

# THE STEENROD ALGEBRA

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The goal of these notes is to show how to use the Steenrod algebra and the Serre spectral sequence to calculate things.

## 1. BROWN REPRESENTABILITY (AS MOTIVATION)

Let  $X$  be a topological space. Suppose that we want to study the cohomology groups  $H^*(X; G)$ . A good way to understand these abelian groups is to define additional algebraic structures on them, like the cup product.

We start this process by studying something that appears to be completely unrelated. Let  $G$  be an abelian group. Let's look for a space  $K(G, n)$  with the property that

$$\pi_i(K(G, n)) = \begin{cases} G & i = n \\ 0 & i \neq n \end{cases}$$

Such a space  $K(G, n)$  is called an *Eilenberg-MacLane space*. Let's assume in addition that  $K(G, n)$  is homotopy equivalent to a CW-complex. It is then not hard to show that  $K(G, n)$  exists, and in fact any two such spaces are homotopy equivalent. Further, if we require that  $K(G, n)$  always comes with a choice of isomorphism

$$\pi_n(K(G, n)) \cong G$$

then any two candidates for  $K(G, n)$  are related by a homotopy equivalence that is unique up to homotopy. So it makes sense to talk about *the* Eilenberg-MacLane space  $K(G, n)$ , as long as we are working primarily with homotopy classes of maps.

Let's look at some examples:

$$\begin{aligned} K(\mathbb{Z}, 1) &= S^1 \\ K(\mathbb{Z}/2, 1) &= \mathbb{RP}^\infty \\ K(\mathbb{Z}, 2) &= \mathbb{CP}^\infty \end{aligned}$$

Any space with contractible universal cover is a  $K(G, 1)$ . This includes all the compact orientable surfaces with genus  $g \geq 1$ .

In general, there is a somewhat geometric model for  $K(G, n)$  using configuration spaces. We may describe  $K(G, n)$  as the space of all finite collections of points in  $S^n - \{*\}$ , together with a label of each point by some non-identity element of  $G$ . The space is topologized so that points may collide, and when they do we add their labels together. A point labelled with the identity element promptly disappears, while a labelled point that travels to the basepoint  $*$  also disappears.

If  $X$  and  $Y$  are spaces, let  $[X, Y]$  denote the set of homotopy classes of based maps  $X \rightarrow Y$ . Let  $\Omega X = \text{Map}_*(S^1, X)$  denote the space of based loops in  $X$ . Applying  $\Omega$  shifts the homotopy groups down by 1:

$$\pi_n(\Omega X) \cong \pi_{n+1}(X)$$

By the uniqueness of  $K(G, n)$ , we get a canonical homotopy equivalence

$$\Omega K(G, n+1) \simeq K(G, n)$$

Every loop space  $\Omega X$  has a multiplication that comes from concatenation of loops. This multiplication is associative up to homotopy, and has a unit and inverses up to homotopy. In addition, each double loop space

$$\Omega\Omega X = \Omega^2 X \cong \text{Map}_*(S^2, X)$$

has a multiplication which is commutative up to homotopy. Therefore the set

$$[X, K(G, n)] \cong [X, \Omega^2 K(G, n+2)]$$

is naturally an abelian group. This defines a functor which sends each space  $X$  to the abelian group  $[X, K(G, n)]$ . It is a contravariant functor, so it reverses the direction of maps:

$$X \xrightarrow{f} Y \rightsquigarrow [X, K(G, n)] \longleftarrow [Y, K(G, n)]$$

Cohomology is another such functor:

$$X \xrightarrow{f} Y \rightsquigarrow H^n(X; G) \longleftarrow H^n(Y; G)$$

Surprisingly, these two functors are naturally isomorphic. In other words, for each space  $X$ , there is an isomorphism of groups

$$[X, K(G, n)] \xrightarrow{\phi} H^n(X; G)$$

and each map  $X \xrightarrow{f} Y$  gives a commuting square

$$\begin{array}{ccc} [X, K(G, n)] & \xleftarrow{-\circ f} & [Y, K(G, n)] \\ \downarrow \phi & & \downarrow \phi \\ H^n(X; G) & \xleftarrow{f^*} & H^n(Y; G) \end{array}$$

This is called *Brown Representability*.

