psi_\infty^{-\infty,\mu,\mathcal{F}}(M;\Lambda) \subset \Psi_c^{-\infty,\mu,\mathcal{F}}(M;\Lambda)$, and since N can be chosen arbitrarily large, it follows that

$$(A-\lambda)^{-1} \in x^\mu \Psi_c^{-\mu,-\mu,\mu}(M;\Lambda) + x^\mu \Psi_c^{-\infty,\mu,\mathcal{F}}(M;\Lambda),$$

According to [29, Theorem 5] or [27, Theorem 4.4], we know that for fixed λ, the resolvent $(A-\lambda)^{-1}$ has expansions at *lb*, *rb*, and *ff*, with index sets $\check{E}^+(\alpha)$, $\check{E}^-(\alpha)$, and $E(\alpha)$, respectively. It follows that $\mathcal{F}$ must equal the index set $\mathcal{E}(\alpha)$ given in (4.10). This proves the first statement of the theorem.

The norm estimate for (4.12) essentially follows from corresponding estimates for $x^\mu B_0(\lambda)$ and $(A_\wedge - \lambda)^{-1}$. First of all, observe that $(A-\lambda)^{-1} \in \mathcal{L}(L_b^2(M), x^\mu H_b^\mu(M))$ is uniformly bounded in λ if and only if $\|(A-\lambda)^{-1}\|_{\mathcal{L}(L_b^2(M))} = \mathcal{O}(|\lambda|^{-1})$ as $|\lambda| \to \infty$.

That $\|x^\mu B_0(\lambda)\|_{\mathcal{L}(L_b^2(M))} = \mathcal{O}(|\lambda|^{-1})$ as $|\lambda| \to \infty$ is a consequence of the fact that the Schwartz kernel of $B_0(\lambda)$ is locally given by the symbol (4.13). On the other hand, the norm estimate for $(A_\wedge - \lambda)^{-1}$ on $L_b^2(Y^\wedge)$ is a direct consequence of its κ-homogeneity properties. More precisely, for every $\varrho > 0$ we have

$$(A_\wedge - \lambda)^{-1} = \varrho^\mu \kappa_\varrho^{-1}(A_\wedge - \varrho^\mu\lambda)^{-1}\kappa_\varrho.$$

Setting $\varrho = |\lambda|^{-1/\mu}$ and using that κ_ϱ is an isometry on $L_b^2(Y^\wedge)$, this gives

$$\|(A_\wedge - \lambda)^{-1}\|_{\mathcal{L}(L_b^2(Y^\wedge))} = |\lambda|^{-1}\|(A_\wedge - \tfrac{\lambda}{|\lambda|})^{-1}\|_{\mathcal{L}(L_b^2(Y^\wedge))}.$$

Hence $\|(A_\wedge - \lambda)^{-1}\|_{\mathcal{L}(L_b^2(Y^\wedge))} = \mathcal{O}(|\lambda|^{-1})$ as $|\lambda| \to \infty$. □

Composing the resolvent with itself N times, we obtain

$$(A-\lambda)^{-N} \in x^{N\mu}\Psi_c^{-N\mu,-N\mu,\mu}(M;\Lambda) + x^{N\mu}\Psi_c^{-\infty,\mu,\mathcal{E}_N(\alpha)}(M;\Lambda)$$

where the index family $\mathcal{E}_N(\alpha)$ is defined inductively from the index family $\mathcal{E}(\alpha)$ using the composition Theorems 3.16, 3.18, and 3.27. For instance, for $N = 2$ we can choose $\varepsilon_2(\alpha) = \varepsilon(\alpha) \cup \varepsilon(\alpha)_\mu \cup \varepsilon(\alpha)_\mu \hat{\circ} \varepsilon(\alpha)$, where $\varepsilon(\alpha)_\mu$ is the shifted index family from Theorem 3.18. Further, composing $(A-\lambda)^{-N}$ with a b-pseudodifferential operator B and using again Theorems 3.17 and 3.18, we obtain the following corollary.

Corollary 4.17 *Let $A \in x^{-\mu}\Psi_b^\mu(M)$, $\mu > 0$, be such that $A - \lambda$ is parameter-elliptic with respect to some α on Λ. Then given any $B \in x^{-\nu}\Psi_b^{\mu'}(M)$, $\nu, \mu' \in \mathbb{R}$, for $\lambda \in \Lambda$ sufficiently large, we have for any $N \in \mathbb{N}$,*

$$B(A-\lambda)^{-N} \in x^{N\mu-\nu}\Psi_c^{\mu'-N\mu,-N\mu,\mu}(M;\Lambda) + x^{N\mu-\nu}\Psi_c^{-\infty,\mu,\mathcal{E}_N(\alpha)}(M;\Lambda).$$

We finish this section showing that the invertibility condition (b) in Definition 4.5 is necessary for the resolvent to be uniformly bounded, cf. [18, Theorem 4.1]. Although we do not discuss here the condition on ${}^b\sigma_\mu(A)$, it can be proved (as in the case of a regular operator on a smooth compact manifold, cf. [40]) that (a) is also a necessary condition. It implies that $A_\wedge - \lambda : \mathcal{K}^{s,\alpha}(Y^\wedge) \to \mathcal{K}^{s-\mu,\alpha-\mu}(Y^\wedge)$ is Fredholm, so its image is closed.

Proposition 4.18 *Let $A \in x^{-\mu}\Psi_b^\mu(M)$, $\mu > 0$, be such that $A - \lambda : x^\mu H_b^\mu \to L_b^2$ is invertible for all $\lambda \in \Lambda$ with $|\lambda| > R$ for some $R > 0$. If the resolvent*

$$(A-\lambda)^{-1} : L_b^2(M) \to x^\mu H_b^\mu(M)$$

is uniformly bounded in λ, then $A_\wedge - \lambda : \mathcal{K}^{\mu,\mu}(Y^\wedge) \to L_b^2(Y^\wedge)$ is invertible for every $\lambda \in \Lambda \setminus \{0\}$.

