Index theory of Dirac operators on manifolds with corners up to codimension two

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Abstract. In this expository article, we survey index theory of Dirac operators using the Gauss-Bonnet formula as the catalyst to discuss index formulas on manifolds with and without boundary. Considered in detail are the Atiyah-Singer and Atiyah-Patodi-Singer index theorems, their heat kernel proofs, and their generalizations to manifolds with corners of codimension two via the method of 'attaching cylindrical ends'.

1. Introduction: The Gauss-Bonnet formula and index theory

The purpose of this paper is to serve as an overview of index theory for Dirac operators on manifolds with corners with emphasis on the b-geometry approach of Melrose [**59**] to such a theory. The underlying theme of this paper is that index formulas are basically generalizations of the classical Gauss-Bonnet formula.

This paper is organized as follows. First, to understand what index theory is and why it is important, we recall the Gauss-Bonnet formula. In particular, we interpret the Gauss-Bonnet formula as an index formula. This interpretation leads us naturally to the Atiyah-Singer index formula for Dirac operators on manifolds without boundary published in 1963 [**6**], which we discuss in Section 2. In 1973, Atiyah, Patodi, and Singer in the seminal paper [**4**] extended the Atiyah-Singer formula for Dirac operators to manifolds with smooth boundary. We present this formula from the 'cylindrical end' point of view in Section 3. We also reformulate the Atiyah-Patodi-Singer (henceforth APS) problem using the language and notation of the b-geometry. In Section 4, we present a b-geometric proof of the APS index formula. Currently, there is no direct analog of the 'APS index formula' for manifolds with corners of codimension two, except under certain restrictive nondegeneracy conditions [**51**], [**67**], which we discuss in Section 5. However, in joint

2000 *Mathematics Subject Classification.* Primary 58J20; Secondary 58J28, 47A53.
Key words and phrases. Index theory; b-pseudodifferential operators; Dirac operators.
Supported by a Ford Foundation Fellowship administered by the National Research Council.

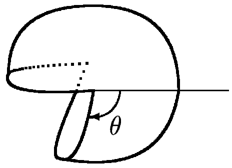

FIGURE 1. The manifold P_θ, where $-\pi < \theta < \pi$, is the unit sphere with a wedge removed.

work with Melrose [**53**], these restrictions are removed by perturbing Dirac operators using b-smoothing operators. We discuss such perturbations and the resulting index formulas for the perturbed Dirac operators in Section 6.

Finally, I thank Gerd Grubb, Rafe Mazzeo, and Richard Melrose for helping to make this research possible. I also thank the referees for valuable suggestions.

1.1. The classical Gauss-Bonnet formula. Let M be a compact, oriented, two-dimensional Riemannian manifold without boundary. Then the Gauss-Bonnet theorem states that

$$\chi(M) = \frac{1}{2\pi} \int_M K, \tag{1.1}$$

where $\chi(M)$ is the Euler characteristic of M and K is the Gaussian curvature of M. The interesting aspect of the Gauss-Bonnet formula is that the left-hand side is a topological/combinatorial object while the right-hand side is a geometric object.[1] This formula was proved by Bonnet in 1848, but is attributed also to Gauss because he proved a special case of it earlier. See [**68**, Ch. 8] for a proof of the Gauss-Bonnet formula.

A natural question to ask is: Does the Gauss-Bonnet formula continue to hold if M has a smooth boundary, or more generally, if M has corners; that is, has a crooked boundary? To answer this question, we consider a concrete example. Cut out a wedge from the unit sphere producing the manifold P_θ, where $-\pi < \theta < \pi$, as shown in Figure 1. This manifold is an example of a manifold with corners of codimension two. Let us check if (1.1) holds verbatim for P_θ. Note that P_θ is topologically equivalent to a disk and hence to a triangle and so has Euler characteristic equal to one (since the number of vertices $-$ edges $+$ faces $= 1$ for a triangle). Since the Gaussian curvature of the unit sphere is one, $\int_{P_\theta} K = \text{Area}(P_\theta)$. Thus $\int_{P_\theta} K$ changes with θ: It is approximately 4π when θ is close to $-\pi$ and it decreases to 0 as θ approaches $+\pi$. Hence,

$$\chi(P_\theta) \neq \frac{1}{2\pi} \int_{P_\theta} K \quad \text{for general } \theta.$$

Thus the Gauss-Bonnet formula (1.1) does not hold verbatim when M has corners. Intuitively, one might guess that the formula does not hold because of the presence

[1]The right-hand side of (1.1) turns out to be topological as well since $-K/2\pi$ defines the Chern class of M.

of the boundary and corners. In fact, one can verify directly that the following formula does hold:

$$\chi(P_\theta) = \frac{1}{2\pi}\int_{P_\theta} K + \frac{1}{2\pi}(2\theta). \tag{1.2}$$

The angle 2θ is called the total exterior angle of the corners. This formula is a special case of the general Gauss-Bonnet formula:

THEOREM 1.1 (Gauss-Bonnet, 1878). *Given a compact, oriented, two-dimensional Riemannian manifold M with corners, we have*

$$\begin{aligned}\chi(M) = \frac{1}{2\pi}&\int_M K \\ &+ \frac{1}{2\pi}(\textit{total geodesic curvature of } \partial M) \\ &+ \frac{1}{2\pi}(\textit{sum of the exterior angles at the corners}).\end{aligned} \tag{1.3}$$

Here, the geodesic curvature of ∂M measures the deviation of the smooth components of ∂M from being geodesics. There is no middle term in the formula (1.2) since the smooth components of ∂P_θ are great circles, which are geodesics on the sphere. Hence, the total geodesic curvature of ∂P_θ is zero.

The Gauss-Bonnet formula in (1.3) is very beautiful as it bridges topology, geometry, and now linear algebra: The left-hand side belongs to combinatorial topology while the first two terms on the right are geometrical and the last term is linear algebraic since it has to do with angles between vectors at the corners. Functional analysis also comes into the picture when we interpret the Gauss-Bonnet formula as an index formula. We remark that comparing the Gauss-Bonnet formula (1.1) for a manifold without boundary to the general formula (1.3), we see that the second and third terms on the right in (1.3) can be thought of as correction terms coming from the smooth boundary components and corners respectively.

1.2. The Gauss-Bonnet formula as an index formula. We now explain how the Gauss-Bonnet formula can be interpreted as an index formula. We first need to introduce the Gauss-Bonnet operator. Let M be a compact, oriented, two-dimensional Riemannian manifold without boundary. Let

$$d : C^\infty(M, \Lambda^k) \to C^\infty(M, \Lambda^{k+1})$$

be the exterior derivative, where $C^\infty(M, \Lambda^k)$ denotes the space of smooth k-forms on M, and let

$$d^* : C^\infty(M, \Lambda^{k+1}) \to C^\infty(M, \Lambda^k)$$

be the adjoint of d with respect to the natural L^2 inner product on k-forms given by integration with respect to the Riemannian volume form. Let $\Lambda^{ev} = \Lambda^0 \oplus \Lambda^2$ be the even form bundle and $\Lambda^{odd} = \Lambda^1$ be the odd form bundle. Then both d and d^* map $C^\infty(M, \Lambda^{ev})$ into $C^\infty(M, \Lambda^{odd})$. The operator

$$D_{GB} = d + d^* : C^\infty(M, \Lambda^{ev}) \to C^\infty(M, \Lambda^{odd})$$

is called the *Gauss-Bonnet operator*. By definition of the (nonnegative) Laplacian,

$$D_{GB}^* D_{GB} = \Delta,$$

where Δ is the Laplacian on the even forms. Thus, D_{GB} represents in some respects a square root of the Laplacian. For this reason D_{GB} is called a *Dirac operator* after the physicist Paul Dirac who in the 1920's was searching for, and found, a square root of the Laplacian in his efforts to quantize the electron. However, in his case, he was working with a Lorentz metric rather than a Riemannian metric.

Before presenting the index formula interpretation of the Gauss-Bonnet formula, we recall two results from Hodge theory. We denote the Sobolev space of order k by H^k. So, $H^k(M, \Lambda^{ev})$ consists of those even-degree forms u on M such that for each $0 \leq j \leq k$, $(d + d^*)^j u$ is square integrable. Then $H^1(M, \Lambda^{ev})$ is the natural domain of D_{GB}.

THEOREM 1.2. *The operator $D_{GB} : H^1(M, \Lambda^{ev}) \to L^2(M, \Lambda^{odd})$ is Fredholm, which means that it is 'almost invertible' in the sense that*

(1) *D_{GB} has a finite dimensional kernel;* $\dim \ker D_{GB} < \infty$.
(2) *D_{GB} has a finite dimensional cokernel;*

$$\dim \operatorname{coker} D_{GB} = \dim \big(L^2(M, \Lambda^{odd}) / \operatorname{Im}(D_{GB}) \big) < \infty.$$

The first condition means that D_{GB} is 'almost injective' in the sense that it is injective up to a finite dimensional space, and the second condition means that D_{GB} is 'almost surjective' in the sense that it is surjective up to a finite dimensional space. The *index* is the difference between the dimensions of the kernel and cokernel:

$$\operatorname{ind} D_{GB} = \dim \ker D_{GB} - \dim \operatorname{coker} D_{GB} \in \mathbb{Z}.$$

Theorem 1.2 follows from the fact that D_{GB} is *elliptic*, and can be proved in a variety of ways, for instance, using pseudodifferential operators [**34**], by embedding properties of Sobolev spaces [**74**], or my favorite proof via the heat operator [**9**]. The second result we need is:

THEOREM 1.3. *The index of D_{GB} is the Euler characteristic of M,*

$$\operatorname{ind} D_{GB} = \chi(M).$$

This result can be proved using the Hodge theorem, which is described as follows. Define the k-th deRham cohomology of M by

$$H^k_{\mathrm{dR}}(M) = \{\alpha \in C^\infty(M, \Lambda^k) \,;\, d\alpha = 0\} / \{d\beta \,;\, \beta \in C^\infty(M, \Lambda^{k-1})\}.$$

The Hodge theorem states that given a deRham cohomology class $[\alpha] \in H^k_{\mathrm{dR}}(M)$ there exists a unique representative of this class $\beta \in [\alpha]$ such that $(d + d^*)\beta = 0$. It is worthwhile mentioning that although the exterior derivative d is canonical, the operator d^* depends on the Riemannian metric chosen on the manifold. The work of Connes with Gromov and Moscovici [**25**] treats a branch of index theory which deals with the analytic difficulties involved in not having a suitable invariant inner product. In any case, the Hodge theorem implies the important fact

$$\ker(d + d^*) \text{ on } k \text{ forms} \cong H^k_{\mathrm{dR}}(M).$$

Now using the description of the Euler characteristic in terms of the cohomology:

$$\begin{aligned}\chi(M) &= \sum_{k \text{ even}} \dim H^k_{\mathrm{dR}}(M) - \sum_{k \text{ odd}} \dim H^k_{\mathrm{dR}}(M) \\ &= \dim H^0_{\mathrm{dR}}(M) + \dim H^2_{\mathrm{dR}}(M) - \dim H^1_{\mathrm{dR}}(M),\end{aligned}$$

and the fact that $\operatorname{coker} D_{GB} \cong \ker D^*_{GB}$, one gets Theorem 1.3. In view of the Gauss-Bonnet formula (1.1), we finally have

Theorem 1.4 (Index version of Gauss-Bonnet). *For the Gauss-Bonnet operator on a compact, oriented, two-dimensional Riemannian manifold M without boundary, we have*

$$\operatorname{ind} D_{GB} = \frac{1}{2\pi} \int_M K.$$

The profound aspect about this version of the Gauss-Bonnet formula is that the left-hand side is a functional analytic object related to the existence and uniqueness of solutions to the equation $D_{GB} u = f$, while the right-hand side is a topological/geometric object. Hence, this formula implies the important fact that the topology/geometry of the manifold can be investigated using functional analysis. We now discuss a far-reaching generalization of this formula to higher dimensional manifolds without boundary. We discuss generalizations of the Gauss-Bonnet formula (1.3) for manifolds with corners in Sections 3, 5, and 6.

2. The Atiyah-Singer index formula

According to Hirzebruch (cf. [**14**, p. vii]) the Atiyah-Singer formula is "one of the deepest and hardest results in mathematics", "probably has wider ramifications in topology and analysis than any other single result". Although the proof of the Atiyah-Singer formula is difficult, understanding it is not if one keeps in mind that it is basically a higher dimensional analog of the Gauss-Bonnet formula.

In this section, we describe the Atiyah-Singer index formula and we outline its proof using the 'heat kernel method'.

2.1. Statement of the Atiyah-Singer index theorem. There are two ingredients to the Atiyah-Singer index formula. The first is topological/geometric data: Let M be an even-dimensional, compact, oriented, Riemannian manifold without boundary, and let E and F be Hermitian vector bundles over M. The second ingredient is functional analytic/geometric data: Let

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

be a *Dirac type operator*. This means that D is an elliptic first-order differential operator such that "$D^*D = \Delta$" in the sense that the principal symbol of D^*D is just the metric $\sigma(D^*D)(\xi) = |\xi|^2$ for all cotangent vectors ξ.[2]

The simplest example of a Dirac type operator is the Cauchy-Riemann operator. Let $M = \mathbb{R}^2$ with its usual Euclidean metric and let $E = F = \mathbb{C}$. Then

$$D_{CR} = \partial_x + i\partial_y$$

is the *Cauchy-Riemann operator*. In this case,

$$D^*_{CR} D_{CR} = \big(- \partial_x + i\partial_y\big)\big(\partial_x + i\partial_y\big) = -\big(\partial_x^2 + \partial_y^2\big)$$

is exactly the Laplacian.