Let's look at the special case  $K(\mathbb{Z}, 1) \simeq S^1$ . Brown Representability gives a natural isomorphism of abelian groups

$$[X, S^1] \cong H^1(X; \mathbb{Z})$$

Geometrically, each map  $X \rightarrow S^1$  sends each cycle in  $X$  to a cycle in  $S^1$ , which has a winding number. This gives a rule that associates each cycle to an integer, which gives a cocycle on  $X$ , which gives a cohomology class. Every degree 1 cohomology class arises this way.

One way to prove Brown Representability is to construct a *fundamental class*

$$\gamma \in H^n(K(G, n); G)$$

which we expect to correspond to the identity map in  $[K(G, n), K(G, n)]$ . To know which class to pick, we use the Hurewicz Theorem and the Universal Coefficient Theorem to get a canonical isomorphism

$$H^n(K(G, n); G) \cong \text{Hom}_{\mathbb{Z}}(G, G) = \text{End}(G)$$

Then we pick the class  $\gamma$  that corresponds to the identity map  $G \rightarrow G$ . Next, we define

$$\begin{aligned} [X, K(G, n)] &\xrightarrow{\phi} H^n(X; G) \\ \phi(f) &= f^*\gamma \end{aligned}$$

In other words, given a map  $X \xrightarrow{f} K(G, n)$ , we pull back the fundamental class  $\gamma$  along  $f$  to get a cohomology class of  $X$ . Then  $\phi$  is clearly an isomorphism when  $X$  is a point. Both cohomology and  $[-, K(G, n)]$  satisfy a form of excision, which implies that if  $X = A \cup B$  is a CW complex with  $A$  and  $B$  subcomplexes, then if  $\phi$  is an isomorphism on  $A$ ,  $B$ , and  $A \cap B$  then it is an isomorphism on  $A \cup B$ . Every finite complex is built inductively from lower-dimensional complexes in this way, so we can show inductively that  $\phi$  is an isomorphism when  $X$  is a finite CW complex. To move to all CW complexes we must compare the cohomology of  $X$  with the cohomology of its finite subcomplexes. This step requires some care and we will gloss over it in our quick treatment.

The point of the Brown Representability theorem is that  $K(G, n)$  is the universal space on which cohomology classes live. Everything that can be done with a degree  $n$  cohomology class with coefficients in  $G$  may be done on  $K(G, n)$  first and then pulled back to any other space. To illustrate this, consider  $K(\mathbb{Z}, 1) \simeq S^1$ . The fundamental class  $\gamma$  is one of the generators of  $H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}$ . Note that  $\gamma \cup \gamma = 0$ . This implies that *any* degree

1 cohomology class  $\alpha \in H^1(X; \mathbb{Z})$  on *any* space  $X$  has the property that  $\alpha \cup \alpha = 0$ . The reason is that there is some map

$$X \xrightarrow{f} S^1$$

such that  $\alpha = f^*\gamma$ , and  $f^*$  preserves the cup product. One might expect this to come out of the skew-commutativity of the cup product, but that only gives the weaker statement

$$2\alpha \cup \alpha = 0$$

In odd degrees higher than 1, only this weaker statement is true.

## 2. DEFINITION OF THE STEENROD SQUARES

We want to understand cohomology. Brown Representability suggests that we should try to understand “universal” cohomology, by which we mean the cohomology of Eilenberg-MacLane spaces

$$H^m(K(G, n); G)$$

By Brown Representability, this group is naturally identified with

$$[K(G, n), K(G, m)]$$

and by the Yoneda Lemma this corresponds to the set of natural transformations

$$H^n(-; G) \longrightarrow H^m(-; G)$$

Each such transformation is called a *cohomology operation*. So when we compute the cohomology of  $K(G, n)$ , we are really enumerating the collection of all cohomology operations.

This is all very interesting, but when we talk this way we are in danger of making circular, content-free statements. We need to make a choice: should we try to compute cohomology classes on  $K(G, n)$  first, or should we try to define the cohomology operations first? We will do the latter.