Proof The assumptions on $A - \lambda$ and $(A-\lambda)^{-1}$ imply that, if $u \in x^\mu H_b^\mu(M)$, then

$$\|(A-\lambda)u\|_0 \geq C\|u\|_\mu \tag{4.19}$$

for some constant $C > 0$, where $\|\cdot\|_\nu$ denotes the norm in $x^\nu H_b^\nu(M)$. From this estimate we will derive the injectivity of $A_\wedge - \lambda : \mathcal{K}^{\mu,\mu}(Y^\wedge) \to L_b^2(Y^\wedge)$.

Let $v \in C_c^\infty(\mathring{Y}^\wedge)$ and pick $\mathcal{U}_Y \subset M$ such that $\mathcal{U}_Y \cong [0,\varepsilon) \times Y$ for some $\varepsilon > 0$. Let $\varrho > 0$ be small enough so that $\kappa_\varrho^{-1} v \in \mathcal{K}^{\mu,\mu}(Y^\wedge)$ is supported in $[0,\varepsilon)\times Y$, so it can be regarded as a function in $x^\mu H_b^\mu(M)$ supported in $\mathcal{U}_Y$. Let $\varphi, \psi \in C_c^\infty(\mathcal{U}_Y)$ be such that $\psi = 1$ on $\operatorname{supp}(\kappa_\varrho^{-1}v)$ and $\varphi\psi = \psi$. We have

$$\begin{aligned}\|(\varrho^\mu\kappa_\varrho\varphi A\psi\kappa_\varrho^{-1} - \lambda)v\|_0 &= \varrho^\mu\|\kappa_\varrho(\varphi A - \varrho^{-\mu}\lambda)\psi\kappa_\varrho^{-1}v\|_0\\ &= \varrho^\mu\|(\varphi A - \varrho^{-\mu}\lambda)\psi\kappa_\varrho^{-1}v\|_0\\ &\geq \varrho^\mu\|(A-\varrho^{-\mu}\lambda)\kappa_\varrho^{-1}v\|_0 - \varrho^\mu\|(1-\varphi)A\psi\kappa_\varrho^{-1}v\|_0\end{aligned}$$

since κ_ϱ is an isometry on L_b^2 and $\varphi = 1-(1-\varphi)$. Note that $(1-\varphi)A\psi$ is a smoothing operator, so the second norm on the right-hand side of the inequality is uniformly bounded in ϱ. On the other hand, for $\varrho < 1$ we can apply (4.19) and get

$$\|(A-\varrho^{-\mu}\lambda)\kappa_\varrho^{-1}v\|_0 \geq C\|\kappa_\varrho^{-1}v\|_\mu = C\varrho^{-\mu}\|v\|_\mu.$$

Thus, for ϱ small,

$$\|(\varrho^\mu\kappa_\varrho\varphi A\psi\kappa_\varrho^{-1} - \lambda)v\|_0 \geq C\|v\|_\mu + \mathcal{O}(\varrho^\mu).$$

Taking the limit as $\varrho \to 0$, by (4.4) we get

$$\|(A_\wedge - \lambda)v\|_0 \geq C\|v\|_\mu \tag{4.20}$$

for every $v \in C_c^\infty(\mathring{Y}^\wedge)$. Since this space is dense in $\mathcal{K}^{\mu,\mu}(Y^\wedge)$, (4.20) also holds for every $v \in \mathcal{K}^{\mu,\mu}(Y^\wedge)$ and we get the injectivity of $A_\wedge - \lambda$ on $\mathcal{K}^{\mu,\mu}(Y^\wedge)$.

Finally, note that the invertibility assumption on $A-\lambda$ implies the invertibility of the formal adjoint $A^\star - \bar{\lambda} : L_b^2(M) \to x^{-\mu}H_b^{-\mu}(M)$. By the previous argument, this implies the injectivity of $A_\wedge^\star - \bar{\lambda} : L_b^2(Y^\wedge) \to \mathcal{K}^{-\mu,-\mu}(Y^\wedge)$, and consequently, the surjectivity of $A_\wedge - \lambda : \mathcal{K}^{\mu,\mu}(Y^\wedge) \to L_b^2(Y^\wedge)$. □

5 Asymptotic expansions

To obtain an asymptotic expansion of $B(A-\lambda)^{-N}$, we will use the following known lemmas whose proofs can be found in [25, Appendix A].

Lemma 5.1 *Suppose that $u(x, y)$ is a compactly supported on $[0, 1)^2$ with expansions at $x = 0$ and $y = 0$ given by index sets not necessarily C^∞ E_{lb} and E_{rb}, respectively. Then the function $v(x)$ defined by*

$$v(x) = \int_0^1 u(x/y, y)\frac{dy}{y} = \int_0^1 u(y, x/y)\frac{dy}{y},$$

can be expanded at $x = 0$ with index set $E_{lb}\overline{\cup}E_{rb}$.

This lemma is a special case of the "Pushforward Theorem" due to Melrose [28]. As discussed in [19], this theorem is related to the "Singular Asymptotics Lemma" due to Brüning and Seeley [4].

Lemma 5.2 *Let $f \in C^\infty(\mathbb{R}_+)$ vanish to infinite order as $x \to \infty$ and suppose that for some $a \in \mathbb{C}$, we have*

$$(x\partial_x - a)f(x) = g(x), \tag{5.3}$$

where $g(x)$ can be expanded at $x = 0$ with index set E, not necessarily a C^∞ index set. Then f has an expansion at $x = 0$ with index set $E\,\overline{\cup}\,\{a\}$.