Another example is the higher-dimensional Gauss-Bonnet operator. Let M be an even-dimensional, compact, oriented, Riemannian manifold without boundary, and let E and F be the even and odd degree form bundles, respectively:

$$E = \Lambda^{ev} = \bigoplus_{k \text{ even}} \Lambda^k \quad \text{and} \quad F = \Lambda^{odd} = \bigoplus_{k \text{ odd}} \Lambda^k.$$

Then,

$$D_{GB} = d + d^* : C^\infty(M, \Lambda^{ev}) \to C^\infty(M, \Lambda^{odd})$$

is called the *Gauss-Bonnet operator*. By definition of the Laplacian on forms, we have $D^*_{GB} D_{GB} = \Delta$, and so D_{GB} is a Dirac operator. Since the even and odd form bundles are real bundles, they of course are technically not Hermitian, but we can always make them so by complexifying them. Regardless, the index formula below still applies to the Gauss-Bonnet operator.

The celebrated Atiyah-Singer index theorem is the following.

THEOREM 2.1 (Atiyah-Singer, 1963). *Let $D : C^\infty(M, E) \to C^\infty(M, F)$ be a Dirac type operator on an even-dimensional, compact, oriented, Riemannian manifold without boundary. Then, $D : H^1(M, E) \to L^2(M, F)$ is Fredholm and*

$$\operatorname{ind} D = \int_M K_{AS}, \tag{2.1}$$

where the 'Atiyah-Singer integrand' K_{AS} is an explicitly defined polynomial in the curvature forms of the manifold M and the vector bundles E and F.

The integral of the polynomial K_{AS} is by definition the integral of the volume form component of K_{AS}. For those readers familiar with characteristic classes, the polynomial K_{AS} is the product of the $\hat{A}$ polynomial of M and the (relative) Chern polynomials of E and F, see [**9**, Ch. 4]. Hence, K_{AS} is both a topological and a geometric object. The reason why we assume that M is even-dimensional is that for odd-dimensional manifolds it turns out that both sides of (2.1) are zero.

Note the similarity between the index version of the Gauss-Bonnet formula given in Theorem 1.4 and the Atiyah-Singer formula for the index of a Dirac type

[2]In some occasions, we will need D to be a *compatible* Dirac operator, which means that it is associated to a 'unitary Clifford connection'. For simplicity, we leave this notion undefined, and at the few places where we actually need this extra hypothesis, we will state so in a footnote.

operator: The Gaussian curvature in the Gauss-Bonnet formula is replaced by a polynomial in the curvature of the manifold and vector bundles. As with the Gauss-Bonnet formula, the profound feature of the Atiyah-Singer formula is that the left-hand side of (2.1) is a functional analytic object related to the existence and uniqueness of solutions to the equation $Du = f$, while the right-hand side is a topological/geometric object. In particular, the Atiyah-Singer formula has the following deep consequence: It implies that the topology/geometry of a manifold can be investigated using functional analytic tools, cf. [**72**], [**55**], and Section 2.3.

For an application of the Atiyah-Singer index theorem, consider the Gauss-Bonnet operator defined above. As in the two-dimensional case explained before Theorem 1.4, for a general even-dimensional, compact, oriented, Riemannian manifold without boundary, Hodge theory implies that

$$\operatorname{ind} D_{GB} = \chi(M).$$

On the other hand, see [**9**, Ch. 4] for the details, working out the explicitly defined polynomial K_{AS} for $E = \Lambda^{ev}$ and $F = \Lambda^{odd}$ gives, after a little bit of algebra,

$$K_{AS} = e(M),$$

where $e(M)$ is the Euler, or Pfaffian, polynomial defined by taking the n-th power of the Riemannian curvature tensor of M and multiplying it by $1/n! \times (-1/2\pi)^n$, where $2n$ is the dimension of M. Thus, the Atiyah-Singer index formula implies

$$\chi(M) = \int_M e(M).$$

This generalization of the Gauss-Bonnet formula is due to Chern [**23**].

Another important corollary of the Atiyah-Singer formula is Hirzebruch's formula for the signature of a manifold. Assume now that M is $4k$ dimensional. Then the map

$$H^{2k}_{\mathrm{dR}}(M) \times H^{2k}_{\mathrm{dR}}(M) \ni ([\,\alpha\,], [\,\beta\,]) \longmapsto \int_M \alpha \wedge \beta \in \mathbb{R}$$

is a well-defined symmetric bilinear map (here, $[\,\cdot\,]$ denotes the corresponding cohomology class). We can represent this map by a matrix by choosing any basis of the finite dimensional real vector space $H^{2k}_{\mathrm{dR}}(M)$. The signature of this matrix, the number of positive eigenvalues minus the number of negative ones, is defined independent of the basis chosen; the signature of the manifold, $\operatorname{sign}(M)$, is by definition the signature of the matrix with respect to any such basis. Hirzebruch [**42**] gives a formula for the signature:

$$\operatorname{sign}(M) = \int \mathcal{L}(M),$$

where $\mathcal{L}(M)$ is the $\mathcal{L}$-class polynomial in the curvature tensor of M. This formula is a simple corollary of the Atiyah-Singer index theorem. In this case, D is the 'signature operator', which is equal to $d + d^*$ like the Gauss-Bonnet operator, but with the vector bundles E and F being essentially certain eigenspaces of the Hodge star operator. The details can be found in [**9**]. For those interested readers, the $\mathcal{L}$-class polynomial is given by

$$\mathcal{L}(M) = \sqrt{\det\Big(\frac{R/2\pi i}{\tanh(R/2\pi i)}\Big)},$$

where R is the Riemannian curvature tensor.

Yet another classical formula that is a simple corollary of the Atiyah-Singer index formula is the Riemann-Roch formula and its generalization to complex manifolds, see [**9**], [**34**], [**74**].

2.2. Outline of the proof of the Atiyah-Singer formula. We outline a proof of the Atiyah-Singer index theorem based on the heat kernel approach of Mckean and Singer [**57**] as exploited by Atiyah, Bott, and Patodi in [**2**]. For accessible versions of the proof, see [**9**], [**34**], [**82**]. I especially like the exposition by Roe [**74**].

To start off, we need the heat operators e^{-tD^*D} and e^{-tDD^*}. Consider for instance e^{-tD^*D}. Then, for each $t > 0$,

$$e^{-tD^*D} : C^\infty(M, E) \to C^\infty(M, E),$$

and it is the solution operator to the heat equation for D^*D in the sense that for each $u \in C^\infty(M, E)$, $u_t = e^{-tD^*D}u$ is the unique solution to the heat equation

$$(\partial_t + D^*D)u_t = 0, \quad t > 0; \qquad u_0 = u.$$

The heat operator e^{-tD^*D} can be defined by means of the resolvent and the functional calculus (cf. [**34**], [**51**]), it can be constructed asymptotically via Hadamard's method (cf. [**9**], [**59**]), or it can be defined using the spectrum as follows. Let $\{\lambda_j\}$ be the eigenvalues of the self-adjoint operator D^*D. Then,

$$e^{-tD^*D} = \sum_j e^{-t\lambda_j}\pi_j,$$

where π_j is the orthogonal projection onto the eigenspace associated to the eigenvalue λ_j. This sum converges uniformly and absolutely and in fact, it can be used to show that the heat operator is, for each $t > 0$, a smoothing operator; that is, for each $t > 0$ the heat operator is an integral operator with a smooth Schwartz kernel [**74**]. In particular, for each $t > 0$, the heat operator is trace class and

$$e^{-tD^*D} = \pi_{\ker D^*D} + F(t), \tag{2.2}$$

where the remainder $F(t) \to 0$ exponentially in the space of smoothing operators as $t \to \infty$. A similar formula holds for e^{-tDD^*}.

The key steps of the Mckean-Singer proof are to consider the function

$$h(t) = \mathrm{Tr}(e^{-tD^*D}) - \mathrm{Tr}(e^{-tDD^*})$$

and to prove the following amazing properties:

(1) $\lim_{t\to\infty} h(t) = \operatorname{ind} D$,

(2) $\lim_{t\to 0} h(t) = \int_M K_{AS}$,

(3) $\frac{d}{dt}h(t) = 0$ so that $h(t)$ is constant.

Equating the values of the constant function $h(t)$ at $t = 0$ and $t = \infty$ proves the index formula. Consider property (1). The formula (2.2) implies that

$$\begin{aligned}\lim_{t\to\infty} h(t) &= \operatorname{Tr}(\pi_{\ker D^*D}) - \operatorname{Tr}(\pi_{\ker DD^*})\\ &= \dim\ker D^*D - \dim\ker DD^*.\end{aligned}$$

Integration by parts shows that $\ker D^*D = \ker D$ and $\ker DD^* = \ker D^*$. Indeed, clearly $\ker D \subset \ker D^*D$ and if $(\cdot,\cdot)$ denotes the L^2 inner product, then

$$D^*Du = 0 \;\Rightarrow\; (D^*Du, u) = 0 \;\Rightarrow\; (Du, Du) = 0 \;\Rightarrow\; Du = 0.$$

Thus $\ker D^*D \subset \ker D$ and so $\ker D^*D = \ker D$. Similarly, $\ker DD^* = \ker D^*$. Thus, as $\operatorname{coker} D \cong \ker D^*$, we obtain

$$\lim_{t\to\infty} h(t) = \operatorname{ind} D.$$

To determine the limit as $t \to 0$ of $h(t)$, we use the trace formulas:

$$\operatorname{Tr}(e^{-tD^*D}) = \int_M \operatorname{tr} e^{-tD^*D}(p,p)\,dg, \quad \operatorname{Tr}(e^{-tDD^*}) = \int_M \operatorname{tr} e^{-tDD^*}(p,p)\,dg,$$

obtained by integrating the pointwise trace of the heat kernels restricted to the diagonal. Now the *local index theorem* states that[3]

$$\lim_{t\to 0}\big\{\operatorname{tr} e^{-tD^*D}(p,p) - \operatorname{tr} e^{-tDD^*}(p,p)\big\} = K_{AS}(p)$$

uniformly in t, where the right-hand side really represents the coefficient of the volume form component of the differential form $K_{AS}(p)$. This result was proved originally by Mckean and Singer [**57**] for dimension two, generalized to higher dimensions by Gilkey [**32**] using invariance theory, and by Patodi [**70**] using a super-symmetry trick which was further developed by Alvarez-Gaumé [**1**] in the setting of path integrals and by Getzler [**30**] in a pseudodifferential setting. Thus,

$$\lim_{t\to 0} h(t) = \int_M K_{AS}. \tag{2.3}$$

Hence, by the fundamental theorem of calculus, we have

$$\operatorname{ind} D - \int_M K_{AS} = \int_0^\infty \frac{d}{dt}h(t)\,dt. \tag{2.4}$$

[3]This formula technically only applies to compatible Dirac operators, and not to arbitrary Dirac type operators. In general, the left-hand side of (2.3) has an asymptotic expansion as $t \to 0$ starting with negative powers of t and the right-hand side is the constant term in the expansion.

We now show that $\frac{d}{dt}h(t) = 0$. We first claim that $D^*De^{-tD^*D} = D^*e^{-tDD^*}D$. To see this, let $u \in C^\infty(M, E)$. Then, $v_t = D^*De^{-tD^*D}u$ and $w_t = D^*e^{-tDD^*}Du$ agree at $t = 0$ and they both satisfy the equation $(\partial_t + D^*D)u_t = 0$. By uniqueness of solutions to the heat equation [**59**, p. 271], we must have $v_t = w_t$; hence, $D^*De^{-tD^*D} = D^*e^{-tDD^*}D$. Thus

$$\begin{aligned}\frac{d}{dt}h(t) &= \operatorname{Tr}\big(-D^*De^{-tD^*D} + DD^*e^{-tDD^*}\big)\\ &= \operatorname{Tr}\big(-D^*e^{-tDD^*}D + DD^*e^{-tDD^*}\big)\\ &= \operatorname{Tr}\big([D, D^*e^{-tDD^*}]\big),\end{aligned} \tag{2.5}$$

where $[D, D^*e^{-tDD^*}]$ is the commutator of D and $D^*e^{-tDD^*}$. Using the well-known fact that the trace vanishes on commutators of pseudodifferential operators when at least one factor is smoothing implies that $\frac{d}{dt}h(t) = 0$. Hence, according to (2.4) we have

$$\operatorname{ind} D = \int_M K_{AS},$$

which is the Atiyah-Singer formula!

2.3. Some remarks on the Atiyah-Singer index theorem. The Atiyah-Singer index formula can be generalized to elliptic pseudodifferential operators using K-theory. However, in this generality, the form K_{AS} occurring on the right-hand side of the index formula is not explicitly defined in terms of the curvature forms. The fact that K_{AS} is explicitly defined in terms of the curvature forms for Dirac type operators is a very special property of Dirac operators and is one of the reasons why Dirac operators are important in applications. The original proof of the Atiyah-Singer index theorem as sketched in [**6**] used cobordism theory, cf. [**69**]. A few years later, the proof was reworked in a series of papers [**7, 8**]. The 'heat kernel proof' appeared in [**2**]. See [**14**] for a comparison of the various proofs. The Atiyah-Singer formula has been generalized to many different contexts, for example, to families of Dirac operators by Bismut [**11**], see [**74**], [**9**], and especially [**34**, Ch. 5] for other generalizations.

