The Reidemeister trace in pictures

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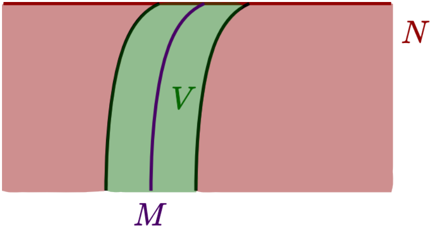
(What is this good for? Computations of $H_*(X)$ are much easier using both f_* and $f_!$.)

First example: $e : M \to N$ a codimension d embedding of closed manifolds, tubular neighborhood $V \subseteq N$.



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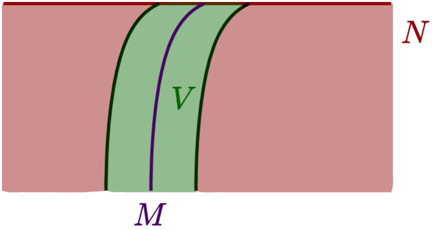

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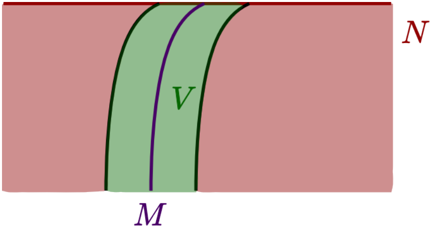

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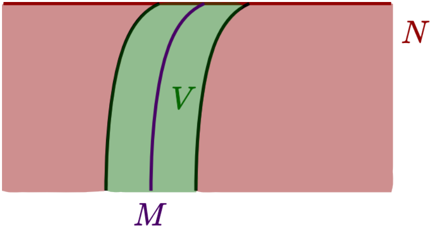

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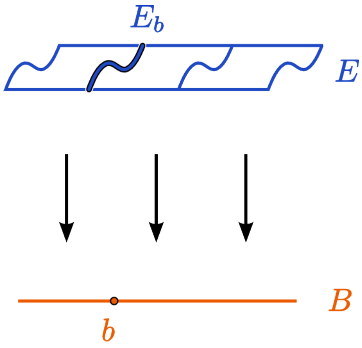
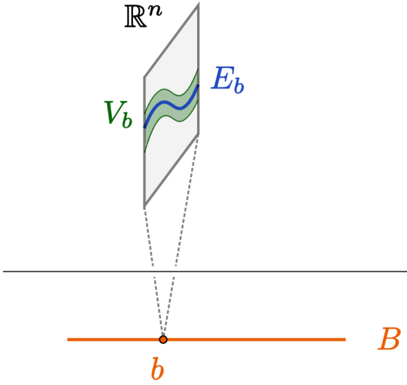
(If M, N are both oriented. Otherwise last term has twisted coefficients.)

Second example: $p : E \to B$ a smooth fiber bundle, fibers are closed manifolds of dimension d.



Embed E fiberwise into $B \times \mathbb{R}^n \cong B \times \mathring{D}^n$, with tubular neighborhood V:



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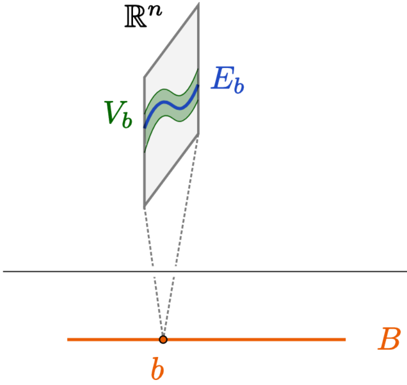

$$(B \times D^n)/(B \times \partial D^n) \longrightarrow V/\partial V$$

$$x \mapsto \begin{cases} x & x \in V \\ * & \text{otherwise} \end{cases}$$

That gives a different transfer

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(Again, last term has twisted coefficients if bundle isn't orientable.)