We are now ready to prove our main result concerning asymptotic expansions of resolvents of pseudodifferential cone operators.

Theorem 5.4 *Let $A \in x^{-\mu}\Psi_b^\mu(M)$, $\mu > 0$, be such that $A - \lambda$ is parameter-elliptic on Λ with respect to some α. Then, given any $B \in x^{-\beta}\Psi_b^{\mu'}(M)$ with $\beta, \mu' \in \mathbb{R}$, for N sufficiently large, $B(A - \lambda)^{-N} : x^{\alpha-\mu}H_b^s(M) \to x^{\alpha-\mu-\beta}H_b^{s-\mu'}(M)$ is trace class for every $s \in \mathbb{R}$, and*

$$\begin{aligned} \operatorname{Tr} B(A-\lambda)^{-N} \;\sim_{|\lambda|\to\infty} &\sum_{k=0}^{\infty}\left\{a_k + b_k\log\lambda + c_k(\log\lambda)^2\right\}\lambda^{(\mu'+n-k)/\mu-N} \\ &+ \sum_{k=0}^{\infty}\left\{d_k + e_k\log\lambda\right\}\lambda^{(\beta-k)/\mu-N} + \sum_{k=0}^{\infty} f_k\lambda^{-k-N}. \end{aligned} \tag{5.5}$$

Moreover, $b_k = 0$ unless $k \in (\mathbb{N}_0 + \mu' + n - \beta) \cup (\mu\mathbb{N}_0 + \mu' + n)$; $c_k = 0$ unless $k \in \mu\mathbb{N}_0 \cap (\mathbb{N}_0 - \beta) + \mu' + n$; and $e_k = 0$ unless $k \in \mu\mathbb{N}_0 + \beta$.

Proof By Corollary 4.17, for $\lambda \in \Lambda$ sufficiently large, we can write

$$B(A-\lambda)^{-N} = F(\lambda) + G(\lambda),$$

where $F \in x^{N\mu-\beta}\Psi_c^{\mu'-N\mu,-N\mu,\mu}(M;\Lambda)$ and $G \in x^{N\mu-\beta}\Psi_c^{-\infty,\mu,\mathcal{E}_N(\alpha)}(M;\Lambda)$. Hence $F(\lambda) \in x^{N\mu-\beta}\Psi_b^{\mu'-N\mu}(M)$ and $G(\lambda) \in x^{N\mu-\beta}\Psi_b^{-\infty,\mathcal{E}_N(\alpha)}(M)$ for every λ. Thus by their mapping properties and Remark 4.2, the operators $F(\lambda)$ and $G(\lambda)$ are both trace class if N is large enough. We assume $\mu' - N\mu < -n$. The expansion (5.5) will be achieved by expanding $\operatorname{Tr} F(\lambda)$ and $\operatorname{Tr} G(\lambda)$.

Step 1: We begin by showing that, as $|\lambda| \to \infty$ in Λ, we have

$$\operatorname{Tr} G(\lambda) \sim \sum_{k=0}^{\infty}\alpha_k\lambda^{(\beta-k)/\mu-N}, \quad \alpha_k \in \mathbb{C}. \tag{5.6}$$

If $\Delta \cong M$ is the diagonal in M^2, then $\operatorname{Tr} G(\lambda) = \int_M K_{G(\lambda)}|_\Delta$. By the definition of $x^{N\mu-\beta}\Psi_c^{-\infty,\mu,\mathcal{E}_N(\alpha)}(M;\Lambda)$, on the interior of Δ, $K_{G(\lambda)}|_\Delta$ vanishes to infinite order as $|\lambda| \to \infty$. Thus we may assume that $K_{G(\lambda)}|_\Delta$ is supported in a neighborhood $[0,1)_x \times Y$ of M near Y. Let $r = |\lambda|^{-1/\mu}$ and $\theta = \lambda/|\lambda|$. Then, integrating out the variables on Y, we can write (for $r \le 1$)

$$\int_M K_{G(\lambda)}|_\Delta = \int_0^{1/r} x^{N\mu-\beta} G(r,\theta,x/r)\frac{dx}{x}$$
$$= r^{N\mu-\beta}\int_0^1 x^{N\mu-\beta} G(r,\theta,x)\frac{dx}{x} \quad (x \mapsto rx),$$

where $G(r,\theta,v)$ is a function smooth in r up to $r=0$, smooth in θ, can be expanded at $v=0$ with index set $E_{N,\mathit{ff}}(\alpha) \ge \mu - N\mu$, and vanishes to infinite order as $v \to \infty$. Since $G(r,\theta,v)$ is smooth at $r=0$, as $r \to 0^+$ we have

$$\operatorname{Tr} G(\lambda) \sim \sum_{k=0}^{\infty} g_k(\theta)\, r^{N\mu-\beta+k}$$

for some $g_k(\theta)$, smooth in θ. Since $r = |\lambda|^{-1/\mu}$ and $G(\lambda)$ is holomorphic in λ, this expansion is really an expansion in λ (cf. [25, Proposition 5.1]), which proves (5.6).

It remains to prove an asymptotic of $\operatorname{Tr} F(\lambda)$ as $|\lambda| \to \infty$. If $\varphi \in C^\infty(M)$ vanishes near the boundary Y, then the trace of $\varphi F(\lambda)$ can be analyzed using techniques similar to [20], for instance. The result is

$$\operatorname{Tr} \varphi F(\lambda) \sim_{|\lambda|\to\infty} \sum_{k=0}^{\infty}\left\{a_k + b_k \log\lambda\right\}\lambda^{(\mu'+n-k)/\mu - N} + \sum_{k=0}^{\infty} f_k \lambda^{-k-N},$$

where $b_k = 0$ unless $k \in (\mathbb{N}_0 + \mu' + n - \beta) \cup (\mu\mathbb{N}_0 + \mu' + n)$.