Let’s restrict attention to  $G = \mathbb{Z}/2$ . Then there is a particularly nice family of cohomology operations called the *Steenrod squares*. Their construction is rather technical, but the full details can be found in [1]. The end result is the following:

**Proposition 2.1.** *There exists for each pair of integers  $i, n \geq 0$  a natural linear map*

$$\text{Sq}^i : H^n(X; \mathbb{Z}/2) \longrightarrow H^{n+i}(X; \mathbb{Z}/2)$$

*called the  $i$ th Steenrod square, with the following properties:*

- $Sq^0$  is the identity map.
- $Sq^1$  is the “Bockstein homomorphism,” the connecting homomorphism in the long exact sequence that arises from

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0$$

- $Sq^n(x) = x^2$ . (Note that  $n$  is the degree of  $x$ .)
- $Sq^i(x) = 0$  when  $i > n$ .
- $Sq^i$  commutes with the connecting homomorphism in the long exact sequence on cohomology. In particular, it commutes with the suspension isomorphism

$$H^n(X) \cong H^{n+1}(\Sigma X)$$

- The Cartan formula:

$$Sq^n(x \cup y) = \sum_{i+j=n} Sq^i(x) \cup Sq^j(y)$$

- The Adem relations hold: when  $a < 2b$ ,

$$Sq^a Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

where  $Sq^a Sq^b$  denotes the composition of the Steenrod squares. The binomial coefficient in this formula is taken mod 2.

The Steenrod Algebra  $\mathcal{A}$  is the free  $\mathbb{Z}/2$ -algebra generated by the symbols  $\{Sq^i : i > 0\}$ , modulo the Adem relations. We can make  $\mathcal{A}$  into a graded algebra by declaring that  $Sq^i$  has degree  $i$ . Then for any space  $X$ , the graded abelian group

$$H^*(X; \mathbb{Z}/2) = \bigoplus_{n=0}^{\infty} H^n(X; \mathbb{Z}/2)$$

is a graded module over  $\mathcal{A}$ . The multiplication map

$$\mathcal{A} \times H^*(X; \mathbb{Z}/2) \longrightarrow H^*(X; \mathbb{Z}/2)$$

is generated by the action of the Steenrod squares  $Sq^i$ .

Let’s do a simple example. Say that we know the cohomology of  $\mathbb{R}P^\infty$  as a ring:

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[\alpha], \quad |\alpha| = 1$$

Then we can compute the action of the Steenrod squares as well. Using the axioms above,

$$\begin{aligned} Sq^0 \alpha &= \alpha \\ Sq^1 \alpha &= \alpha^2 \\ Sq^{>1} \alpha &= 0 \end{aligned}$$

From this we can calculate  $Sq^i(\alpha^n)$  using the Cartan formula. This is a bit tedious, but we can take a shortcut if we define the operation

$$Sq := Sq^0 + Sq^1 + Sq^2 + \dots$$

Now, technically, this isn't an element of the Steenrod algebra. Only finite sums are allowed. However, it still has a well-defined action on any one cohomology class  $x$ , because only finitely many of the terms will give nonzero results. Moreover, the Cartan formula is equivalent to the statement that

$$H^*(X) \xrightarrow{Sq} H^*(X)$$

is a homomorphism of rings. (Here and afterwards we will continue to take  $\mathbb{Z}/2$  coefficients, but we will drop them from the notation.) Therefore

$$\begin{aligned} Sq(\alpha^n) &= (Sq \alpha)^n \\ &= (\alpha + \alpha^2)^n \\ &= \sum_{i=0}^n \binom{n}{i} \alpha^{n+i} \end{aligned}$$

Then  $Sq^i \alpha^n$  is the only component of this sum of degree  $(n+i)$ , so we conclude that

$$Sq^i \alpha^n = \binom{n}{i} \alpha^{n+i}$$

This finishes our determination of  $H^*(\mathbb{R}P^\infty) = H^*(K(\mathbb{Z}/2, 1))$  as a module over the Steenrod algebra. Since this case is universal, we now understand the action of the Steenrod squares, the cup product, and all combinations of these things when they act on a cohomology class of degree 1. Our goal now is to continue this calculation to  $H^*(K(\mathbb{Z}/2, n))$ . For this task we need more tools.