The Atiyah-Singer index theorem has far-reaching applications (see [**45**, Ch. 4]) that include group actions on manifolds, immersions into Euclidean space, integrality and divisibility of certain characteristic numbers, existence of metrics with positive scalar curvature [**37**], twisted signature and Riemann-Roch-Hirzebruch formulas, and formal dimensions of certain moduli spaces [**27, 36**].

3. The Atiyah-Patodi-Singer index formula

Now we ask a similar question concerning the Atiyah-Singer index formula as we did for the Gauss-Bonnet formula in the introduction: Does the Atiyah-Singer formula, $\operatorname{ind} D = \int_M K_{AS}$, continue to hold verbatim if M has a smooth boundary? From our experience with the Gauss-Bonnet formula, we expect that the answer is "no"; there should be a correction term added to the right-hand side

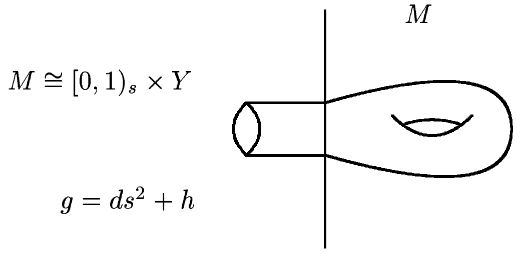

FIGURE 2. The manifold M with a collar neighborhood near its boundary over which all geometric structures are of product type.

due to the presence of the boundary. This is in fact the case. It turns out that the correction term is a *spectral invariant* of the boundary.

In this section, we describe the Atiyah-Patodi-Singer (or APS) index formula [**5**], which extends the Atiyah-Singer formula to manifolds with boundary. For manifolds with boundary, there are various ways to develop an index theory, for instance, introducing boundary conditions or 'attaching a cylindrical end' to the boundary. We focus on the latter method. For the BVP point of view, see [**15**]. Finally, we reformulate the index problem in terms of Melrose's b-geometric objects.

3.1. Attaching a cylindrical end. The ingredients of the APS index formula include topological/geometric data: Let M be an even-dimensional, compact, oriented, Riemannian manifold with boundary and let E and F be Hermitian vector bundles over M. For simplicity, we assume that M has a collar neighborhood $M \cong [0,1)_s \times Y$ where the metric is a product $g = ds^2 + h$ with h a metric on $Y = \partial M$, and where E and F are isomorphic to their restrictions E_0 and F_0 respectively to Y over this collar. See Figure 2.

We are also given functional analytic/geometric data: Let

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

be a *Dirac type operator*, a first-order elliptic differential operator such that the principal symbol of D^*D is the metric $\sigma(D^*D)(\xi) = |\xi|^2$ for all cotangent vectors ξ. We assume that D is of product type on the collar of the following sort:

$$D = \Gamma(\partial_s + D_Y),$$

where

$$D_Y : C^\infty(Y, E_0) \to C^\infty(Y, E_0)$$

is a self-adjoint Dirac type operator on the odd-dimensional manifold Y, and where Γ is a unitary isomorphism from E_0 onto F_0. With these hypotheses, one might think that $D : H^1(M, E) \to L^2(M, F)$ is Fredholm. This however is not the case.

THEOREM 3.1. *The Dirac type operator*

$$D : H^1(M, E) \to L^2(M, F)$$

is never Fredholm. In fact, its kernel is infinite dimensional!

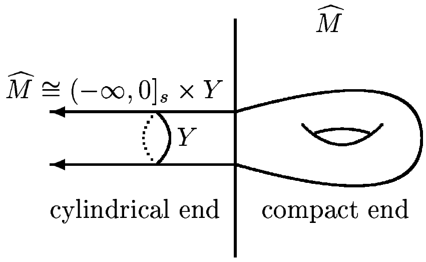

FIGURE 3. Attaching an infinite cylinder to M produces the manifold with cylindrical end $\widehat{M}$.

A proof of Theorem 3.1 can be found in [**15**]. To see why this theorem holds, consider the following simple example. Let $M_0 = [0,1] \times \mathbb{S}^1$ with metric $g = ds^2 + d\theta^2$, let $E = F = \mathbb{C}$, and let

$$D_0 = \partial_s + i\partial_\theta.$$

Certainly, the manifold and operator are of product type. Moreover, $\ker D_0$ consists of all functions $f(s,\theta)$ that are holomorphic in $z = s + i\theta$ for $0 \leq s \leq 1$ and periodic in θ with period 2π. Of course, there are infinitely many such functions, for example e^{kz} where $k \in \mathbb{Z}$. Thus, $\dim \ker D_0 = \infty$.

Since D is not Fredholm, it might look like our hopes for an index formula are crushed. By the way, it turns out that in general, $D : H^1(M,E) \to L^2(M,F)$ is surjective [**15**]. Thus, the problem with D is its kernel on $H^1(M,E)$. There are various ways that have been developed to 'tame' the infinite dimensional kernel. One successful method is the theory of boundary value problems pioneered by Calderón [**19**] and Seeley [**77**] as explained in [**15**]. However, we will focus on the method of attaching a cylindrical end, which is described as follows.

Consider $\widehat{D}_0 = \partial_s + i\partial_\theta$ on the enlarged manifold $\widehat{M}_0 = (-\infty,\infty)_s \times \mathbb{S}^1$ rather than on $M_0 = [0,1]_s \times \mathbb{S}^1$. Here, $\widehat{M}_0$ has the naturally extended metric $g = ds^2 + d\theta^2$. We claim that on $\widehat{M}_0$, we have $\ker \widehat{D}_0 = 0$ on $H^1(\widehat{M}_0)$. Indeed, $\ker \widehat{D}_0$ consists of all functions $f(s,\theta) \in H^1(\widehat{M}_0)$ that are holomorphic in $z = s + i\theta$ for $s \in \mathbb{R}$ and periodic in θ with period 2π. By Sobolev embedding, f is bounded in s and hence is a bounded holomorphic function on $\mathbb{C}$, so is constant by Liouville's theorem. Since $f \in H^1(\widehat{M}_0)$, the constant must be zero.

With this example as motivation, in the general case we enlarge the compact manifold with boundary M to a noncompact manifold $\widehat{M}$ as follows: Let $\widehat{M}$ be the manifold formed by taking the infinite cylinder $(-\infty,0]_s \times Y$ and gluing it onto the end of the collar $[0,1)_s \times Y$ of M as shown in Figure 3:

$$\widehat{M} = (-\infty, 0]_s \times Y \sqcup_{\partial M} M.$$

Since all the geometric structures and the Dirac operator were of product type on the collar of M, they all have natural extensions to the manifold $\widehat{M}$. We denote these extended structures on $\widehat{M}$ using the same notations for the original objects on

M; however, since the extended Dirac operator on $\widehat{M}$ has a completely different domain than the Dirac operator on M, we denote the extension of the Dirac operator by $\widehat{D}$. Note that the natural domain of $\widehat{D}$ is $H^1(\widehat{M}, E)$, which consists of those sections u on $\widehat{M}$ such that $\widehat{D}u$ is square integrable with respect to the measure dg on $\widehat{M}$. Now we ask: Does this idea work? Is the operator $\widehat{D}$ Fredholm on its natural domain? The answer is: sometimes. It turns out that the boundary operator D_Y, which can be considered the model operator for $\widehat{D}$ 'at infinity' on the cylindrical end, determines the Fredholm condition.

THEOREM 3.2. *The Dirac type operator*

$$\widehat{D} : H^1(\widehat{M}, E) \to L^2(\widehat{M}, F)$$

is Fredholm if and only if the boundary operator $D_Y : H^1(Y, E_0) \to L^2(Y, E_0)$ *is invertible; that is, if it has zero kernel.*

It turns out that the kernel of $\widehat{D}$ is always finite dimensional, so the enlargement of M to $\widehat{M}$ did tame the infinite dimensional kernel of D as expected, but the cokernel of $\widehat{D}$ is infinite dimensional unless D_Y is invertible. For a proof of Theorem 3.2, see [**59**, Th. 5.60]. There is a general principle underlying the Fredholm properties of Dirac operators on noncompact manifolds:

(3.1) **General Principle:** A Dirac operator on a noncompact manifold is Fredholm if and only if it is invertible 'at infinity'.

Recall that a Fredholm operator is an operator that is 'almost invertible'. Roughly speaking, a Dirac operator is always 'almost invertible' on the 'compact end' of a noncompact manifold simply because a Dirac operator is elliptic so we can always construct a parametrix for it on the compact end; however, to construct a global parametrix for the Dirac operator, we need to invert the Dirac operator 'at infinity'.

We now show that Dirac operators can *always* be made Fredholm on weighted Sobolev spaces. To see this, extend the coordinate function s on the cylindrical end of $\widehat{M}$ into the compact end of $\widehat{M}$ to be a positive function there. Let $\alpha \in \mathbb{R}$. Then observe that on the cylindrical end we have

$$e^{-\alpha s} \widehat{D}\, e^{\alpha s} = \Gamma(\partial_s + D_Y + \alpha),$$

and $D_Y + \alpha$ is invertible for $|\alpha| > 0$ less than the smallest absolute value of a nonzero eigenvalue of D_Y. Hence, the 'General Principle' implies that

$$e^{-\alpha s} \widehat{D} e^{\alpha s} : H^1(\widehat{M}, E) \to L^2(\widehat{M}, F)$$

is Fredholm for all $|\alpha| > 0$ sufficiently small, which is equivalent to

$$\widehat{D} : e^{\alpha s} H^1(\widehat{M}, E) \to e^{\alpha s} L^2(\widehat{M}, F)$$

is Fredholm on weighted Sobolev spaces. Thus we have the following:

THEOREM 3.3. *There exists a $\delta > 0$ such that for all $0 < |\alpha| < \delta$, the Dirac type operator*

$$\widehat{D} : e^{\alpha s} H^1(\widehat{M}, E) \to e^{\alpha s} L^2(\widehat{M}, F)$$

is Fredholm.

For the proof of this theorem, see [**59**, Th. 5.60]. We next state the APS formula for the index of the operator $\widehat{D}$ on weighted Sobolev spaces.

3.2. Statement of the Atiyah-Patodi-Singer index theorem. Before stating the APS index theorem for the operator $\widehat{D}$, we first need to define the eta invariant. Since D_Y is a self-adjoint elliptic operator on the closed compact manifold Y, it has discrete spectrum $\{\lambda_j\} \subset \mathbb{R}$. The eta function, $\eta(z)$, is the holomorphic function

$$\eta(z) = \sum_{\lambda_j \neq 0} \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^z}.$$

One of the main results of [**5**] was that $\eta(z)$ defines a meromorphic function on $\mathbb{C}$ that is regular at $z = 0$. The *eta invariant* of D_Y is the value of the eta function at zero, $\eta(D_Y) = \eta(0)$, which represents a formal signature of the operator D_Y:

$$\text{“}\,\eta(D_Y) = \sum_{\lambda_j \neq 0} \operatorname{sign} \lambda_j = \#\{\lambda_j > 0\} - \#\{\lambda_j < 0\}\,\text{”}.$$

Thus, $\eta(D_Y)$ is a measurement of the spectral asymmetry of D_Y. Another way to express the eta function is through the heat operator:

$$\eta(z) = \frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{\frac{z-1}{2}} \operatorname{Tr}(D_Y\, e^{-tD_Y^2})\, dt, \tag{3.2}$$

where $\Gamma(z)$ is the Gamma function. This formula follows from the fact that

$$\operatorname{Tr}(D_Y\, e^{-tD_Y^2}) = \sum_{\lambda_j \neq 0} \lambda_j\, e^{-t\lambda_j^2}$$

and that

$$\frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{\frac{z-1}{2}} \lambda_j\, e^{-t\lambda_j^2}\, dt = \frac{\lambda_j}{|\lambda_j|^{z+1}} \frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{\frac{z-1}{2}} e^{-t}\, dt = \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^z},$$

where we made the change of variables $t \mapsto t/|\lambda_j|^2$. Moreover, the *local index theorem for odd-dimensional manifolds* proved by Bismut and Freed [**10**] states that $\operatorname{Tr}(D_Y e^{-tD_Y^2})$ is a smooth function of $t^{1/2}$ vanishing at $t = 0$, and so the formula (3.2) can be used to prove that[4]

$$\eta(D_Y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}(D_Y\, e^{-tD_Y^2})\, dt. \tag{3.3}$$

We are now ready to state the Atiyah-Patodi-Singer index theorem.

THEOREM 3.4 (Atiyah-Patodi-Singer, 1973). *Let D be a Dirac type operator on an even-dimensional, compact, oriented, Riemannian manifold with boundary with product type structures specified. Then there exists a $\delta > 0$ such that for all $0 < |\alpha| < \delta$, the Dirac type operator*

$$\widehat{D} : e^{\alpha s} H^1(\widehat{M}, E) \to e^{\alpha s} L^2(\widehat{M}, F)$$

is Fredholm and if its index is denoted by $\operatorname{ind}_\alpha \widehat{D}$, then

$$\operatorname{ind}_\alpha \widehat{D} = \int_M K_{AS} - \frac{1}{2}\Big\{ \eta(D_Y) + \operatorname{sign}\alpha \cdot \dim\ker D_Y \Big\},$$

where K_{AS} is the Atiyah-Singer integrand and $\eta(D_Y)$ is the eta invariant.