Thus it suffices to assume that $F(\lambda)$ is supported in a collar $[0,1)_x \times Y$. By taking a partition of unity of Y, we may assume that $F(\lambda)$ is supported in a coordinate neighborhood in the Y factor. Also, as with the expansion for $\operatorname{Tr} G(\lambda)$, we only need to prove an expansion of the form (5.5) with λ replaced by $r^{-\mu}$ and coefficients that depend smoothly on $\theta = \lambda/|\lambda|$. In other words, we will prove the expansion

$$\operatorname{Tr} F(\lambda) \sim_{r\to 0^+} \sum_{k=0}^{\infty}\left\{a_k(\theta) + b_k(\theta)\log r + c_k(\theta)(\log r)^2\right\} r^{N\mu-\mu'-n+k}$$
$$+ \sum_{k=0}^{\infty}\left\{d_k(\theta) + e_k(\theta)\log r\right\} r^{N\mu-\beta+k} + \sum_{k=0}^{\infty} f_k(\theta) r^{(k+N)\mu}. \tag{5.7}$$

Note that θ appears only as a parameter, so we may assume without loss of generality that $\Lambda = [0,\infty)$. We will complete our proof in two more steps.

Step 2: We reduce (5.7) to an application of Lemma 5.1. Using the definition of $x^{N\mu-\beta}\Psi_c^{\mu'-N\mu,-N\mu,\mu}(M;\Lambda)$ and integrating out the Y factor of $[0,1)\times Y$, we can write

$$\operatorname{Tr} F(\lambda) = \int_0^1\int_{\mathbb{R}^n} x^{N\mu-\beta} a(x,\xi,x^\mu\lambda)\, đ\xi \frac{dx}{x},$$

where $a(x,\xi,\lambda) \in C_c^\infty\big([0,1)_x, S_{r,c\ell}^{\mu'-N\mu,-N\mu,\mu}(\mathbb{R}^n_\xi;\Lambda)\big)$. By assumption, $\mu' - N\mu < -n$, so the integral in ξ is absolutely convergent. If $r = \lambda^{-1/\mu}$, then

$$\operatorname{Tr} F(\lambda) = \int_0^1 A(x, r/x)\frac{dx}{x},$$

where $A(x,z) = x^{N\mu-\beta}\int_{\mathbb{R}^n} a(x,\xi,z^{-\mu})\,đ\xi$. Let $\varphi \in C^\infty(\mathbb{R}_+)$ be such that $\varphi(z) = 1$ for $z \le 1$ and $\varphi(z) = 0$ for $z \ge 2$. Then, for $r \le 1$,

$$\operatorname{Tr} F(\lambda) = \int_0^1 \varphi(r/x)\, A(x,r/x)\frac{dx}{x} + \int_0^{1/r} (1-\varphi(r/x))\, A(x,r/x)\frac{dx}{x}. \tag{5.8}$$

We analyze the asymptotics of each integral as $r \to 0^+$. For the second integral, we make the change of variables $x \mapsto rx$, which gives

$$\int_0^{1/r} (1-\varphi(r/x))\, A(x,r/x)\frac{dx}{x} = \int_0^1 (1-\varphi(1/x))\, A(rx,1/x)\frac{dx}{x}. \tag{5.9}$$

Since $A(rx,1/x) = (rx)^{N\mu-\beta}\int_{\mathbb{R}^n} a(rx,\xi,x^\mu)\,đ\xi$ and $N\mu - \beta > 0$, the integral (5.9) converges absolutely. Moreover, since $a(x,\xi,\lambda)$ is smooth at $x = 0$, (5.9) has an expansion at $r = 0$ with index set $N\mu - \beta + \mathbb{N}_0$. Thus the second integral in (5.8) contributes an expansion of the form given by the second sum in (5.7).

It remains to analyze the asymptotics of the first integral in (5.8). Note that $A(x,z)$ has an expansion at $x = 0$ with index set $N\mu - \beta + \mathbb{N}_0$ since $a(x,\xi,\lambda)$ is smooth at $x = 0$. Thus, as $\varphi(z)\,A(x,z)$ is compactly supported in z and x, we can apply Lemma 5.1: if $A(x,z)$ has an expansion at $z = 0$ with some index set E, then the first integral in (5.8) has an expansion as $r = \lambda^{-1/\mu} \to 0^+$ with index set $E\overline{\cup}(N\mu - \beta + \mathbb{N}_0)$ (see (3.25) for the definition of $\overline{\cup}$). In the following step we will show that

$$E = (\mu N + \mu\mathbb{N}_0)\overline{\cup}(N\mu - \mu' - n + \mathbb{N}_0).$$

Step 3: Since the asymptotics of $A(x,z)$ at $z = 0$ do not depend on x, we may omit the x variable. Thus it suffices to determine the asymptotics of

$$A(z) = \int_{\mathbb{R}^n} a(\xi, z^{-\mu})\,đ\xi \quad \text{at } z = 0,$$

where $a(\xi,\lambda) \in S_{r,c\ell}^{\mu'-N\mu,-N\mu,\mu}(\mathbb{R}^n;\Lambda)$. Let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 0$ near $\xi = 0$ and $\chi(\xi) = 1$ outside a neighborhood of 0. Then, given $L \in \mathbb{N}$, expanding $a(\lambda,\xi)$ in its homogeneous components, we can write