### 3. THE SERRE SPECTRAL SEQUENCE

Recall that there are two notions of “short exact sequence of spaces.” There is the cofibration sequence

$$A \longrightarrow X \longrightarrow X/A$$

where  $X$  is (for example) a CW complex and  $A$  is a subcomplex. This gives a long exact sequence of cohomology groups

$$\dots \longrightarrow H^{n-1}(A) \longrightarrow H^n(X/A) \longrightarrow H^n(X) \longrightarrow H^n(A) \longrightarrow H^{n+1}(X/A) \longrightarrow \dots$$

There is also the fibration sequence

$$F \longrightarrow E \longrightarrow B$$

in which  $p : E \rightarrow B$  is a *fibration* and the fiber is  $F = p^{-1}(b_0)$ . Fibrations are nice surjective maps in which nearby fibers have the same homotopy type; they are rigorously defined by a lifting property. Notable examples of fibrations include covering spaces, fiber bundles, and the *path-loop fibration*

$$\Omega X \rightarrow \text{Map}_*([0, 1], X) \rightarrow X$$

In particular, there is always a fibration sequence

$$K(G, n-1) \rightarrow E \rightarrow K(G, n)$$

in which  $E$  is contractible.

A fibration sequence does not give a long exact sequence on cohomology groups (except in a stable range). Instead it gives a *cohomology spectral sequence*

$$E_2^{p,q} = H^p(B; H^q(F)) \cong H^p(B) \otimes H^q(F) \Rightarrow H^{p+q}(E)$$

For simplicity and ease of calculation, we will continue to take  $\mathbb{Z}/2$  coefficients, and we will also assume that  $\pi_1(B)$  acts trivially on  $H^q(F)$ . (In particular, in all of our examples  $\pi_1(B) = 0$ .)

To draw this spectral sequence, we put a grid of abelian groups on the  $xy$  plane. For each pair of integer coordinates  $(p, q)$ , we draw at the point  $(p, q)$  the abelian group

$$E_2^{p,q} := H^p(B) \otimes H^q(F)$$

This is a vector space over  $\mathbb{Z}/2$ , whose dimension is the product of the dimensions of  $H^p(B)$  and  $H^q(F)$ . In practice,  $H^0(B)$  and  $H^0(F)$  will be  $\mathbb{Z}/2$ . Therefore, on the  $x$ -axis we write the groups  $H^p(B)$ , and on the  $y$ -axis we write the groups  $H^q(F)$ .

Next we draw some homomorphisms (called *differentials*) between these abelian groups. Typically we don't actually have an explicit formula for the differentials; we just know that they exist. The differentials on the  $E_2$  are all called  $d_2$  for simplicity.  $d_2$  takes each group on the  $E_2$  page to the group that is one slot down and two slots to the right:

$$d_2 : E_2^{p,q} \rightarrow E_2^{p+2, q-1}$$

If we compose two differentials then we get zero. So we really have a collection of chain complexes

$$\begin{array}{ccccccc} \dots & \longrightarrow & E_2^{p-2, q} & \longrightarrow & E_2^{p, q+1} & \longrightarrow & E_2^{p+2, q+2} & \longrightarrow & \dots \\ \dots & \longrightarrow & E_2^{p-2, q-1} & \longrightarrow & E_2^{p, q} & \longrightarrow & E_2^{p+2, q+1} & \longrightarrow & \dots \\ \dots & \longrightarrow & E_2^{p-2, q-2} & \longrightarrow & E_2^{p, q-1} & \longrightarrow & E_2^{p+2, q} & \longrightarrow & \dots \end{array}$$

At each point in the plane, we take the kernel of the map going out, modulo the image of the map coming in. This gives a new group  $E_3^{p,q}$  for each pair  $(p, q)$ . These groups form the  $E_3$  page.

$$E_3^{p,q} := \ker d_2^{p,q} / \text{im} d_2^{p-2,q-1}$$

The  $E_3$  page has more differentials  $d_3$ , which go 3 to the right and 2 down. Taking homology of  $d_3$  gives the  $E_4$  page. This process has a well-defined limit, called the  $E_\infty$  page. Then the cohomology of the total space  $E$  is simply the direct sum

$$H^n(E) \cong \bigoplus_{p+q=n} E_\infty^{p,q}$$

We're interested in using this machine to calculate  $H^*(K(\mathbb{Z}/2, 2))$ . We have the fibration sequence

$$K(\mathbb{Z}/2, 1) \longrightarrow E \longrightarrow K(\mathbb{Z}/2, 2)$$

where  $E$  is contractible, and  $K(\mathbb{Z}/2, 1) \simeq \mathbb{R}P^\infty$  has known cohomology. Therefore the Serre spectral sequence has  $E_2$  page

