A Derivation of Taylor's Formula with Integral Remainder

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Taylor's formula with integral remainder is usually derived using integration by parts [**4, 5**], or sometimes by differentiating with respect to a parameter [**1, 2**]. According to M. Spivak [**7**, p. 390], integration by parts is applied in a "rather tricky way" to derive Taylor's formula, using a substitution that "one might discover after sufficiently many similar but futile manipulations". In this MAGAZINE, Lampret [**3**] derived both Taylor's formula and the Euler-Maclaurin summation formula using a rather heroic application of integration by parts.

We derive the remainder formula in a way that avoids tricks and heroics. The key step is changing the order of integration in multiple integrals, a topic that many students in an analysis class will benefit from reviewing. This derivation has almost certainly been found many times before [**6**], however, most people seem to be unaware of it.

The Taylor formula Suppose that a function $f(x)$ and all its derivatives up to $n+1$ are continuous on the real line. Then Taylor's formula for $f(x)$ about 0 is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R(x), \qquad (1)$$

where the remainder, $R(x)$, is given by

$$R(x) = \frac{1}{n!}\int_0^x (x-u)^n f^{(n+1)}(u)\,du.$$

Our derivation is based on the following simple idea: Try to reconstruct f by integrating $f^{(n+1)}$, $n+1$ times. This approach is suggested by the case $n=0$, when (1) is merely the fundamental theorem of calculus. For notational simplicity, we prove (1) for only $n=2$; however, the general case is similar. Thus, consider

$$\widetilde{R}(x) := \int_0^x \int_0^w \int_0^v f^{(3)}(u)\,du\,dv\,dw. \qquad (2)$$

Now let's evaluate this integral in two ways. The first way is by direct integration using the fundamental theorem of calculus three times:

$$\widetilde{R}(x) = f(x) - f(0) - xf'(0) - \frac{x^2}{2!}f''(0). \qquad (3)$$

The second way to integrate (2) is by interchanging the order of integration:

$$\int_0^w \int_0^v f^{(3)}(u)\,du\,dv = \int_0^w \int_u^w f^{(3)}(u)\,dv\,du = \int_0^w (w-u) f^{(3)}(u)\,du.$$

Interchanging the order of integration again gives

$$\begin{aligned}
\int_0^x \left\{ \int_0^w \int_0^v f^{(3)}(u)\,du\,dv \right\} dw &= \int_0^x \left\{ \int_0^w (w-u) f^{(3)}(u)\,du \right\} dw \\
&= \int_0^x \int_u^x (w-u) f^{(3)}(u)\,dw\,du \\
&= \frac{1}{2} \int_0^x (x-u)^2 f^{(3)}(u)\,du. \qquad (4)
\end{aligned}$$

Equating (3) and (4) yields the Taylor formula (1) for $n = 2$.

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