$$A(z) = \sum_{k=0}^{L-1} A_k(z) + R_L(z),$$

where for each k,

$$A_k(z) = \int_{\mathbb{R}^n} \chi(\xi)\, a_k(\xi, z^{-\mu})\,đ\xi \tag{5.10}$$

with $a_k(\xi,\lambda)$ anisotropic homogeneous of degree $\mu' - N\mu - k$, and where $R_L(z) = \int r_L(\xi, z^{-\mu})\, đ\xi$ with $r_L(\xi,\lambda) \in S_r^{\mu'-N\mu-L,-N\mu,\mu}(\mathbb{R}^n;\Lambda)$. In particular, $r_L(\xi, z^{-\mu}) = z^{\mu N}\widetilde{r}_L(\xi, z^{\mu})$ where $\widetilde{r}_L(\xi, w)$ is smooth at $w = 0$ and satisfies estimates of the form (3.1). These estimates imply that $R_L(z)$ can be expanded to higher and higher order at $z = 0$ with index set $\mu N + \mu\mathbb{N}_0$ as L is chosen larger and larger. Thus it suffices to analyze the asymptotics of each $A_k(z)$ at $z = 0$.

Recall that $a_k(\xi,\lambda)$ has the following properties:

- $a_k(\delta\xi, \delta^{\mu}\lambda) = \delta^{\mu'-N\mu-k} a_k(\xi,\lambda)$ for $\delta > 0$,
- $a_k(\xi, z^{-\mu}) = z^{\mu N}\widetilde{a}_k(\xi, z^{\mu})$ with $\widetilde{a}_k(\xi, w)$ smooth at $w = 0$.

Now, making the change of variables $\xi \mapsto z^{-1}\xi$ in (5.10) and using the homogeneity properties of a_k, we get

$$A_k(z) = z^{N\mu-\mu'-n+k} \int_{\mathbb{R}^n} \chi(\xi/z)\, a_k(\xi, 1)\, đ\xi.$$

Let $\gamma = N\mu - \mu' - n + k$. Since $\big(z\partial_z - \gamma\big)z^{\gamma} = 0$ and $z\partial_z\chi(\xi/z) = -(\xi\cdot\partial_\xi\chi)(\xi/z)$ where $\xi\cdot\partial_\xi = \sum \xi_j\partial_{\xi_j}$, we have

$$\begin{aligned}\big(z\partial_z - \gamma\big)A_k(z) &= -z^{\gamma}\int_{\mathbb{R}^n} (\xi\cdot\partial_\xi\chi)(\xi/z)\, a_k(\xi,1)\, đ\xi \\ &= -\int_{\mathbb{R}^n} (\xi\cdot\partial_\xi\chi)(\xi)\, a_k(\xi, z^{-\mu})\, đ\xi \\ &= -z^{\mu N}\int_{\mathbb{R}^n} (\xi\cdot\partial_\xi\chi)(\xi)\, \widetilde{a}_k(\xi, z^{\mu})\, đ\xi\end{aligned}$$

by means of the change $\xi \mapsto z\xi$ and due to the properties of a_k. Since the function $(\xi\cdot\partial_\xi\chi)(\xi)$ is supported in a compact subset of $\mathbb{R}^n \setminus \{0\}$, the last integral is absolutely convergent and so it can be expanded at $z = 0$ with index set $\mu N + \mu\mathbb{N}_0$. Hence, Lemma 5.2 implies that $A_k(z)$ can be expanded at $z = 0$ with index set $(\mu N + \mu\mathbb{N}_0)\overline{\cup}(N\mu - \mu' - n + k)$. Thus, as $A(z)$ is an asymptotic sum of the A_k's, $A(z)$ itself can be expanded at $z = 0$ with index set $(\mu N + \mu\mathbb{N}_0)\overline{\cup}(N\mu - \mu' - n + \mathbb{N}_0)$. This completes the proof of the theorem. □

Let Λ be a sector of the form

$$\Lambda = \{\lambda \in \mathbb{C} \,|\, \varepsilon_0 \le \arg(\lambda) \le 2\pi - \varepsilon_0 \text{ for some } 0 < \varepsilon_0 < \pi/2\}.$$

Let $A \in x^{-\mu}\Psi_b^{\mu}(M)$, $\mu > 0$, be such that $A - \lambda$ is parameter-elliptic on Λ with respect to some α. Then the heat operator of A can defined as the Cauchy integral

$$e^{-tA} = \frac{i}{2\pi}\int_{\Upsilon} e^{-t\lambda}\,(A-\lambda)^{-1}\, d\lambda, \tag{5.11}$$

where Υ is an counter-clockwise contour in Λ of the form

$$\Upsilon = a + \{\lambda \in \mathbb{C} \,|\, \arg(\lambda) = \delta \text{ or } \arg(\lambda) = 2\pi - \delta\},\quad a < 0,\ \ \varepsilon_0 < \delta < \pi/2.$$

Integrating by parts $N - 1$ times, we can rewrite (5.11) as

$$e^{-tA} = \frac{i}{2\pi}\frac{(-t)^{-N+1}}{(N-1)!}\int_{\Upsilon} e^{-t\lambda}\,(A-\lambda)^{-N}\, d\lambda. \tag{5.12}$$

The asymptotic analysis from [1, Sect. 4.6] applied to the expansion (5.5) induces the following theorem.

Theorem 5.13 *Let $A \in x^{-\mu}\Psi_b^{\mu}(M)$, $\mu > 0$, be such that $A - \lambda$ is parameter-elliptic on Λ with respect to some α. Then, given any $B \in x^{-\beta}\Psi_b^{\mu'}(M)$ with $\beta, \mu' \in \mathbb{R}$, the operator Be^{-tA} is trace class for $t > 0$, and*

$$\operatorname{Tr} Be^{-tA} \sim_{t\to 0^+} \sum_{k=0}^{\infty} \Big\{\alpha_k + \beta_k \log t + \gamma_k (\log t)^2\Big\}\, t^{(k-\mu'-n)/\mu} + \sum_{k=0}^{\infty} \Big\{\delta_k + \varepsilon_k \log t\Big\} t^{(k-\beta)/\mu} + \sum_{k=0}^{\infty} \kappa_k\, t^k. \tag{5.14}$$

Moreover, $\beta_k = 0$ unless $k \in (\mathbb{N}_0 + \mu' + n - \beta) \cup (\mu\mathbb{N}_0 + \mu' + n)$; $\gamma_k = 0$ unless $k \in \mu\mathbb{N}_0 \cap (\mathbb{N}_0 - \beta) + \mu' + n$; and $\varepsilon_k = 0$ unless $k \in \mu\mathbb{N}_0 + \beta$.