$$E_2^{p,q} = H^p(K(\mathbb{Z}/2, 2)) \otimes H^q(\mathbb{R}P^\infty)$$

and converges to the cohomology of a point:

$$\bigoplus_{p+q=n} E_\infty^{p,q} \cong H^*(E) \cong \begin{cases} \mathbb{Z}/2 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Therefore  $E_\infty^{p,q} = 0$  so long as  $(p, q) \neq (0, 0)$ . So we know that everything on the  $E_2$  page must eventually die. Elements die by landing inside the image of a differential, or outside the kernel of a differential. Using the fact that everything dies eventually, we can start with the known cohomology

$$H^*(\mathbb{R}P^\infty) \cong (\mathbb{Z}/2)[\alpha], \quad |\alpha| = 1$$

and reason out the unknown cohomology  $H^*(K(\mathbb{Z}/2, 2))$ .

Start with  $\alpha \in E_2^{0,1}$ .  $\alpha$  must die, and the only thing that can kill it is  $d_2$ . Therefore

$$d_2\alpha \in E_2^{2,0} \cong H^2(K(\mathbb{Z}/2, 2))$$

is a nonzero element in the cohomology of  $K(\mathbb{Z}/2, 2)$ . Call it  $\beta$ . If there were anything else in  $H^1$  or  $H^2$  of  $K(\mathbb{Z}/2, 2)$ , there would be nothing on the  $E_2$  page to kill it, so there cannot be anything else:

$$\begin{aligned} H^1(K(\mathbb{Z}/2, 2)) &= 0 \\ H^2(K(\mathbb{Z}/2, 2)) &= \langle \beta \rangle \end{aligned}$$



Since  $\beta$  and the fundamental class are both nonzero cohomology classes in  $H^2(K(\mathbb{Z}/2, 2))$ ,  $\beta$  must be the fundamental class.

We can now calculate what happens to all the monomials  $\alpha^i \beta^j$  using the *Leibniz rule*:

$$d_n(xy) = (d_n x)y + x(d_n y)$$

What do the products mean? On the  $E_2$  page, they are simply the cup product on the cohomology  $H^*(F) \otimes H^*(B)$ . The Leibniz rule guarantees that this product induces a well-defined product on all subsequent pages. So now we get

$$\begin{aligned} d_2(\alpha^2) &= (d_2\alpha)\alpha + \alpha(d_2\alpha) &= 0 \\ d_2(\alpha^3) &= (d_2\alpha)\alpha^2 + \alpha(d_2\alpha^2) &= \alpha^2\beta \\ &\vdots \\ d_2(\alpha^n) &= \alpha^{n-1}\beta & n \text{ odd} \\ &0 & n \text{ even} \\ d_2(\alpha\beta) &= (d_2\alpha)\beta + \alpha(d_2\beta) &= \beta^2 \\ &\vdots \\ d_2(\alpha^n\beta^k) &= \alpha^{n-1}\beta^{k+1} & n \text{ odd} \\ &0 & n \text{ even} \end{aligned}$$

Inspecting the family of elements  $\alpha^i \beta^j$ , we see that they are all killed on the  $E_2$  page except for the even powers of  $\alpha$ , namely  $\alpha^2, \alpha^4, \alpha^6$ , etc. So we haven't captured everything yet.

The last chance for  $\alpha^2$  to die is on the  $E_3$  page. So

$$d_3(\alpha^2) \in E_3^{3,0} \cong H^3(K(\mathbb{Z}/2, 2))$$

is a new nonzero cohomology class. I claim that it's  $\text{Sq}^1\beta$ . This is a result of the following *transgression theorem*:

A *transgression* is a differential that goes all the way from the  $y$ -axis to the  $x$ -axis:

$$d_n : E_n^{0,n-1} \longrightarrow E_n^{n,0}$$

Note that  $E_n^{0,n-1}$  is a subgroup of  $E_2^{0,n-1}$ , while  $E_n^{n,0}$  is a quotient of  $E_2^{n,0}$ . If  $x \in E_2^{0,n-1}$  is in this subgroup, and its image under  $d_n$  agrees with the image of  $y \in E_2^{n,0}$ , we say that  $x$  *transgresses* to  $y$ . Looking into the guts of the Serre spectral sequence, the transgression is just the zig-zag

$$H^{n-1}(F) \xrightarrow{\delta} H^n(E, F) \xleftarrow{p^*} H^n(B, *)$$

which is also a homomorphism from a subgroup of  $H^{n-1}(F)$  to a quotient of  $H^n(B, *)$ . But both maps in the zig-zag commute with  $\text{Sq}^i$ . Therefore, if  $x$  transgresses to  $y$ , then  $\text{Sq}^i x$  transgresses to  $\text{Sq}^i y$ .