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$$p_* \circ p_! = n \cdot \mathrm{id}_{H_*(B)}.$$

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Answer: (Lind-M, 2016) Yes! (extends Schlichtkrull, 1998)

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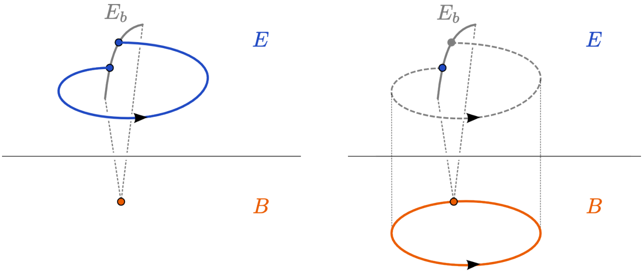
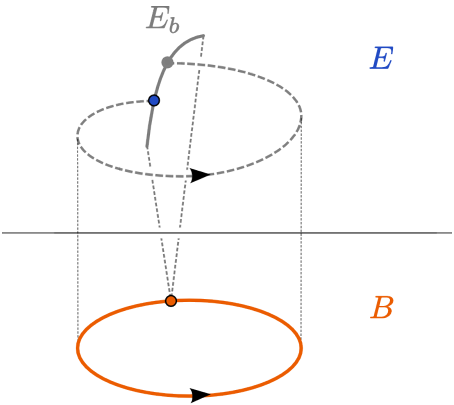
P has two descriptions:

1. Space of paths in E whose endpoints lie in the same fiber E_b
2. Space of choices of point $e \in E$, and loop in B based at $p(e)$.



These describe homotopy-equivalent spaces.



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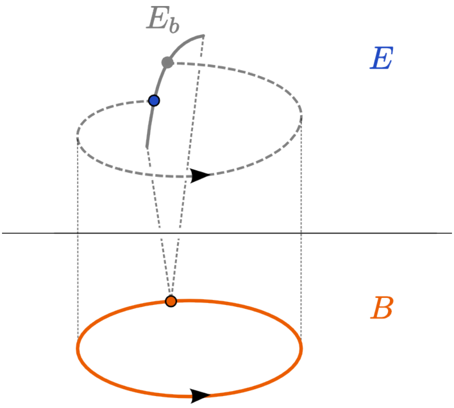

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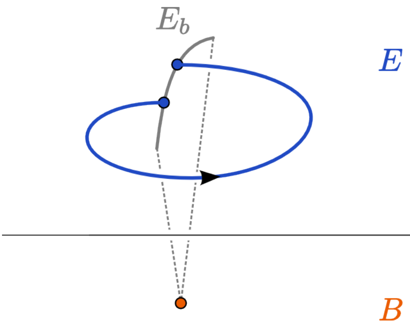
$$H_q(LB) \longrightarrow H_{q+d}(P)$$



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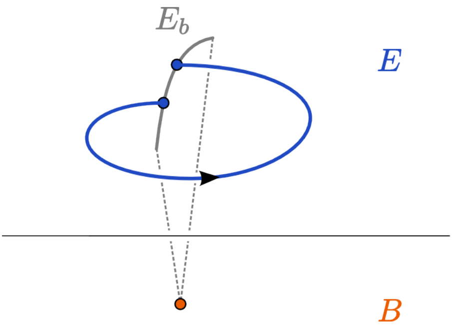

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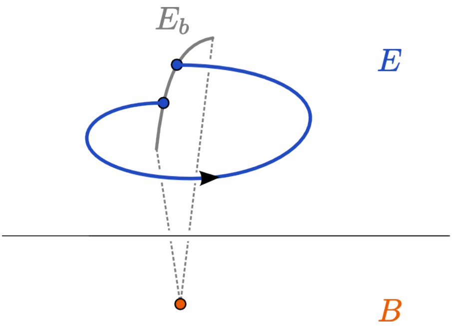

First description makes $LE \longrightarrow P$ like an inclusion with tubular neighborhood(*), so we get a shifting-down transfer

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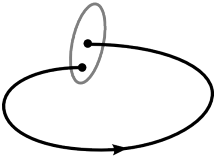
Putting these together recovers our mystery map $p_?$.