Now suppose that $(A-\lambda)^{-1}$ exists on a neighborhood of Λ. Then as in [15] one can show that the complex power A^z of A exists and defines an entire family of b-pseudodifferential operators satisfying $A^z A^w = A^{z+w}$ for $z, w \in \mathbb{C}$. Using the following formula for the complex powers in terms of the heat operator

$$A^z = \frac{1}{\Gamma(-z)} \int_0^{\infty} t^{-z} e^{-tA}\, \frac{dt}{t}, \quad \Re z << 0,$$

we can write

$$\zeta_A(z) := \operatorname{Tr} A^z = \frac{1}{\Gamma(-z)} \mathcal{M}(f)(-z),$$

where $\mathcal{M}(f)(z)$ is the Mellin transform of the function $f(t) = \operatorname{Tr}(e^{-tA})$. Applying the results on the poles of Mellin transforms found in [1, Sect. 4.3], using the expansion (5.14) of $\operatorname{Tr}(e^{-tA})$ as $t \to 0$, plus the fact that $1/\Gamma(-z)$ vanishes for $z \in \mathbb{N}_0$, we obtain the following theorem that generalizes Seleey's results, cf. [39].

Theorem 5.15 (Analyticity of the Zeta function) *The zeta function $\zeta_A(z)$ is holomorphic for $\Re z < -n/\mu$ and extends to be meromorphic on $\mathbb{C}$, with (possible) simple poles on the set $\big\{\frac{k-n}{\mu} \mid k \in \mathbb{N}_0\big\}$ and (possible) triple poles on the set $\big\{\frac{k}{\mu} \mid k \in \mathbb{N}_0,\ \frac{k}{\mu} \notin \mathbb{N}_0\big\}$.*

6 Properties of the index

At last we consider the problem of finding the index of the closed extensions of a b-elliptic differential cone operator A and give a formula for the index of its closure. To this end we first prove that, for the purpose of index calculations, some significant simplifications can be made. In fact, one can reduce the problem to the case where the operator has coefficients independent of x near ∂M, and even more, one can assume $\mathcal{D}_{\min}(A)$ to be a weighted Sobolev space. These results show that simplifying assumptions made by various authors in the past can indeed be used without lost of generality.

Invariance properties of the index Let M be a smooth compact manifold with boundary. Let $A \in x^{-\mu}\operatorname{Diff}_b^m(M)$ be b-elliptic, $\mu > 0$. We regard A as an unbounded operator $A : C_c^{\infty}(M) \subset x^{\nu}L_b^2(M) \to x^{\nu}L_b^2(M)$ and denote by $\mathcal{D}_{\min}(A)$ the domain of the closure of

A. It is convenient to assume $\nu = -\mu/2$; we can always reduce to this case by conjugation with $x^{\nu+\mu/2}$. It is known (cf. [16,22]) that every closed extension $A_{\mathcal{D}}$ of A on $x^{-\mu/2}L_b^2(M)$ is Fredholm with index

$$\operatorname{ind} A_{\mathcal{D}} = \operatorname{ind} A_{\mathcal{D}_{\min}} + \dim \mathcal{D}/\mathcal{D}_{\min}.$$

Note that $\dim \mathcal{D}/\mathcal{D}_{\min}$ is completely determined by the boundary spectrum of A. In this section we will give an analytic formula for the index of $A_{\mathcal{D}_{\min}}$ using the heat trace asymptotics obtained in the previous section.

We shall need the following lemma which also establishes the notation.

Lemma 6.1 *On $\mathcal{D}_{\min}(A)$, for $\varepsilon > 0$ small enough, the operator norm*

$$\|u\|_A = \|u\|_{x^{-\mu/2}L_b^2} + \|Au\|_{x^{-\mu/2}L_b^2}$$

and the norm

$$\|u\|_{A,\varepsilon} = \|u\|_{x^{\mu/2-\varepsilon}L_b^2} + \|Au\|_{x^{-\mu/2}L_b^2}.$$

are equivalent.

Proof Recall that the embedding $x^{\mu/2-\varepsilon}L_b^2 \hookrightarrow x^{-\mu/2}L_b^2$ is continuous for $\varepsilon < \mu$. The equivalence of the norms follows from the continuity of $(\mathcal{D}_{\min}(A), \|\cdot\|_A) \hookrightarrow x^{\mu/2-\varepsilon}L_b^2$ which is a consequence of the closed graph theorem. □

Write $D_x = -i\frac{\partial}{\partial x}$. The operator $A = x^{-\mu}P$ is said to have coefficients independent of x near the boundary if $(xD_x)P = P(xD_x)$ near ∂M. Write $A = A_0 + xA_1$ with A_0 having coefficients independent of x near ∂M. Let $\varphi \in C_c^\infty(\mathbb{R})$, $\varphi = 1$ near 0. Furthermore, for $\tau > 0$ let $\varphi_\tau = \varphi(x/\tau)$ and let

$$A_{[\tau]} = \varphi_\tau A_0 + (1-\varphi_\tau)A.$$

Clearly, A and $A_{[\tau]}$ have the same conormal symbol (indicial family).