Since  $\alpha^2 = \text{Sq}^1 \alpha$ , we conclude that  $\alpha^2$  transgresses to  $\text{Sq}^1 \beta$ . Notice that  $\text{Sq}^1$  is the cup product square for  $\alpha$ , but not for  $\beta$ ; this is because  $\alpha$  and  $\beta$  have different degrees. We have now computed

$$H^3(K(\mathbb{Z}/2, 2)) = \langle \text{Sq}^1 \beta \rangle$$

We can continue to apply the Leibniz rule and the transgression theorem to get more of the cohomology. It becomes hard to pin things down precisely, but we start to observe that all the monomials  $\beta^i (\text{Sq}^1 \beta)^j$  seem to show up on the  $x$ -axis and take care of all the powers  $\alpha^n$  where  $n$  is even but not a multiple of 4.

After that,  $\alpha^4 = \text{Sq}^2 \text{Sq}^1 \alpha$  transgresses to  $\text{Sq}^2 \text{Sq}^1 \beta$ , and we get all the monomials in

$$\beta, \text{Sq}^1 \beta, \text{Sq}^2 \text{Sq}^1 \beta$$

That takes care of all the powers of  $\alpha$  that are not multiples of 8. We can start to guess where this is going. We expect all these cohomology classes to be nonzero:

$$\beta, \text{Sq}^1 \beta, \text{Sq}^2 \text{Sq}^1 \beta, \text{Sq}^4 \text{Sq}^2 \text{Sq}^1 \beta, \text{Sq}^8 \text{Sq}^4 \text{Sq}^2 \text{Sq}^1 \beta, \dots$$

and we expect  $H^*(K(\mathbb{Z}/2, 2))$  to be freely generated as a commutative  $\mathbb{Z}/2$ -algebra by these classes.

#### 4. ADMISSIBLE SEQUENCES

To verify our guess we need to talk more about the Steenrod algebra. Let  $I = (i_1, i_2, \dots, i_r, 0, 0, \dots)$  be any sequence of nonnegative integers which is 0 except for finitely many places. Then define

$$\text{Sq}^I = \text{Sq}^{i_1} \text{Sq}^{i_2} \dots \text{Sq}^{i_r}$$

Recall that the Steenrod algebra  $\mathcal{A}$  is generated as a ring by the  $\text{Sq}^i$ . Therefore the  $\text{Sq}^I$  are an additive spanning set for  $\mathcal{A}$ . However, they do not form a basis. We still have the Adem relations: when  $a < 2b$ ,

$$\text{Sq}^a \text{Sq}^b = \sum_c \binom{b-c-1}{a-2c} \text{Sq}^{a+b-c} \text{Sq}^c$$

By an inductive argument, we can use the Adem relations to take any monomial  $\text{Sq}^I$  and express it in terms of monomials  $\text{Sq}^J$  for which

$$\begin{aligned} J &= (j_1, j_2, \dots, j_r, 0, \dots) \\ j_1 &\geq 2j_2 \\ j_2 &\geq 2j_3 \\ &\vdots \end{aligned}$$

Call such a sequence  $J$  *admissible*. So  $\{\text{Sq}^I : I \text{ admissible}\}$  forms a spanning set for  $\mathcal{A}$ . By looking at  $K(\mathbb{Z}/2, 1)^n$ , it is possible to show that it is a basis. This is called the *Serre-Cartan basis* for the Steenrod algebra  $\mathcal{A}$ .