The APS theorem in [**5**] technically only applies to the case of $\alpha > 0$; as presented above, the theorem is due to Melrose [**59**]. We prove this theorem using Melrose's b-geometry approach in Section 4. An important corollary is the notable generalization of Hirzebruch's signature formula to manifolds with boundary: If D is the signature operator, then using the fact that $\operatorname{ind}_\alpha \widehat{D} + \operatorname{ind}_{-\alpha} \widehat{D} = 2 \operatorname{sign}(M)$ for $\alpha > 0$ sufficiently small (see [**59**, Sec. 9.3]) gives

$$\begin{aligned} 2 \operatorname{sign}(M) = \int_M \mathcal{L}(M) - \frac{1}{2}\Big\{ \eta(D_Y) + \dim\ker D_Y \Big\} \\ + \int_M \mathcal{L}(M) - \frac{1}{2}\Big\{ \eta(D_Y) - \dim\ker D_Y \Big\}, \end{aligned}$$

or

$$\operatorname{sign}(M) = \int_M \mathcal{L}(M) - \frac{1}{2}\eta(D_Y).$$

Hirzebruch's conjecture for the signature-defect, the difference between the signature of M and the integral of the $\mathcal{L}$-class polynomial, was the original motivation of Atiyah, Patodi, and Singer in the discovery of the eta invariant [**4, 5**].

[4]Actually, the local index theorem for odd-dimensional manifolds is not true for arbitrary D_Y but only for those associated to a 'unitary Clifford connection', cf. the discussion in footnote (3) concerning the local index theorem for even-dimensional manifolds. In general, $\operatorname{Tr}(D_Y e^{-tD_Y^2})$ only has an asymptotic expansion as $t \to 0$ starting from negative powers of t. In this case, the lower limit 0 in the integral (3.3) must be replaced by $\varepsilon > 0$ and the resulting integral has an asymptotic expansion as $\varepsilon \to 0$. The right-hand side of the above equation represents the constant term in the expansion.

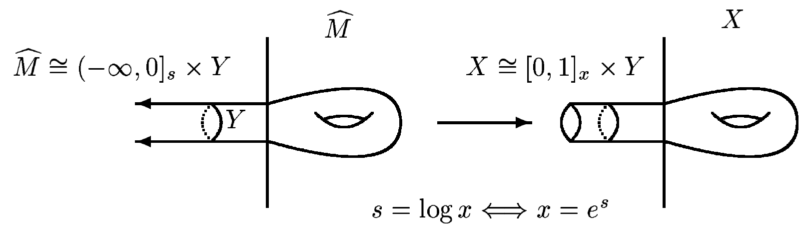

FIGURE 4. The compact manifold with boundary X is the compactification of the manifold with cylindrical end $\widehat{M}$.

3.3. Interpretation as b-objects. One of the primary tools used to prove Fredholm properties of Dirac operators, or elliptic differential operators in general, on compact manifolds without boundary is the algebra of pseudodifferential operators. Such an algebra would be useful on a manifold with cylindrical end in order to prove Fredholm properties of Dirac operators on such manifolds. For various classes of operators defined on noncompact manifolds, see Lockhart and McOwen [**49**], Rabinovič [**73**], or Schrohe [**76**]. One usually requires the noncompact manifold to have a finite atlas with control at infinity of the coordinate changes and special estimates at infinity on the symbols of the operators considered on the manifold. In particular, a direct definition of pseudodifferential operators on manifolds with a cylindrical end might be considered unbalanced as the analysis is treated in distinctly differently ways on the cylindrical end and the compact end.

Melrose's novel idea was to unify the analysis on these two ends by making the whole manifold compact; that is, compactifying the cylindrical end forming a *compact* manifold with boundary. On this new compact manifold with boundary, he defines a space of pseudodifferential operators, imitating as close as possible, the *global* geometric definition of pseudodifferential operators on compact manifolds without boundary in terms of their Schwartz kernels as discussed in, for example, Hörmander [**43**]. The resulting operators are called b-pseudodifferential operators. For excellent introductions to this subject, see Grieser [**35**], Mazzeo [**56**], or Melrose [**59**].

We now explain the compactification. On the cylindrical end $(-\infty, 0]_s \times Y$ of $\widehat{M}$ we make the change of variables $x = e^s$. As $s \to -\infty$, $x \to 0$. Thus, under this change of variables, $\widehat{M}$ becomes the interior of the compact manifold with boundary X, where X has the same compact end as $\widehat{M}$ but with the cylindrical end $(-\infty, 0]_s \times Y$ replaced with the compact manifold $[0,1]_x \times Y$, see Figure 4. Since $x = e^s$, we have $ds = dx/x$ and $\partial_s = x\partial_x$. Thus the geometric objects on the manifold with cylindrical end transform into corresponding singular geometric 'b-objects' on the compact manifold with boundary:

$$g = ds^2 + h \;\rightsquigarrow\; g = \Big(\frac{dx}{x}\Big)^2 + h \quad (b\text{-metric}),$$

$$dg = ds\, dh \;\rightsquigarrow\; dg = \frac{dx}{x}\, dh \quad (b\text{-measure}),$$

and consequently,

$$H^k(\widehat{M}) \rightsquigarrow H^k_b(X) \quad (b\text{-Sobolev space}),$$
$$\widehat{D} = \Gamma(\partial_s + D_Y) \rightsquigarrow \widehat{D} = \Gamma(x\partial_x + D_Y) \quad (b\text{-differential operator}).$$

Although the manifold X is topologically compact, its interior is geometrically a manifold with cylindrical end since X inherited all its geometric structures from $\widehat{M}$. In particular, the boundary of X is geometrically at infinity. The fact that X is compact is key to the definition of b-pseudodifferential operators since these operators are defined using only the usual classes of smooth functions and distributions on compact manifolds with boundary. Of course, there is a trade off: The distributions defining the Schwartz kernels of b-pseudodifferential operators are required to have a special structure, which takes some time getting used to [**59**, Ch. 4].

We repeat the statement of the APS index theorem in the current context.

Theorem 3.5. *With the same hypotheses as in Theorem* 3.4*, but now in the b-geometry context, there exists a $\delta > 0$ such that for all $0 < |\alpha| < \delta$, the Dirac type operator*

$$\widehat{D} : x^\alpha H^1_b(X, E) \to x^\alpha L^2_b(X, F)$$

is Fredholm and if its index is denoted by $\operatorname{ind}_\alpha \widehat{D}$*, then*

$$\operatorname{ind}_\alpha \widehat{D} = \int_M K_{AS} - \frac{1}{2}\Big\{\eta(D_Y) + \operatorname{sign}\alpha \cdot \dim\ker D_Y\Big\},$$

where K_{AS} is the Atiyah-Singer integrand and $\eta(D_Y)$ is the eta invariant

$$\eta(D_Y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}(D_Y\, e^{-tD_Y^2})\, dt.$$

In Section 4, we prove this theorem.

3.4. Some remarks on the Atiyah-Patodi-Singer index theorem. The original Atiyah-Patodi-Singer index theorem was proved in the context of (pseudo-differential) boundary value problems. A nice introduction to these methods can be found in the book by Booß-Bavnbek and Wojciechowski [**15**]. The 'direct approach' to the APS boundary value problem based on asymptotic expansions of the heat operator was initiated by Grubb and Seeley [**39**], see also Grubb's book [**38**]. The approach of 'attaching cylinders' was mentioned in [**5**], but was not developed. Besides attaching a cylinder to the boundary, another way to develop an index theory on manifolds with boundary is by attaching a cone, see Cheeger [**22**]; for other generalizations, see Atiyah, Donnelly, and Singer [**3**], Müller [**64**], Stern [**81**], Brüning [**16**], Fedosov and Schulze [**29**], Schulze, Sternin, and Shatalov [**79**], and Carron [**21**]. Melrose introduced the b-geometry in the seminal paper [**58**], and these ideas were developed by Melrose and Mendoza in [**60**]. The APS index formula was generalized to Fredholm b-pseudodifferential operators by Piazza [**71**].

The eta invariant itself has become a topic of much interest. In particular, the extension of the eta function and its regularity properties for pseudodifferential operators has been examined by Gilkey [**33**] and Wodzicki [**85**] among others. For a survey of various topics on the eta invariant, including its decomposition under gluing of manifolds, see Mazzeo and Piazza [**55**].

The index theorem for families of Dirac operators was extended to the case when the fibers are manifolds with boundary by Bismut and Cheeger [**12, 13**] under the assumption that the Dirac operators on the fibers of the boundary fibration are invertible. Later, this result was generalized by Melrose and Piazza [**62, 63**] with no assumptions on the boundary Dirac operators. For further extensions of the index formula, see Getzler [**31**], Wu [**86**], Melrose and Nistor [**61**], and Leichtnam, Lott, and Piazza [**46**].

Other expository articles incorporating recent developments on the APS index theorem include Müller [**66**], Piazza [**72**], and Seeley [**78**]. Because of space constraints, we have only touched the surface on some of the many extensions and generalizations of the Atiyah-Patodi-Singer problem, a list of references can be found in the book by Gilkey [**34**, Ch. 5].

4. Melrose's b-geometry proof of the Atiyah-Patodi-Singer theorem

Let D be a Dirac type operator on an even-dimensional, compact, oriented, Riemannian manifold with boundary M with product type structures near the boundary as described in Section 3.1. For simplicity, we assume that the boundary Dirac operator D_Y is invertible. Now form the manifold with cylindrical end $\widehat{M}$ and then compactify it under the change of variables $x = e^s$, where s is the variable on the cylinder, to form the manifold X as described in Section 3.3. Then $\widehat{D}$ defines an operator on X such that

$$\widehat{D} : H^1_b(X, E) \to L^2_b(X, F)$$

is Fredholm. We now give the b-geometry proof of the APS index formula in Theorem 3.5 modeling, as close as possible, the proof the Atiyah-Singer formula given in Section 2.2. We shall see that there are certain variations to the proof that need to be fleshed out in order to make the proof work.

4.1. The proof of APS with details left out. As with the Atiyah-Singer proof given in Section 2.2, we would like to define the Mckean-Singer function

$$\text{“}\operatorname{Tr}(e^{-t\widehat{D}^*\widehat{D}}) - \operatorname{Tr}(e^{-t\widehat{D}\widehat{D}^*})\text{”}.$$

Here, we meet our first variation to the Atiyah-Singer proof – the reason for the quotation marks is that the heat operators are not trace class, and so the traces are not even defined! Basically, the heat operators are not trace class because X has infinite volume (is geometrically not compact) which implies that the heat kernels restricted to the diagonal are not integrable. Thus, we cannot prove the APS formula by imitating the proof of the Atiyah-Singer formula verbatim. However, in Section 4.3, we define a natural extension of the trace called the b-trace, denoted by ${}^b\mathrm{Tr}$, such that the heat operators are b-trace class. We can now define a modified Mckean-Singer function

$$h(t) = {}^b\mathrm{Tr}(e^{-t\widehat{D}^*\widehat{D}}) - {}^b\mathrm{Tr}(e^{-t\widehat{D}\widehat{D}^*}),$$

and continue as in Section 2.2. As in the manifold without boundary case, the following limits hold:

$$\lim_{t\to\infty} h(t) = \operatorname{ind}\widehat{D},$$

and[5]

$$\lim_{t\to 0} h(t) = \int_M K_{AS}.$$

In fact, using b-pseudodifferential operators, the proofs of these two results are not much different from the corresponding proofs in the manifold without boundary case, see Chapters 7 and 8 of [**59**] for the proofs. Continuing as in Section 2.2, we find that

$$\operatorname{ind}\widehat{D} = \int_M K_{AS} + \int_0^\infty \frac{d}{dt} h(t)\, dt,$$

where repeating the same algebraic calculation as before, we have

$$\frac{d}{dt} h(t) = {}^b\mathrm{Tr}\left([\widehat{D}, \widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}]\right).$$

Here, we meet our second variation – in the proof of the Atiyah-Singer index formula, this expression is zero, in this present case it is not. Figuratively speaking, the b-trace is a trace on the interior of X and only fails to be a trace on the boundary of X. Thus intuitively, ${}^b\mathrm{Tr}\left([\widehat{D}, \widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}]\right)$ should be a boundary integral of some sort. This is in fact the case; in Section 4.4 we compute that

$${}^b\mathrm{Tr}\left([\widehat{D}, \widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}]\right) = -\frac{1}{\sqrt{4\pi t}}\,\mathrm{Tr}(D_Y\, e^{-tD_Y^2}) = \frac{1}{2\sqrt{\pi}}\, t^{-1/2}\,\mathrm{Tr}(D_Y\, e^{-tD_Y^2}).$$

Hence,

$$\operatorname{ind}\widehat{D} = \int_X K_{AS} - \frac{1}{2}\eta(D_Y),$$

where

$$\eta(D_Y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\,\mathrm{Tr}(D_Y\, e^{-tD_Y^2})\, dt,$$

and the Atiyah-Patodi-Singer index formula is proved!

[5]The same discussion as in footnote (3) concerning the local index theorem on manifolds without boundary applies in this situation too. The integral of K_{AS} is over M because the product type assumption implies that the volume form component of K_{AS} is supported on the manifold M regarded as a subset of X.