Proposition 6.2 *For small enough $\tau > 0$ the operator $A_{[\tau]}$ is also b-elliptic and therefore $\mathcal{D}_{\min}(A_{[\tau]}) = \mathcal{D}_{\min}(A)$. Moreover, as $\tau \to 0$, $A_{[\tau]} \to A$ in the graph norm of A. Thus, on $\mathcal{D}_{\min}(A)$,*

$$\operatorname{ind} A_{[\tau]} = \operatorname{ind} A$$

for every small $\tau > 0$.

Proof Let ${}^b\sigma_m(A)$ denote the totally characteristic principal symbol of A. Then,

$$\begin{aligned}
{}^b\sigma_m(A_{[\tau]}) &= \varphi_\tau\, {}^b\sigma_m(A_0) + (1-\varphi_\tau)\, {}^b\sigma_m(A)\\
&= \varphi_\tau\, {}^b\sigma_m(A) + (1-\varphi_\tau)\, {}^b\sigma_m(A) - x\varphi_\tau\, {}^b\sigma_m(A_1)\\
&= {}^b\sigma_m(A) - \tau\tilde{\varphi}_\tau\, {}^b\sigma_m(A_1)
\end{aligned}$$

with $\tilde{\varphi}_\tau = (x/\tau)\varphi(x/\tau)$. Since $\tilde{\varphi}_\tau$ is bounded, $\tau\tilde{\varphi}_\tau$ is small for τ small, and thus the invertibility of ${}^b\sigma_m(A)$ implies that of ${}^b\sigma_m(A) - \tau\tilde{\varphi}_\tau\, {}^b\sigma_m(A_1)$ for such τ. Hence $A_{[\tau]}$ is b-elliptic too. Since A and $A_{[\tau]}$ have the same conormal symbol, we have from [16, Proposition 4.1] that $\mathcal{D}_{\min}(A_{[\tau]}) = \mathcal{D}_{\min}(A)$.

Further, from the b-ellipticity of A it follows that there is a bounded parametrix $Q : x^\gamma H_b^s \to x^{\gamma+\mu}H_b^{s+m}$ such that

$$R = I - QA : x^\gamma H_b^s \to x^\gamma H_b^\infty$$

is bounded for all s and γ. Write

$$A - A_{[\tau]} = x\varphi_\tau A_1 = x\varphi_\tau A_1 QA + x\varphi_\tau A_1 R$$
$$= \tau\tilde{\varphi}_\tau A_1 QA + x\varphi_\tau A_1 R.$$

Now, $A_1 Q : x^{-\mu/2}L_b^2 \to x^{-\mu/2}L_b^2$ is bounded, so if $u \in \mathcal{D}_{\min}(A)$, then

$$\|\tau\tilde{\varphi}_\tau A_1 QAu\|_{x^{-\mu/2}L_b^2} \le c\,\tau\|Au\|_{x^{-\mu/2}L_b^2} \le c\,\tau\|u\|_A.$$

Write $x\varphi_\tau A_1 R = \tau^{1-\varepsilon}(\frac{x}{\tau})^{1-\varepsilon}\varphi_\tau\, x^\varepsilon A_1 R$ and note that

$$x^\varepsilon A_1 R : x^{\mu/2-\varepsilon}L_b^2 \to x^{-\mu/2}L_b^2$$

in continuous. Then using Lemma 6.1 we get

$$\|x\varphi_\tau A_1 Ru\|_{x^{-\mu/2}L_b^2} \le \tilde{c}\,\tau^{1-\varepsilon}\|u\|_{x^{\mu/2-\varepsilon}L_b^2} \le c\,\tau^{1-\varepsilon}\|u\|_A.$$

Altogether,

$$\|(A - A_{[\tau]})u\|_{x^{-\mu/2}L_b^2} \le C\,\tau^{1-\varepsilon}\|u\|_A$$

and thus $A_{[\tau]} \to A$ as $\tau \to 0$. □

Remark 6.3 Norm estimates related to those obtained in the previous proof can be found in the book by Lesch [22, Lemma 1.3.10].

In general, $\mathcal{D}_{\min}$ is not a Sobolev space. The problem lies in the possible presence of elements of $\operatorname{spec}_b(A)$ along the line $\Im\sigma = -\mu/2$. However, for index purposes, one can conveniently reduce the analysis to a slightly modified operator whose closure has a Sobolev space as its domain.

Proposition 6.4 *Let A be b-elliptic. Let $A_\varepsilon = x^\varepsilon A$, and regard it as an unbounded operator on $x^{-(\mu-\varepsilon)/2}L_b^2(M)$. If $\varepsilon > 0$ is sufficiently small, then*

$$A_\varepsilon : x^{(\mu-\varepsilon)/2}H_b^m(M) \to x^{-(\mu-\varepsilon)/2}L_b^2(M)$$

is Fredholm, and

$$\operatorname{ind} A_\varepsilon = \operatorname{ind} A_{\mathcal{D}_{\min}}.$$