If  $I$  is admissible, define the *excess* of  $I$  to be

$$e(I) = \sum_k (i_k - 2i_{k+1}) = 2i_1 - \sum_k i_k$$

As the name implies, this is a measure of how much  $I$  exceeds the minimum requirements to be admissible. We may sort the admissible sequences by excess:

$e(I)$	sequences with this excess
0	(0)
1	(1), (2, 1), (4, 2, 1), (8, 4, 2, 1), ...
2	(2), (3, 1), (4, 2), (5, 2, 1), (6, 3, 1), (8, 4, 2), ...
$\vdots$	$\vdots$

This table leads us to the following guess:

**Proposition 4.1.**      • *Let  $\beta \in H^2(K(\mathbb{Z}/2, 2))$  be the fundamental class. Then  $H^*(K(\mathbb{Z}/2, 2))$  is the free commutative  $\mathbb{Z}/2$ -algebra on the set*

$$\{\text{Sq}^I \beta : I \text{ admissible, } e(I) < 2\}$$

- *Let  $\gamma_n \in H^n(K(\mathbb{Z}/2, n))$  be the fundamental class. Then  $H^*(K(\mathbb{Z}/2, n))$  is the free commutative  $\mathbb{Z}/2$ -algebra on the set*

$$\{\text{Sq}^I \gamma_n : I \text{ admissible, } e(I) < n\}$$

Note that when  $n = 1$  this gives

$$H^*(K(\mathbb{Z}/2, 1)) \cong (\mathbb{Z}/2)[\gamma_1]$$

as expected.

## 5. BOREL'S THEOREM

Now that we have made our guess in Prop. 4.1 above, we will verify that it is correct. Here is the main step. Suppose that  $E_*^{*,*}$  is a multiplicative spectral sequence of  $\mathbb{Z}/2$ -modules that converges to  $H^*(\text{pt})$ . Let  $A = E_2^{0,*}$  denote the ring on the  $y$ -axis of the  $E_2$  page, and let  $B = E_2^{*,0}$  denote the ring on the  $x$ -axis. Suppose that  $A$  has a *simple system of generators*  $\{x_1, x_2, \dots\}$ . This means that there are finitely many  $x_i$  in each degree, and the set

$$\{x_{i_1}x_{i_2}\dots x_{i_r} : i_1 < i_2 < \dots < i_r\}$$

forms an additive basis. Suppose finally that each  $x_i$  transgresses to some element  $y_i \in B$ .

**Theorem 5.1** (Borel). *Under these assumptions, the obvious map*

$$(\mathbb{Z}/2)[y_1, y_2, \dots] \longrightarrow B$$

*is an isomorphism of rings.*

Note that the ring  $(\mathbb{Z}/2)[\alpha]$  has a simple system given by  $\{\alpha, \alpha^2, \alpha^4, \dots\}$ . More generally,  $(\mathbb{Z}/2)[\alpha_i : i \in S]$  has a simple system  $\{\alpha_i^{2^k}\}_{i \in S, k > 0}$ . In our earlier calculation of  $H^*(K(\mathbb{Z}/2, 2))$ , we have already observed that the simple system  $\{\alpha^{2^k}\}$  transgresses to  $\text{Sq}^{2^{k-1}} \dots \text{Sq}^2 \text{Sq}^1 \beta$ , so Borel's theorem is enough to prove that our guess is correct.

*Proof.* (of Borel's Theorem)

Let  $\Lambda = (\mathbb{Z}/2)[x_1, x_2, \dots]/(x_1^2, x_2^2, \dots)$  denote the exterior  $\mathbb{Z}/2$ -algebra on the elements  $x_i$ . Then there is an obvious map  $\Lambda \rightarrow A$  that preserves addition but not multiplication. This map is an isomorphism of graded  $\mathbb{Z}/2$ -modules.

Let  $P = (\mathbb{Z}/2)[y_1, y_2, \dots]$  denote the polynomial algebra on the  $y_i$ . As we have already seen, there is an obvious map of graded rings  $P \rightarrow B$ , and our goal is to prove that this map is an isomorphism.