4.2. Some facts about the heat kernels. To implement the proof in Section 2.2, we need the heat operators

$$e^{-t\widehat{D}^*\widehat{D}} : L^2_b(X,E) \to H^2_b(X,E) \quad \text{and} \quad e^{-t\widehat{D}\widehat{D}^*} : L^2_b(X,F) \to H^2_b(X,F).$$

It turns out that these heat operators are b-smoothing operators; that is, they are b-pseudodifferential operators of order $-\infty$ [**59**, Ch. 7], which implies a couple of useful results. These results can be proved using other methods, but the theory of b-pseudodifferential operators gives these results more or less 'for free'. First, the Schwartz kernels of these heat operators are smooth on the interior of X^2 vanishing to infinite order at ∂X^2 except at $\partial X \times \partial X$. The second result is that these heat operators have a simple structure on the collar of X described as follows. For concreteness, we focus on $e^{-t\widehat{D}\widehat{D}^*}$. On the collar $[0,1]_x \times Y$ of X we have

$$\widehat{D} = \Gamma(x\partial_x + D_Y),$$

where Γ is a unitary isomorphism of E_0 onto F_0. Thus, on the collar,

$$\widehat{D}\widehat{D}^* = \Gamma(x\partial_x + D_Y)(-x\partial_x + D_Y)\Gamma^* = \Gamma((xD_x)^2 + D_Y^2)\Gamma^*,$$

where $D_x = i^{-1}\partial_x$. This suggests that near ∂X we have

$$e^{-t\widehat{D}\widehat{D}^*} = \Gamma e^{-t(xD_x)^2} e^{-tD_Y^2}\Gamma^* + \mathcal{O}(x),$$

where $e^{-t(xD_x)^2}$ is the heat operator for xD_x on $[0,\infty)_x$, and where $\mathcal{O}(x)$ is an operator smooth in x and vanishing at $x=0$. In fact, even a stronger result is true. Under the change of variables $s = \log x$, which takes the interior of $[0,\infty)_x$ onto $(-\infty,\infty)_s$, we have $xD_x = D_s$. Since the Schwartz kernel of $e^{-tD_s^2}$ on $(-\infty,\infty)_s$ is given by the well-known formula

$$\widetilde{K}_1(s,s',t) = \frac{1}{\sqrt{4\pi t}}\, e^{-(s-s')^2/4t},$$

the Schwartz kernel of $e^{-t(xD_x)^2}$ is obtained by setting $s = \log x$:

$$K_1(x,x',t) = \frac{1}{\sqrt{4\pi t}}\, e^{-(\log x - \log x')^2/4t}.$$

The second result is that the Schwartz kernel of $e^{-t\widehat{D}\widehat{D}^*}$ near $\partial X \times \partial X$ is given by

$$e^{-t\widehat{D}\widehat{D}^*}(x,y,x',y') = \Gamma(y) K_1(x,x',t)\, e^{-tD_Y^2}(y,y')\,\Gamma(y')^* + \mathcal{O}(x), \tag{4.1}$$

where $\mathcal{O}(x)$ is a smooth function of the variables x, $\log x - \log x'$, y, and y' that vanishes at $x = 0$.

4.3. Filling in the details for the b-trace. The simple fact that dx/x is not integrable over $[0,1]_x$ implies that the heat operators are not trace class. Indeed, consider the heat operator $e^{-t\widehat{D}\widehat{D}^*}$. By (4.1), on the collar $[0,1]_x \times Y$ we have

$$\operatorname{tr} e^{-t\widehat{D}\widehat{D}^*}(x,y,x,y) = \frac{1}{\sqrt{4\pi t}} \operatorname{tr} e^{-tD_Y^2}(y,y) + \mathcal{O}(x), \tag{4.2}$$

where we used that $\Gamma\Gamma^* = \mathrm{Id}$ since Γ is a unitary isomorphism, and where $\mathcal{O}(x)$ is smooth in x and vanishes at $x = 0$. Since $dg = (dx/x)\, dh$ on this collar and

$$\int\limits_{[0,1]\times Y} \operatorname{tr} e^{-tD_Y^2}(y,y)\, \frac{dx}{x} dh = \left(\int_0^1 \frac{dx}{x} \right) \cdot \int_Y \operatorname{tr} e^{-tD_Y^2}(y,y)\, dh,$$

the following trace formula does *not* make sense:

$$\operatorname{Tr}(e^{-t\widehat{D}\widehat{D}^*}) = \int_X \operatorname{tr} e^{-t\widehat{D}\widehat{D}^*}(p,p)\, dg.$$

Similarly, the corresponding integral for $e^{-t\widehat{D}^*\widehat{D}}$ does not exist. Although the trace formula above does not make sense, we can 'force' it to make sense by the considering another notion of trace and integral as we now describe. Note that for $\operatorname{Re} z > 0$, x^z is integrable with respect to dx/x over $[0,1]_x$. Extend the coordinate function x on the collar of X to be a smooth function on X which is positive off the collar. Then it follows that $x^z e^{-t\widehat{D}\widehat{D}^*}$ is trace class for $\operatorname{Re} z > 0$ with trace given by

$$\operatorname{Tr}(x^z e^{-t\widehat{D}\widehat{D}^*}) = \int_X x^z \operatorname{tr} e^{-t\widehat{D}\widehat{D}^*}(p,p)\, dg, \quad \operatorname{Re} z > 0.$$

This argument is the basis for defining a new functional called the b-trace, which we introduce after the following lemma.

Lemma 4.1. *Let $f \in C^\infty(X)$. Then for all complex numbers z with $\operatorname{Re} z > 0$, the integral*

$$F(z) = \int_X x^z f\, dg$$

exists, and it extends from $\operatorname{Re} z > 0$ to define a meromorphic function on all of $\mathbb{C}$. The b-integral of f is by definition the regular value of $F(z)$ at $z = 0$:

$$ {}^b\!\!\int_X f\, dg = \operatorname{Reg}_{z=0} F(z). \tag{4.3}$$

Finally, the residue of $F(z)$ at $z = 0$ given by

$$\operatorname{Res}_{z=0} F(z) = \int_Y f(0,y)\, dh.$$

To understand why this lemma is true, note that $x^z = e^{z \log x}$ is an entire function of z for $x > 0$. Thus, we may assume that f is supported on the collar $[0,1]_x \times Y$ of X. Then $F(z)$ is well defined for $\operatorname{Re} z > 0$ since $x^z f(x,y)$ is integrable with respect to the measure $(dx/x)\, dh$ as long as $\operatorname{Re} z > 0$. Now expand $f(x,y)$ in Taylor series at $x = 0$: $f(x,y) \sim \sum_{k=0}^\infty x^k f_k(y)$. Since

$$\int\limits_{[0,1]\times Y} x^{z+k} f_k(y) \frac{dx}{x} dh = \frac{1}{z+k} \int_Y f_k(y)\, dh$$

it follows that $F(z)$ extends from $\operatorname{Re} z > 0$ to be a meromorphic function on $\mathbb{C}$ with only simple poles at $z = \{0, -1, -2, \ldots\}$ with residue at $z = 0$ given by

$$\int_Y f_0(y)\, dh = \int_Y f(0, y)\, dh.$$

To see why the notion of the b-integral is natural, note that if $f(0, y) = 0$, then $F(z)$ is regular at $z = 0$, and

$$ {}^b\!\!\int_X f\, dg = \operatorname{Reg}_{z=0} F(z) = F(0) = \int_X f\, dg,$$

which is the usual integral of f.

The *b-trace* of the heat operator $e^{-t\widehat{D}\widehat{D}^*}$ is by definition

$$ {}^b\mathrm{Tr}(e^{-t\widehat{D}\widehat{D}^*}) = {}^b\!\!\int_X \operatorname{tr} e^{-t\widehat{D}\widehat{D}^*}(p,p)\, dg,$$

where the b-integral of the function $\operatorname{tr} e^{-t\widehat{D}\widehat{D}^*}(p,p)$ is defined by (4.3):

$$ {}^b\!\!\int_X \operatorname{tr} e^{-t\widehat{D}\widehat{D}^*}(p,p)\, dg = \operatorname{Reg}_{z=0} \mathrm{Tr}(x^z e^{-t\widehat{D}\widehat{D}^*}).$$

The b-trace of $e^{-t\widehat{D}^*\widehat{D}}$ is defined similarly.

4.4. Filling in the details for the eta invariant. In this section, we show that

$$ {}^b\mathrm{Tr}\left([\widehat{D}, \widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}]\right) = -\frac{1}{\sqrt{4\pi t}}\, \mathrm{Tr}(D_Y\, e^{-tD_Y^2}).$$

The proof is just a computation using the definition of the b-trace,

$$ {}^b\mathrm{Tr}\left([\widehat{D}, \widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}]\right) = \operatorname{Reg}_{z=0} \mathrm{Tr}\, x^z [\widehat{D}, \widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}],$$

where $\mathrm{Tr}\, x^z [\widehat{D}, \widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}]$ is meromorphically extended from $\operatorname{Re} z > 0$. Observe that

$$x^z [\widehat{D}, \widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}] = [x^z, \widehat{D}]\widehat{D}^* e^{-t\widehat{D}\widehat{D}^*} + [\widehat{D}, x^z \widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}].$$

Since the trace vanishes on commutators, we have $\mathrm{Tr}[\widehat{D}, x^z \widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}] = 0$ for $\operatorname{Re} z > 0$, and thus its meromorphic extension to all of $\mathbb{C}$ is also zero. Hence,

$$ {}^b\mathrm{Tr}\left([\widehat{D}, \widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}]\right) = \operatorname{Reg}_{z=0} \mathrm{Tr}[x^z, \widehat{D}]\widehat{D}^* e^{-tDD^*} = \operatorname{Reg}_{z=0} \mathrm{Tr}\, x^z a(z),$$

where

$$a(z) = x^{-z}[x^z, \widehat{D}]\widehat{D}^* e^{-t\widehat{D}\widehat{D}^*} = \widehat{D}\widehat{D}^* e^{-t\widehat{D}\widehat{D}^*} - x^{-z}\widehat{D} x^z \widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}.$$

Note that $a(0) = 0$. We claim that $a(z)$ is an entire function of z. Indeed, since x^z is an entire function of z for $x > 0$, we may consider $a(z)$ on the collar $[0, 1]_x \times Y$ over which $\widehat{D} = \Gamma(x\partial_x + D_Y)$. In this case, $x^{-z}\widehat{D} x^z = \Gamma(x\partial_x + D_Y + z)$ which is entire, so $a(z)$ is entire. Since $a(z)$ is entire and vanishes at $z = 0$, we can write

$$a(z) = zA + \mathcal{O}(z^2),$$

where A is independent of z. It follows that

$$^b\mathrm{Tr}\left([\widehat{D},\widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}]\right)=\mathrm{Reg}_{z=0}\,\mathrm{Tr}\,x^z a(z)=\mathrm{Res}_{z=0}\,\mathrm{Tr}\,x^z A,$$

and so by Lemma 4.1,

$$^b\mathrm{Tr}\left([\widehat{D},\widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}]\right)=\int_Y \mathrm{tr}\,A(x,y,x,y)|_{x=0}\,dh. \tag{4.4}$$

To calculate this integral, we work over the collar $[0,1]_x\times Y$. Here,

$$x^{-z}\widehat{D}x^z=\Gamma(x\partial_x+D_Y+z)=\widehat{D}+z\,\Gamma,$$

and thus

$$a(z)=\widehat{D}\widehat{D}^*e^{-t\widehat{D}\widehat{D}^*}-x^{-z}\widehat{D}x^z\widehat{D}^*e^{-t\widehat{D}\widehat{D}^*}=-z\Gamma\widehat{D}^*e^{-t\widehat{D}\widehat{D}^*},$$

which implies that over the collar,

$$A=-\Gamma\widehat{D}^*e^{-t\widehat{D}\widehat{D}^*}.$$

As $\widehat{D}^*=(-x\partial_x+D_Y)\Gamma^*$ and $\Gamma^*\Gamma=\mathrm{Id}$, by (4.1) we have

$$A=-\Gamma(y)(-x\partial_x+D_Y)\frac{1}{\sqrt{4\pi t}}e^{-(\log x-\log x')^2/4t}e^{-tD_Y^2}(y,y')\,\Gamma(y')^*$$

modulo a term that vanishes on the boundary. It follows that

$$A(x,y,x,y)|_{x=0}=-\frac{1}{\sqrt{4\pi t}}\,\Gamma(y)D_Y e^{-tD_Y^2}(y,y)\,\Gamma(y)^*,$$

which in view of (4.4) gives

$$\begin{aligned} ^b\mathrm{Tr}\left([\widehat{D},\widehat{D}^* e^{-t\widehat{D}\widehat{D}^*}]\right)&=-\frac{1}{\sqrt{4\pi t}}\int_Y \mathrm{tr}\,D_Y e^{-tD_Y^2}(y,y)\,dh\\ &=-\frac{1}{\sqrt{4\pi t}}\,\mathrm{Tr}(D_Y\,e^{-tD_Y^2}). \end{aligned}$$

5. Index theory on manifolds with corners of codimension two

In this section, we describe an extension of the APS index formula to manifolds with corners of codimension two. As we discussed for manifolds with boundary, there were various ways to develop an index theory for Dirac operators, e.g., introducing boundary conditions or attaching a ‘cylindrical end’ to the boundary. For manifolds with corners, it turns out that there is *no* well-developed theory of boundary value problems for Dirac operators. However, we can still formulate an index problem by attaching ‘multi-cylindrical ends’ and considering an L^2-index problem.