Further explanation of (*).

path with close endpoints | loop with tangent vector



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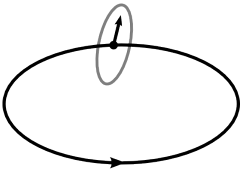

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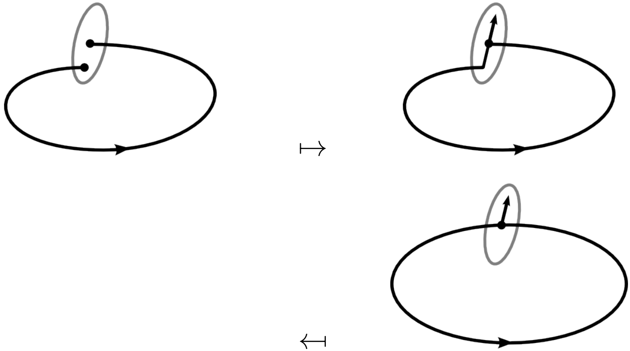

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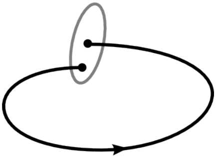

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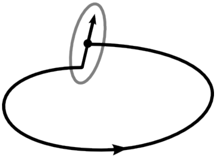

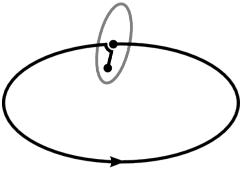

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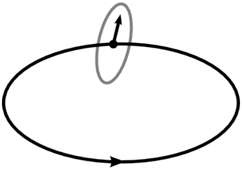

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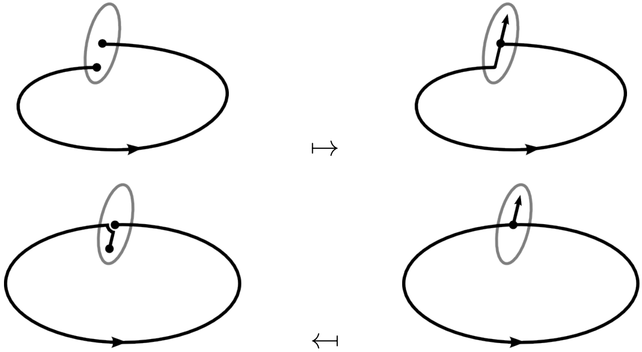

(Gives homotopy equivalence between neighborhood of LE in P and vertical tangent bundle over LE.)

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- (do this first) Just compute $p_?$ in some examples. Relatively unexplored.

For $p : LBS^1 \to LBS^3$, we have

$$
\begin{array}{ccccccc}
H_q(LBS^1) & \xrightarrow{p_*} & H_q(LBS^3) & & H_q(LBS^3) & \xrightarrow{p_?} & H_q(LBS^1) \\
\mathbb{Z} & \xrightarrow{\quad 1 \quad} & \mathbb{Z} & q=0 & \mathbb{Z} & \xrightarrow{\quad 2 \quad} & \mathbb{Z} \\
\mathbb{Z} & & 0 & q=1 & 0 & & \mathbb{Z} \\
\mathbb{Z} & & 0 & q=2 & 0 & & \mathbb{Z} \\
\mathbb{Z} & \xrightarrow{\quad 2 \quad} & \mathbb{Z} & q=3 & \mathbb{Z} & \xrightarrow{\quad 1 \quad} & \mathbb{Z} \\
\mathbb{Z} & \xrightarrow{\quad 1 \quad} & \mathbb{Z} & q=4 & \mathbb{Z} & \xrightarrow{\quad 2 \quad} & \mathbb{Z} \\
\mathbb{Z} & & 0 & q=5 & 0 & & \mathbb{Z} \\
\mathbb{Z} & & 0 & q=6 & 0 & & \mathbb{Z} \\
\mathbb{Z} & \xrightarrow{\quad 2 \quad} & \mathbb{Z} & q=7 & \mathbb{Z} & \xrightarrow{\quad 1 \quad} & \mathbb{Z} \\
\mathbb{Z} & \xrightarrow{\quad 1 \quad} & \mathbb{Z} & q=8 & \mathbb{Z} & \xrightarrow{\quad 2 \quad} & \mathbb{Z} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots
\end{array}
$$