Now  $\Lambda \otimes P$  is a bigraded ring. Let

$$\delta : \Lambda \otimes P \longrightarrow \Lambda \otimes P$$

be the unique derivation with

$$\delta(x_i) = y_i, \quad \delta(y_i) = 0$$

In other words, we use the Leibniz rule to determine what  $\delta$  does on everything else. We may think of the bigraded ring  $\Lambda \otimes P$  as simply a graded ring,

by thinking of degree  $(i, j)$  elements as degree  $(i + j)$ . With this convention,  $(\Lambda \otimes P, \delta)$  is a chain complex. It is actually a tensor product over each  $i$  of

$$\Lambda[x_i] \otimes (\mathbb{Z}/2)[y_i]$$

which has the cohomology of a point. By the Kunneth formula, the cohomology of  $(\Lambda \otimes P, \delta)$  is also that of a point.

Now filter our chain complex  $(\Lambda \otimes P, \delta)$  by

$$(\Lambda \otimes P)_p := \bigoplus_{i \geq p} \Lambda \otimes P_i$$

A filtered complex gives rise to a spectral sequence  $\hat{E}$ . Since we know all the generators, relations, and levels of the filtration, we can give this spectral sequence and all of its differentials explicitly. The differentials on the  $\hat{E}_1$  page are trivial. The  $\hat{E}_2$  page is the bigraded ring  $\Lambda \otimes P$ , as we would expect. The differentials agree with  $\delta$  when this is possible, and when it is not they are zero. The  $\hat{E}_\infty$  page has the cohomology of a point. Using this information, it is straightforward to make an additive map of spectral sequences  $\hat{E} \rightarrow E$ . In other words, every group  $\hat{E}_r^{p,q}$  maps to the corresponding group  $E_r^{p,q}$ , and the map commutes with all the differentials, though it does *not* preserve the multiplication. Then we use the following lemma:

**Lemma 5.2.** (*[1], p.91*) *Suppose  $f : \hat{E} \rightarrow E$  is a map of spectral sequences, both of which have the property that the groups on the  $E_2$  page are the tensor products of the groups on the  $x$ -axis and  $y$ -axis, and the  $E_\infty$  pages are trivial. If  $f$  is an isomorphism on every group on the  $y$ -axis of the  $E_2$ -page, then it is an isomorphism on the  $x$ -axis as well.*

In our case, this means our map of spectral sequences is an isomorphism on the  $x$ -axis. By construction, this is simply the map

$$P = (\mathbb{Z}/2)[y_1, y_2, \dots] \rightarrow B$$

so we have proven that it is an isomorphism.  $\square$

We have finished checking Prop. 4.1 in the special case when  $n = 2$ . We may do the general case now without too much more effort. Suppose that  $\gamma_n$  is a degree  $n$  cohomology class and  $I$  is an admissible sequence. If  $e(I) > n$ , then

$$\text{Sq}^I \gamma_n = \text{Sq}^{i_1} (\text{Sq}^{i_2} \dots \text{Sq}^{i_r} \gamma_n) = 0$$

because the term in parentheses has degree

$$n + \left( \sum_k i_k \right) - i_1 = n - e(I) + i_1 < i_1$$

Similarly, when  $e(I) = n$ , the term in parentheses has degree  $i_1$ , so the outside  $Sq^{i_1}$  simply squares:

$$Sq^I \gamma_n = (Sq^{i_2} \dots Sq^{i_r} \gamma_n)^2$$

If  $(i_2, \dots, i_r)$  has excess  $n$  then we may reduce it further:

$$Sq^I \gamma_n = (Sq^{i_3} \dots Sq^{i_r} \gamma_n)^4$$

and so on until the remainder of  $I$  has excess less than  $n$ . So if  $H^*(K(\mathbb{Z}/2, n))$  is the free commutative  $\mathbb{Z}/2$ -algebra on

$$\{Sq^I \gamma_n : I \text{ admissible, } e(I) < n\}$$

then it has a simple system of generators

$$\{Sq^I \gamma_n : I \text{ admissible, } e(I) \leq n\}$$

In the spectral sequence for

$$K(\mathbb{Z}/2, n) \longrightarrow * \longrightarrow K(\mathbb{Z}/2, n+1)$$

this simple system transgresses to

$$\{Sq^I \gamma_{n+1} : I \text{ admissible, } e(I) < n+1\}$$

and so  $H^*(K(\mathbb{Z}/2, n+1))$  is the free commutative  $\mathbb{Z}/2$ -algebra on this set. Therefore Prop. 4.1 is true by induction.

## REFERENCES

- [1] Mosher and Tangora, *Cohomology Operations and Applications in Homotopy Theory*.