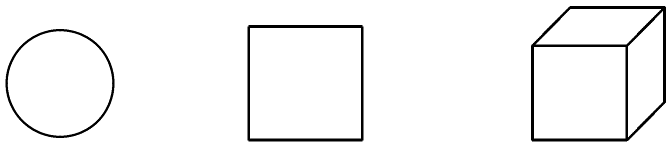

FIGURE 5. Examples of manifolds with corners of codimension one, two, and three respectively.

5.1. Dirac operators on manifolds with corners. We begin by defining manifolds with corners. An n-dimensional manifold with corners X is a paracompact Hausdorff topological space with local charts of the form $[0,1)^k \times (-1,1)^{n-k}$, where k can run anywhere between 0 and n depending on where the chart is located on the manifold, such that the transition maps between any two charts is smooth. A codimension k face Z is a connected closed subset of X such that given any interior point of Z there is a coordinate patch on X centered at the point of the form $[0,1)^k \times (-1,1)^{n-k}$. The largest codimension face that occurs is called the codimension of X. A boundary hypersurface is the same as a codimension one face. For technical reasons we assume that each boundary hypersurface has a boundary defining function; that is, for each hypersurface H of X, there is a nonnegative smooth function $\rho_H \in C^\infty(X)$ which vanishes only on H where it has a nonzero differential. Note that a manifold with corners of codimension one is just a manifold with boundary. Examples of manifolds with corners are found in Figure 5. The disk is a manifold with corners of codimension one. The square is a manifold with corners of codimension two; its edges are boundary hypersurfaces and its corners are codimension two faces. Lastly, the solid cube is a manifold with corners of codimension three; its sides are boundary hypersurfaces, its edges are codimension two faces, and its corners are codimension three faces.

To build a geometric index theory, we first need topological/geometric data. We focus on manifolds with corners of codimension two. Thus let M be an even-dimensional, compact, oriented, Riemannian manifold with corners of codimension two, and let E and F be Hermitian vector bundles over M. For simplicity, we assume that M has exactly two boundary hypersurfaces that intersect in exactly one codimension two face Y. We fix a labeling M_1 and M_2 of the hypersurfaces. Near each hypersurface M_i, we assume that M has a collar neighborhood $M \cong [0,1)_{s_i} \times M_i$ where the metric is a product $g = ds_i^2 + h_i$ with h_i a metric on M_i, and where E and F are isomorphic to their restrictions E_i and F_i respectively to M_i. For compatibility we assume that the product decompositions near each M_i give a common decomposition $M \cong [0,1)_{s_1} \times [0,1)_{s_2} \times Y$ near the corner where the metric is a product $g = ds_1^2 + ds_2^2 + h$ with h a metric on Y, and where E and F are isomorphic to their restrictions E_0 and F_0, respectively, to Y. See Figure 6.

Next, we need analytic/geometric data: Let

$$D : C^\infty(M,E) \to C^\infty(M,F)$$

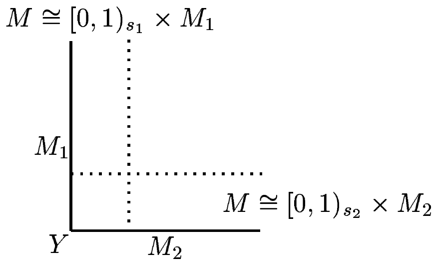

FIGURE 6. The manifold M near the corner Y. On the common intersection of the collars, M is isomorphic to $[0,1)_{s_1} \times [0,1)_{s_2} \times Y$.

be a *Dirac type operator*, a first-order elliptic differential operator such that the principal symbol of D^*D is the metric $\sigma(D^*D)(\xi) = |\xi|^2$ for all cotangent vectors ξ, which is of product type near each hypersurface:

$$D = \Gamma_i(\partial_{s_i} + D_i) \tag{5.1}$$

on the collar $M \cong [0,1)_{s_i} \times M_i$, where Γ_i is a unitary isomorphism from E_i onto F_i, and where

$$D_i : C^\infty(M_i, E_i) \to C^\infty(M_i, E_i)$$

is a (formally) self-adjoint Dirac type operator on the odd-dimensional manifold with boundary M_i. We assume that on the product decomposition near the corner, $M \cong [0,1)_{s_1} \times [0,1)_{s_2} \times Y$, the Dirac operator takes the form

$$D = \Gamma_1 \partial_{s_1} + \Gamma_2 \partial_{s_2} + B, \tag{5.2}$$

where

$$B : C^\infty(Y, E_0) \to C^\infty(Y, F_0)$$

is a Dirac type operator on the even-dimensional manifold without boundary Y.

Hidden in these assumptions are some interesting algebraic consequences at the corner as we now describe. Comparing (5.1) and (5.2), we find that on the collar $M \cong [0,1)_{s_1} \times [0,1)_{s_2} \times Y$, we have

$$\Gamma_i(\partial_{s_i} + D_i) = \Gamma_1 \partial_{s_1} + \Gamma_2 \partial_{s_2} + B, \quad i = 1, 2.$$

Multiplying each side by $\Gamma_i^* = \Gamma_i^{-1}$ and solving for D_i gives

$$D_1 = \Gamma_1^* \Gamma_2 \partial_{s_2} + \Gamma_1^* B \quad \text{and} \quad D_2 = \Gamma_2^* \Gamma_1 \partial_{s_1} + \Gamma_2^* B. \tag{5.3}$$

Since each D_i is assumed (formally) self-adjoint: $D_i^* = D_i$, we must have $\Gamma_1^* \Gamma_2 = -\Gamma_2^* \Gamma_1$ and $\Gamma_i^* B = B^* \Gamma_i$; that is,

$$\Gamma_i^* \Gamma_j + \Gamma_j^* \Gamma_i = 2\,\delta_{ij}, \quad B^* \Gamma_i = \Gamma_i^* B, \tag{5.4}$$

where δ_{ij} is the Kronecker delta. The reader familiar with Clifford multiplication might recognize the left equality as representing a 'Clifford two structure' at the corner. Factoring out the $\Gamma_1^* \Gamma_2$ from the right-hand side of D_1 in (5.3) gives

$$D_1 = \Gamma(\partial_{s_2} + D_Y), \quad \text{where } \Gamma = \Gamma_1^* \Gamma_2, \ D_Y = \Gamma_2^* B.$$

We call D_Y the *induced Dirac operator* on Y. We really should call D_Y the induced operator from D_1. However, the induced operator from D_2 is related to D_Y in a simple way. Indeed, one can verify that

$$D_2 = -\Gamma(\partial_{s_1} + \widetilde{D}_Y), \quad \widetilde{D}_Y = \Gamma D_Y.$$

The induced Dirac operator on Y refers only to D_Y and not $\widetilde{D}_Y$.

As part of the 'Clifford two package', the induced operator D_Y has a nice splitting property as we now describe. First, the left-hand identity in (5.4) implies that

$$\Gamma^2 = -\mathrm{Id}.$$

Hence, $\Gamma : E_0 \to E_0$ has eigenvalues $\pm i$. Let $E_0^{\pm}$ denote the eigenspaces corresponding to the eigenvalues $\pm i$; these are subbundles of E_0 and

$$E_0 = E_0^+ \oplus E_0^-$$

is an orthogonal decomposition since Γ is unitary. Also, a short computation utilizing (5.4) gives

$$D_Y\,\Gamma = -\Gamma\,D_Y.$$

Thus D_Y is odd with respect to Γ; hence odd with respect to the $\mathbb{Z}_2$-grading $E_0 = E_0^+ \oplus E_0^-$. We summarize this property in the following lemma.

LEMMA 5.1. *With respect to the orthogonal decomposition $E_0 = E_0^+ \oplus E_0^-$, where $E_0^{\pm}$ are the $\pm i$ eigenspaces of $\Gamma = \Gamma_1^* \Gamma_2$, the induced Dirac operator $D_Y = \Gamma_2^* B$ takes the following form*

$$\begin{bmatrix} 0 & D_Y^- \\ D_Y^+ & 0 \end{bmatrix} : C^\infty(Y, E_0^+ \oplus E_0^-) \to C^\infty(Y, E_0^+ \oplus E_0^-),$$

where $D_Y^{\pm}$ are the restrictions of D_Y to $C^\infty(Y, E_0^{\pm})$.

Note that since D_Y is self-adjoint, we have $(D_Y^+)^* = D_Y^-$. The following theorem follows from the cobordism theorem of Atiyah-Singer, which is published in Palais' book [**69**]. The cobordism theorem was one of the key steps in the original proof of the Atiyah-Singer index theorem [**6**].

THEOREM 5.2. *The index of the Dirac type operator*

$$D_Y^+ : H^1(Y, E_0^+) \to L^2(Y, E_0^-)$$

on the even-dimensional manifold without boundary Y is zero: $\operatorname{ind} D_Y^+ = 0$. *Since $(D_Y^+)^* = D_Y^-$, it follows that* $\dim\ker D_Y^+ = \dim\ker D_Y^-$.

5.2. Attaching multi-cylindrical ends. As in the manifold with boundary case, we cannot build an index theory of D with its natural domain on the manifold with corners M:

THEOREM 5.3. *The Dirac type operator*

$$D : H^1(M, E) \to L^2(M, F)$$

is never Fredholm. In fact, $\dim\ker D = \infty$.

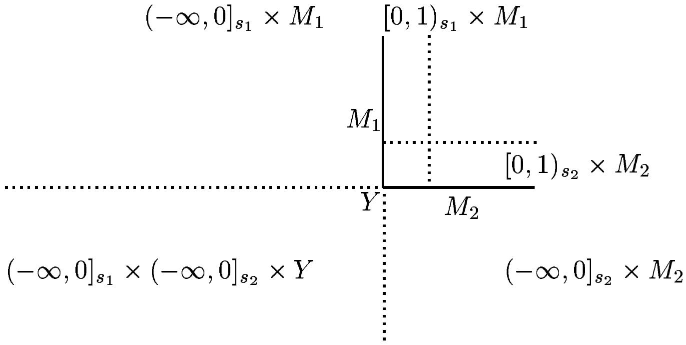

FIGURE 7. A view of $\widehat{M}$ near Y. The manifold with 'multi-cylindrical ends' $\widehat{M}$ is obtained by gluing multiple cylinders onto M.

The corresponding theorem for manifolds with boundary (Theorem 3.1) can be used to prove this result. For a concrete example consider the Cauchy-Riemann operator $D_{CR} = \partial_x + i\partial_y$ on the square $[-1,1]_x \times [-1,1]_y$. Certainly, the manifold and operator are both of product type. Then, $\ker D_{CR}$ is infinite dimensional since the kernel of the D_{CR} consists of all holomorphic functions on the square.[6]

Currently there is no suitable theory of elliptic boundary value problems for manifolds with corners of codimension two because the Calderón projector [**19**] in this context is not understood. However, in analogy with the case of a manifold with boundary, we can build an index theory through attaching multi-cylinders and then compactifying again forming the corresponding b-theory.

We attach multi-cylinders as follows: Let $\widehat{M}$ be the manifold formed by taking the infinite cylinder $(-\infty,0]_{s_1} \times M_1$ and attaching it onto the collar $[0,1)_{s_1} \times M_1$ of M, then taking $(-\infty,0]_{s_2} \times M_2$ and attaching it onto the collar $[0,1)_{s_2} \times M_2$, and finally taking $(-\infty,0]_{s_1} \times (-\infty,0]_{s_2} \times Y$ and attaching it onto the remaining open quadrant; see Figure 7. Since all the geometric structures and the Dirac operator are of product type near the boundary of M, they all have natural extensions to the manifold $\widehat{M}$. We denote these extended structures on $\widehat{M}$ using the same notations as were used for the original objects on M, except for the Dirac operator which we denote by $\widehat{D}$.

The 'General Principle' (3.1) gives the following theorem (see [**53**] for a proof).

THEOREM 5.4. *The Dirac type operator*

$$\widehat{D} : H^1(\widehat{M}, E) \to L^2(\widehat{M}, F)$$

is Fredholm if and only if $\widehat{D}_i : H^1(\widehat{M}_i, E_i) \to L^2(\widehat{M}_i, E_i)$ *for* $i = 1, 2$*, and the corner operator* $D_Y : H^1(Y, E_0) \to L^2(Y, E_0)$ *are each invertible.*

---

[6]Although the square has four corners instead of one, this example illustrates the point of the theorem.

Here, $\widehat{M_i}$ is the manifold with cylindrical end formed by attaching an infinite cylinder to the odd-dimensional compact manifold with boundary M_i, and $\widehat{D}_i$ is the natural extension of the Dirac operator D_i to $\widehat{M}_i$.

From Theorem 3.3, we know that the Dirac operator on a manifold with a cylindrical end formed from a manifold with boundary can *always* be made Fredholm by considering it on weighted Sobolev spaces. For a manifold with corners of codimension two, this is not the case.

THEOREM 5.5. *There exists a $\delta > 0$ such that for all $0 < |\alpha| < \delta$, the Dirac type operator*

$$\widehat{D} : e^{\alpha s} H^1(\widehat{M}, E) \to e^{\alpha s} L^2(\widehat{M}, F)$$

is Fredholm if and only if the corner operator $D_Y : H^1(Y, E_0) \to L^2(Y, E_0)$ is invertible (*has zero kernel*).