Proof Write $A = x^{-\mu}P$ with $P \in \operatorname{Diff}_b^m(M)$. Let $\eta > 0$ be so small that there is no $\sigma \in \operatorname{spec}_b(A)$ with $\mu/2 - \eta \le \Im\sigma < \mu/2$ or $-\mu/2 < \Im\sigma \le -\mu/2 + \eta$. The kernel of A on tempered distributions $x^{-\infty}H_b^{-\infty}(M)$ is the same as that of P, which we denote by $K(P)$. Recall that $\mathcal{D}_{\max}(A) = \{u \in x^{-\mu/2}L_b^2 \,|\, Au \in x^{-\mu/2}L_b^2\}$. The kernel $K_{\max}(A)$ of $A : \mathcal{D}_{\max} \subset x^{-\mu/2}L_b^2 \to x^{-\mu/2}L_b^2$ consists of those elements of $K(P)$ whose Mellin transforms are holomorphic in $\Im\sigma \ge \mu/2$; since these elements belong to $x^{-\mu/2}L_b^2$ and $Au \in x^{-\mu/2}L_b^2$. That is, their Mellin transforms are holomorphic on $\Im\sigma > \mu/2 - \eta$. Thus $K_{\max}(A) = K_{\max}(A_\varepsilon)$ if $0 < \varepsilon < \eta$. On the other hand, the kernel $K_{\min}(A)$ of $A : \mathcal{D}_{\min} \subset x^{-\mu/2}L_b^2 \to x^{-\mu/2}L_b^2$ consists of those elements of $K(P)$ whose Mellin transforms are holomorphic in $\Im\sigma > -\mu/2$; indeed in [16, Proposition 3.6] it is shown show that $\mathcal{D}_{\min} = \mathcal{D}_{\max} \cap x^{\mu/2-\eta}H_b^m$. Thus if $\varepsilon < \eta$ then $K_{\min}(A) = K_{\min}(A_\varepsilon)$. Consequently, $\dim K_{\min}(A) = \dim K_{\min}(A_\varepsilon)$.

Finally, note that the formal adjoint of A in $x^{-\mu/2}L_b^2$ is $A^\star = x^{-\mu}P^\star$, where $P^\star$ is the formal adjoint of P in L_b^2, and likewise $A_\varepsilon^\star = x^{-\mu+\varepsilon}P^\star$. Now recall that the Hilbert adjoint of $A_{\mathcal{D}_{\min}}$ is $A^\star$ with domain $\mathcal{D}_{\max}(A^\star)$, so the first part of the argument yields $\dim K_{\max}(A^\star) = \dim K_{\max}(A_\varepsilon^\star)$. □

Index formula According to the previous discussion, we can reduce the computation of the index of the closure of a b-elliptic differential operator A to the case where A has coefficients independent of x near ∂M and such that

$$A : x^{\mu/2} H_b^m(M) \to x^{-\mu/2} L_b^2(M) \tag{6.5}$$

is Fredholm. Under these assumptions, we will give a formula for the index of A in the spirit of [3,9,12,13,22,23,32,34,38] that holds even when A is pseudodifferential.

Recently, Witt [42] proved a factorization theorem for operator-valued elliptic Mellin symbols. Using his result, it follows that there is a cone pseudodifferential operator B with empty boundary spectrum, and a smoothing Mellin operator H, such that $A - B(1 + H)$ is compact. This implies

$$\operatorname{ind} A = \operatorname{ind}(B(1 + H)) = \operatorname{ind} B + \operatorname{ind}(1 + H).$$

Note that ${}^b\sigma_m(A) = {}^b\sigma_m(B)$ and $\operatorname{spec}_b(A) = \operatorname{spec}_b(1 + H)$. In other words, this formula separates the index contributions from the totally characteristic principal symbol and the boundary spectrum of A.

We first discuss the index of $B : x^{\mu/2} H_b^m(M) \to x^{-\mu/2} L_b^2(M)$.

Lemma 6.6 *If $B \in x^{-\mu}\Psi_b^m(M)$ is b-elliptic with* $\operatorname{spec}_b(B) = \varnothing$, *then*

$$\operatorname{ind} B = \operatorname{Tr} e^{-tB^\star B} - \operatorname{Tr} e^{-tBB^\star} \quad \textit{for } t > 0, \tag{6.7}$$

where $B^\star$ is the formal adjoint of B.

Proof In general, the Hilbert space adjoint B^* of B on $x^{-\mu/2} L_b^2$ is not equal to but rather a closed extension of the formal adjoint $B^\star$. However, since $\operatorname{spec}_b(B) = \varnothing$, we also have $\operatorname{spec}_b(B^\star) = \varnothing$ and therefore $\mathcal{D}_{\min}(B^\star) = \mathcal{D}_{\max}(B^\star) = x^{\mu/2} H_b^m$. Thus B^* must be equal to $B^\star$ with domain $x^{\mu/2} H_b^m$, so (6.7) is nothing but the well-known Mckean–Singer identity. □

The identity (6.7) is not always true because $B^\star$ may be different from the Hilbert space adjoint B^*. The condition on the boundary spectrum of B is what makes it work. The consequence of the previous lemma is that since $B^\star$ is a cone pseudodifferential operator, we can apply our results from Sect. 5 to get an asymptotic expansion of the right-hand side of (6.7) as $t \to 0$, and obtain

$$\operatorname{ind} B = \omega(B, B^\star),$$

where $\omega(B, B^\star)$ is the constant term in the expansion.

On the other hand, it follows from Piazza [34] that

$$\operatorname{ind}(1 + H) = -\eta_{\mu/2}(0, 1 + \hat{H}),$$

where (cf. also [30,32])

$$\eta_{\mu/2}(0, 1 + \hat{H}) = \frac{1}{2\pi i} \int\limits_{\Im\sigma = -\mu/2} \operatorname{Tr}\left(\frac{d}{d\sigma}\hat{H}(\sigma)\,(1 + \hat{H}(\sigma))^{-1}\right) d\sigma.$$

As a consequence, we obtain the following index formula.

Theorem 6.8 *Let $A = B(1 + H)$ as above. Then the index of* (6.5) *is given by*

$$\operatorname{ind} A = \omega(B, B^\star) - \frac{1}{2\pi i} \int\limits_{\Im\sigma = -\mu/2} \operatorname{Tr}\left(\frac{d}{d\sigma}\hat{H}(\sigma)\,(1 + \hat{H}(\sigma))^{-1}\right) d\sigma.$$

Acknowledgements We thank Gerardo Mendoza for many valuable discussions.

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