See [**53**] or [**52**] for a proof. Here, each coordinate function s_i is extended into the rest of $\widehat{M}$ to be a positive bounded function there, $\alpha = (\alpha_1, \alpha_2)$ is a pair of real numbers, $0 < |\alpha| < \delta$ means that $0 < |\alpha_i| < \delta$ for $i = 1, 2$, and finally, $e^{\alpha s} = e^{\alpha_1 s_1} e^{\alpha_2 s_2}$. We remark that in many cases the kernels of Dirac operators represent topological quantities. In these cases, the invertibility of the corner Dirac operator would require certain cohomology groups of the corner Y to vanish. Thus the Fredholm condition in Theorem 5.5 is actually very restrictive.

5.3. Müller's generalization of the APS index formula. We now explain Müller's generalization [**67**] of the APS formula in Theorem 3.4 to manifolds with corners of codimension two under the assumption that the corner Dirac operator D_Y is invertible. We remove this assumption in Section 6.

We first need to introduce the b-eta invariants of the operators $\widehat{D}_1$ and $\widehat{D}_2$, cf. [**59**, Sec. 9.7]. Consider the operator $\widehat{D}_1$ on $\widehat{M_1}$. Here, $\widehat{M_1}$ is the manifold with cylindrical end formed by attaching an infinite cylinder to the odd-dimensional compact manifold with boundary M_1. The operator $\widehat{D}_1$ turns out to have continuous spectrum, and not discrete spectrum, due to the fact that $\widehat{M_1}$ has infinite volume. Thus its eta invariant cannot be defined as a regularized signature in the same way as in the case of a manifold with boundary considered in Section 3.2. However, since the heat operator of $\widehat{D}_1^2$ does exist, we can still try to define the eta invariant via the integral (3.3):

$$\text{“}\,\eta(\widehat{D}_1) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}(\widehat{D}_1 \, e^{-t\widehat{D}_1^2})\, dt\,\text{”}.$$

Unfortunately, the operator $\widehat{D}_1 \, e^{-t\widehat{D}_1^2}$ is not trace class, cf. Section 4.3, so the right-hand side is not defined, which is the reason for the quotes. However, the b-trace of $\widehat{D}_1 \, e^{-t\widehat{D}_1^2}$ is defined.[7] Replacing Tr with ${}^b\mathrm{Tr}$ in the above formula defines

[7]We first compactify $\widehat{M_1}$, and then define the b-trace as in Section 4.3.

the *b-eta invariant*[8],

$$ {}^b\eta(\widehat{D}_1) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\, {}^b\mathrm{Tr}(\widehat{D}_1\, e^{-t\widehat{D}_1^2})\, dt. $$

The b-eta invariant of $\widehat{D}_2$ is defined in the same way. The APS formula in Theorem 3.4 generalizes as follows.

Theorem 5.6 (Müller, 1996). *Let D be a Dirac type operator on an even-dimensional, compact, oriented, Riemannian manifold with corners of codimension two with exactly two boundary hypersurfaces intersecting in exactly one corner and with product type structures specified. Then there exists a $\delta > 0$ such that for all $0 < |\alpha| < \delta$, the Dirac type operator*

$$ \widehat{D} : e^{\alpha s} H^1(\widehat{M}, E) \to e^{\alpha s} L^2(\widehat{M}, F) $$

is Fredholm if and only if the corner operator $D_Y : H^1(Y, E_0) \to L^2(Y, E_0)$ is invertible (has zero kernel); in which case,

$$ \mathrm{ind}_\alpha \widehat{D} = \int_M K_{AS} - \frac{1}{2} \sum_{i=1,2} \Big\{ {}^b\eta(\widehat{D}_i) + \mathrm{sign}\,\alpha \cdot \dim\ker \widehat{D}_i \Big\}. $$

Müller's theorem in [**67**] technically only applies to the case of $\alpha > 0$; in the generality presented above, the theorem is due to Melrose, cf. [**53**]. The formula for $\mathrm{ind}_\alpha \widehat{D}$ is almost exactly the same as the APS formula in Theorem 3.4. In fact, using the b-calculus, the proof of Theorem 5.6 proceeds in almost identical fashion as the proof of Theorem 3.4. The only 'hard' part is defining the appropriate generalization of b-pseudodifferential operators and the b-trace to manifolds with corners of codimension two. Once this machinery is set up, the proof of the APS index formula can be used to prove Theorem 5.6.

5.4. b-version of Müller's theorem. In analogy with the case of a manifold with boundary, we now compactify the manifold $\widehat{M}$ by introducing the change of variables $x_1 = e^{s_1}$ and $x_2 = e^{s_2}$. As $s_i \to -\infty$, $x_i \to 0$, and so this change of variables compactifies $\widehat{M}$ to be the interior of a compact manifold with corners of codimension two, which we denote by X. Moreover, since $ds_i = dx_i/x_i$ and $\partial_{s_i} = x_i \partial_{x_i}$, the geometric objects on $\widehat{M}$ transform into corresponding singular geometric 'b-objects' on the compact manifold with corners:

$$ \begin{aligned} g = ds_1^2 + ds_2^2 + h &\rightsquigarrow g = \Big(\frac{dx_1}{x_1}\Big)^2 + \Big(\frac{dx_2}{x_2}\Big)^2 + h \quad (b\text{-metric}), \\ dg = ds_1\, ds_2\, dh &\rightsquigarrow dg = \frac{dx_1}{x_1}\frac{dx_2}{x_2}\, dh \quad (b\text{-measure}), \\ H^k(\widehat{M}) &\rightsquigarrow H_b^k(X) \quad (b\text{-Sobolev space}), \end{aligned} $$

[8]The same discussion as in footnote (4) concerning the local index theorem on odd-dimensional manifolds without boundary applies in this situation too.

and finally,

$$\widehat{D} = \Gamma_1 \partial_{s_1} + \Gamma_2 \partial_{s_2} + B \rightsquigarrow \widehat{D} = \Gamma_1\, x_1 \partial_{x_1} + \Gamma_2\, x_2 \partial_{x_2} + B,$$

a b-differential operator. We repeat the statement of Müller's theorem in the present context.

THEOREM 5.7. *Let D be a Dirac type operator on an even-dimensional, compact, oriented, Riemannian manifold with corners of codimension two with exactly two boundary hypersurfaces intersecting in exactly one corner and with product type structures specified. Then there exists a $\delta > 0$ such that for all $0 < |\alpha| < \delta$, the Dirac type operator*

$$\widehat{D} : x^\alpha H^1_b(X, E) \to x^\alpha L^2_b(X, F)$$

is Fredholm if and only if the corner operator $D_Y : H^1(Y, E_0) \to L^2(Y, E_0)$ is invertible (has zero kernel); in which case,

$$\operatorname{ind}_\alpha \widehat{D} = \int_M K_{AS} - \frac{1}{2} \sum_{i=1,2} \Big\{ {}^b\eta(\widehat{D}_i) + \operatorname{sign} \alpha \cdot \dim \ker \widehat{D}_i \Big\}.$$

As already mentioned, with a proper generalization of b-pseudodifferential operators and the b-trace to manifolds with corners of codimension two, the proof of Theorem 5.7 proceeds in almost identical fashion as the proof of Theorem 3.4. In fact, the above theorem and its proof generalize to not only Dirac type operators but also b-pseudodifferential on manifolds with corners of *arbitrary* codimension (see [**50, 51, 52, 53**]). These generalizations are due to Melrose (for Dirac operators) and the author (for b-pseudodifferential operators), cf. Lauter and Moroianu [**44**] for the cusp case. The ability to handle arbitrary codimensions is a nice feature of using b-pseudodifferential operators to attack index problems on manifolds with corners.

5.5. Some remarks on index theory on manifolds with corners. In [**75**] Salomenson builds an index theory for Dirac operators on manifolds with corners of codimension two by attaching cylinders in a very different way than considered here. Instead of attaching separate cylinders to each hypersurface M_i and then filling in the lower quadrant with a product cylinder as shown in Figure 7, he notes that ∂M has a natural smooth structure and attaches the cylinder $(-\infty, 0] \times \partial M$ onto M. This creates a manifold with cylindrical end like in the case of a manifold with boundary, except that it has a 'wedge singularity' at the original corner Y. Results of Cheeger [**22**] or Chou [**24**] can be used to handle the wedge singularity.

In a different direction, Hassel, Mazzeo, and Melrose [**41**] prove a signature formula for manifolds with corners of codimension two. Unlike the signature formulas for manifolds with and without boundary, which are direct corollaries of index formulas on such manifolds, the HMM formula is *not* a consequence of an index formula on manifolds with corners of codimension two. Instead, they round off the corner and consider X as a limit as $\varepsilon \to 0$ of manifolds with smooth boundary X_ε.

The resulting signature formula is obtained by a careful analysis of the limit of the APS signature formulas of each X_ε. They rely on 'analytic surgery' techniques in [**40**] to identify the limiting formula. Wall [**84**] considers a manifold with boundary divided into two parts, each a manifold with corners of codimension two (e.g., a disk divided into two half wedges). Although not an index formula *per se*, Wall gives a formula for the signature of the manifold with boundary in terms of the signatures of the two manifolds with corners of codimension two and a correction term given by the Maslov index of certain Lagrangian subspaces, cf. Section 6.

6. Perturbations of Dirac operators on manifolds with corners

We now consider the APS index formula for manifolds with corners of codimension two dropping the invertibility assumption on the corner Dirac operator. From our experience with the Gauss-Bonnet formula in the introduction, we expect there to be a correction term added to the right-hand side of the APS formula due to the presence of the corners. Theorem 5.6 did not have a corner contribution, essentially because the invertibility of the corner Dirac operator D_Y makes the Dirac operator $\widehat{D}$ not 'notice' the presence of the corners. For the Gauss-Bonnet formula, the correction term was given by the exterior angles of the corners. For the APS formula without the invertibility assumption on the corner operator, there is a correction term in the index formula and it represents an 'exterior angle' of sorts between certain Lagrangian vector spaces. In this section, we use the same notation as in Section 5.

6.1. Fredholm perturbation of Dirac operators. By Theorem 5.5, there exists a $\delta > 0$ such that for all $0 < |\alpha| < \delta$, the Dirac type operator

$$\widehat{D} : x^\alpha H^1_b(X, E) \to x^\alpha L^2_b(X, F)$$

is Fredholm if and only if the corner operator $D_Y : H^1(Y, E_0) \to L^2(Y, E_0)$ is invertible (has zero kernel). This nondegeneracy condition is actually very restrictive since in many cases the kernels of Dirac operators represent cohomology. However, we now show that it is *always* possible to make $\widehat{D}$ Fredholm on weighted Sobolev spaces by perturbation with b-smoothing operators.

To define these perturbations we recall some notation from Section 5.1. The manifold with corners of codimension two M is assumed to have exactly two boundary hypersurfaces M_1 and M_2 that intersect in exactly one codimension two face Y. Near the corner Y the Dirac type operator D takes the form

$$D = \Gamma_1 \partial_{s_1} + \Gamma_2 \partial_{s_2} + B,$$

where

$$B : C^\infty(Y, E_0) \to C^\infty(Y, F_0)$$

is a Dirac type operator on the even-dimensional manifold without boundary Y. The induced operator D_1 on the hypersurface M_1 takes the form

$$D_1 = \Gamma(\partial_{s_2} + D_Y), \quad \Gamma = \Gamma_1^* \Gamma_2, \quad D_Y = \Gamma_2^* B, \tag{6.1}$$

and the operator D_2 on M_2 takes the form

$$D_2 = -\Gamma(\partial_{s_1} + \widetilde{D}_Y), \quad \widetilde{D}_Y = \Gamma D_Y. \tag{6.2}$$

The minus sign in front of Γ and the fact that $\widetilde{D}_Y = \Gamma D_Y$ will come into play later. Also, see Lemma 5.1, $E_0 = E_0^+ \oplus E_0^-$ where $E_0^\pm$ are the $\pm i$ eigenspaces of $\Gamma = \Gamma_1^* \Gamma_2$, and the induced Dirac operator $D_Y = \Gamma_2^* B$ is odd with respect to Γ and so decomposes as

$$\begin{bmatrix} 0 & D_Y^- \\ D_Y^+ & 0 \end{bmatrix} : C^\infty(Y, E_0^+ \oplus E_0^-) \to C^\infty(Y, E_0^+ \oplus E_0^-),$$

where $D_Y^\pm$ are the restrictions of D_Y to $C^\infty(Y, E_0^\pm)$. Moreover, see Theorem 5.2,

$$D_Y^+ : H^1(Y, E_0^+) \to L^2(Y, E_0^-)$$

has index zero; that is, $\dim \ker D_Y^+ = \dim \ker D_Y^-$.

We now define the perturbations. Since the kernel of the Dirac type operator D_Y is exactly the obstruction to $\widehat{D}$ being Fredholm on weighted Sobolev spaces, the perturbations are chosen to be isomorphisms on the kernel. Since D_Y is odd with respect to Γ, we only consider isomorphisms on $\ker D_Y$ having the same property. Thus, let $T : \ker D_Y \to \ker D_Y$ be a self-adjoint unitary isomorphism that is odd with respect to Γ. Hence T decomposes as an odd matrix

$$\begin{bmatrix} 0 & T^- \\ T^+ & 0 \end{bmatrix} : \ker D_Y^+ \oplus \ker D_Y^- \to \ker D_Y^+ \oplus \ker D_Y^-,$$

where $T^\pm : \ker D_Y^\pm \to \ker D_Y^\mp$ are unitary isomorphisms with respect to the L^2 inner product on $\ker D_Y \subset L^2(Y, E_0)$. Such an operator T exists because $\dim \ker D_Y^+ = \dim \ker D_Y^-$. We can define T explicitly as follows. Let $\{u_j\}_{j=1}^N$ and $\{v_j\}_{j=1}^N$ be orthonormal bases of $\ker D_Y^+$ and $\ker D_Y^-$, respectively. By elliptic regularity, $u_j,\ v_j \in C^\infty(Y, E_0)$ for every j. Then,

$$T = \sum_{j=1}^N u_j \otimes \overline{v}_j + \sum_{j=1}^N v_j \otimes \overline{u}_j$$

defines a self-adjoint unitary isomorphism on $\ker D_Y$ that is odd with respect to Γ and any such T can be written in this way for some choice of bases. Moreover, this formula shows that T is a smoothing operator on Y. Obviously,

$$D_Y - T : H^1(Y, E_0) \to L^2(Y, E_0)$$

is invertible. This suggests that if we can extend T to an operator $\widehat{T}$ on X, then

$$\widehat{D} - \widehat{T} : x^\alpha H_b^1(X, E) \to x^\alpha L_b^2(X, F)$$

is Fredholm for all $0 < |\alpha| < \delta$ for some $\delta > 0$. To extend T, we first define $\widehat{T}$ on the manifold with multi-cylindrical ends $\widehat{M}$. Let $\chi \in C_c^\infty([0,1)^2)$ be such that

$\chi(x) = 1$ for x near zero. Then $\chi(e^s) = \chi(e^{s_1}, e^{s_2})$ can be regarded as a smooth function on $\widehat{M}$ supported on the cylindrical end (cf. Figure 7)

$$(-\infty, 0]_{s_1} \times (-\infty, 0]_{s_2} \times Y.$$

Let $\varphi(\xi_1, \xi_2) = e^{-\xi_1^2 - \xi_2^2}$. Then, given any $u \in C_c^\infty(\widehat{M}, E)$, we define

$$\widehat{T}u(s,y) = \chi(e^s)\,\Gamma_2\,\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{is\cdot\xi}\,\varphi(\xi)\,T\,\widehat{\chi u}(\xi,y)\,d\xi, \tag{6.3}$$

where $\widehat{\chi u}$ is the Fourier transform of $\chi(e^s)\,u(s,y)$ with respect to s:

$$\widehat{\chi u}(\xi,y) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-is\cdot\xi}\,\chi(e^s)\,u(s,y)\,ds.$$

The reason for the factor of Γ_2 on the right-hand side of $\widehat{T}u$ is that $\widehat{T}$ is required to map sections of E to sections of F and the Γ_2 factor provides this property. Note that the operator T in the definition of $\widehat{T}$ only acts on the y variable of $\widehat{\chi u}(\xi, y)$. Regarded as an operator on the compactified manifold X under the change of variables $x_i = e^{s_i}$, the operator $\widehat{T}$ is an example of a b-smoothing operator; that is, a b-pseudodifferential operator of order $-\infty$. The mapping properties of such operators (cf. [**59**, Ch. 5]) imply that

$$\widehat{T} : x^\alpha H_b^k(X,E) \to x^\alpha H_b^\ell(X,F) \quad \text{for all } \alpha, k, \ell.$$

The next result follows from the properties of b-pseudodifferential operators and the fact that $D_Y - T : H^1(Y, E_0) \to L^2(Y, E_0)$ is invertible, cf. [**52, 53**].

Lemma 6.1. *There exists a $\delta > 0$ such that for all $0 < |\alpha| < \delta$,*

$$\widehat{D} - \widehat{T} : x^\alpha H_b^1(X,E) \to x^\alpha L_b^2(X,F)$$

is Fredholm.

6.2. An index formula for perturbed Dirac operators. We now give a formula for the index of $\widehat{D} - \widehat{T}$. Before doing so, we need to review the 'scattering Lagrangian' of each operator $\widehat{D}_i$. Consider the operator $\widehat{D}_1$ on $\widehat{M}_1$. Recall that $\widehat{M}_1$ is formed by attaching an infinite cylinder $(-\infty, 0]_{s_2} \times Y$ to the odd-dimensional compact manifold with boundary M_1. The set

$$\Lambda_{C_1} = \Big\{ \lim_{s_2 \to -\infty} u(s_2, y)\,;\, u \in C^\infty(\widehat{M}_1, E) \text{ is bounded, and } \widehat{D}_1 u = 0 \Big\}$$

is called the *scattering Lagrangian* of $\widehat{D}_1$. It turns out that $\Lambda_{C_1} \subset \ker D_Y$ and the dimension of Λ_{C_1} is exactly one-half the dimension of $\ker D_Y$, cf. [**59**, Sec. 6.5]. The *scattering matrix* of $\widehat{D}_1$ is the operator $C_1 : \ker D_Y \to \ker D_Y$ defined by $C_1 = +1$ on Λ_{C_1} and $C_1 = -1$ on $\Lambda_{C_1}^\perp$, where '$\perp$' means the orthogonal complement with respect to the L^2 inner product. Then C_1 is odd with respect to Γ, cf. [**65**, Sec. 4]. The scattering Lagrangian Λ_{C_2} and the matrix C_2 of $\widehat{D}_2$ are defined in the same way. In [**53**] we give the following formula for the index of $\widehat{D} - \widehat{T}$:

Theorem 6.2 (Loya-Melrose, 2002). *Let D be a Dirac type operator on an even-dimensional, compact, oriented, Riemannian manifold with corners of codimension two with exactly two boundary hypersurfaces intersecting in exactly one corner and with product type structures specified. Let $T : \ker D_Y \to \ker D_Y$ be a self-adjoint unitary isomorphism that is odd with respect to Γ and let $\widehat{T}$ be the perturbation defined by* (6.3). *Then there exists a $\delta > 0$ such that for all $0 < |\alpha| < \delta$, the perturbed Dirac operator*

$$\widehat{D} - \widehat{T} : x^\alpha H^1_b(X, E) \to x^\alpha L^2_b(X, F)$$

is Fredholm. Moreover, if its index is denoted by $\operatorname{ind}_\alpha(\widehat{D} - \widehat{T})$, then

$$\operatorname{ind}_\alpha(\widehat{D} - \widehat{T}) = \int_M K_{AS} - \frac{1}{2} \sum_{i=1,2} \Big\{ {}^b\eta(\widehat{D}_i) + \operatorname{sign}\alpha \cdot \dim\ker \widehat{D}_i \Big\} - \frac{1}{2} c_\alpha(\Lambda_T, \Lambda_{C_1}, \Lambda_{C_2}). \tag{6.4}$$

The first line on the right-hand side is the same as in Theorem 5.7; the third 'corner correction term' is described as follows. First, $\Lambda_T \subset \ker D_Y$ is the $+1$ eigenspace of the matrix T (since T is a self-adjoint unitary isomorphism, $T^2 = \mathrm{Id}$, so T has eigenvalues ± 1). Then,

$$c_\alpha(\Lambda_T, \Lambda_{C_1}, \Lambda_{C_2}) = \operatorname{sign}\alpha \cdot \big\{ \dim(\Lambda_T \cap \Lambda_{C_1}) + \dim(\Lambda_{\Gamma T} \cap \Lambda_{C_2}) \big\} + \eta(D_{\mathcal{G}}), \tag{6.5}$$

where $\Lambda_{\Gamma T} \subset \ker D_Y$ is the $+1$ eigenspace of the self-adjoint unitary isomorphism ΓT and $\eta(D_{\mathcal{G}})$ is the eta invariant of a Dirac operator on a directed graph $\mathcal{G}$ defined as follows, cf. [**41**], [**55**], [**20**], [**47**]. This graph has two vertices v_1 and v_2 representing the hypersurfaces M_1 and M_2, respectively, and two edges e_{12} and e_{21} connecting the vertices representing the single corner Y. To put a manifold structure on this graph, we identify e_{jk} with the interval $[-1, 1]_t$, where the vertex v_j corresponds to $t = -1$ and the vertex v_k to $t = +1$. We consider $V = \ker D_Y \oplus \ker D_Y$ as a 'vector bundle' over $\mathcal{G}$ where the first and second factors of $\ker D_Y$ are 'fibers' over the edges e_{12} and e_{21}, respectively. Thus a section of this vector bundle is a sum $s_{12} \oplus s_{21}$, where $s_{jk} : e_{jk} = [-1, 1] \to \ker D_Y$. We define a Dirac operator $D_{\mathcal{G}}$ acting on sections of $\mathcal{G}$ by

$$D_{\mathcal{G}} = \Gamma \frac{d}{dt} \oplus \big(- \Gamma \frac{d}{dt} \big).$$

The minus sign in the second term stems from the minus sign in (6.2). The domain of $D_{\mathcal{G}}$ consists of those sections $s_{12} \oplus s_{21}$ such that $s_{12}(v_1) \in \Lambda_T$, $s_{12}(v_2) \in \Lambda_{C_1}$, and $s_{21}(v_2) \in \Lambda_{\Gamma T}$, $s_{21}(v_1) \in \Lambda_{C_2}$. The Lagrangian $\Lambda_{\Gamma T}$ paired with the scattering Lagrangian Λ_{C_2} stems from the fact that $\widetilde{D}_Y = \Gamma D_Y$ in (6.2). The term $\eta(D_{\mathcal{G}})$ appearing in (6.5) is then the eta invariant of $D_{\mathcal{G}}$.

Lesch and Wojciechowski [**47**] give the following linear-algebraic form for the eta term:

$$\eta(D_{\mathcal{G}}) = m(\Lambda_T, \Lambda_{C_1}) - m(\Lambda_{\Gamma T}, \Lambda_{C_2}),$$

where

$$m(\Lambda_{L_1}, \Lambda_{L_2}) = -\frac{1}{i\pi} \sum_{\substack{e^{i\theta} \in \operatorname{spec}(-L_1^- L_2^+) \\ \theta \in (-\pi, \pi)}} i\theta$$

for any given self-adjoint unitary isomorphisms L_1, L_2 on $\ker D_Y$ that are odd with respect to Γ, and with $\Lambda_{L_i} \subset \ker D_Y$ denoting the $+1$ eigenspace of L_i. The number $m(\Lambda_{L_1}, \Lambda_{L_2})$ can be interpreted as an 'exterior angle' of sorts between Λ_{L_1} and Λ_{L_2}, cf. [**48**], [**20**], [**18**]. Hassel, Mazzeo, and Melrose [**40, 41**] give a somewhat more sophisticated linear-algebraic description of the eta term.

Theorem 6.2 is proved as follows: First, following the proof of Theorem 5.7, which uses similar arguments found in Section 4, produces the formula

$$\operatorname{ind}_\alpha(\widehat{D} - \widehat{T}) = \int_M K_{AS} - \frac{1}{2} \sum_{i=1,2} \left\{ {}^b\eta(\widehat{D}_i - \widehat{T}_i) + \operatorname{sign}\alpha \cdot \dim\ker(\widehat{D}_i - \widehat{T}_i) \right\},$$

where $\widehat{T}_i$ is an operator naturally induced by $\widehat{T}$ on $\widehat{M}_i$. The second and most difficult part of the proof is to show that the terms involving $\widehat{D}_i - \widehat{T}_i$ in this formula decompose as in (6.4). To do this, we show that

$$\dim\ker(\widehat{D}_i - \widehat{T}_i) = \dim\ker \widehat{D}_i + \dim(\Lambda_{T_i} \cap \Lambda_{C_i}),$$

where $T_1 = T$ and $T_2 = \Gamma T$, and that

$${}^b\eta(\widehat{D}_i - \widehat{T}_i) = {}^b\eta(\widehat{D}_i) \pm m(\Lambda_{T_i}, \Lambda_{C_i}),$$

where the sign is positive or negative if $i = 1$ or $i = 2$, respectively. The decomposition of the b-eta invariants uses techniques that have been developed by many authors concerning gluing/splitting formulas for eta invariants, e.g. Brüning and Lesch [**17**] (cf. Vishik [**83**]), Douglas and Wojciechowski [**28**], Lesch and Wojciechowski [**47**], and Müller [**65**]. For related works on the eta invariant, see Singer [**80**], Bunke [**18**], Dai and Freed [**26**], and Hassel, Mazzeo, and Melrose [**40**].

6.3. Some concluding remarks. If M has more than one corner, the result from Theorem 6.2 still holds (with minor changes in the index formula accounting for the various faces and corners) as long as we *assume* that each corner Dirac operator has index zero [**53**]. This assumption allows us to construct separate perturbations for each corner, then sum these perturbations producing a b-smoothing operator giving a Fredholm perturbation of the Dirac operator. Melrose and Nistor show that it is in fact necessary that each corner operator have index zero for the existence of a b-smoothing Fredholm perturbation. However, using a slightly larger class of perturbations called 'overblown' b-smoothing operators, it is possible to make Fredholm perturbations without any assumptions at the corners [**54**].

Current plans include relating the index of the perturbed Dirac operator to an index of the Dirac operator on a domain depending on the choice of perturbation. The index should be a type of Carron index [**21**]. In future work, we expect to generalize the program of 'overblown' b-smoothing Fredholm perturbations of Dirac operators to manifolds with corners of arbitrary codimension. Finally, one of the ultimate goals of this project is to derive via an index computation an analytic formula for the topological signature of any compact manifold with corners of arbitrary codimension in terms of geometric and other types of invariants of the manifold and its boundary faces.

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