# Scissors congruence, K-theory, Thom spectra, and homological stability

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June 14, 2024

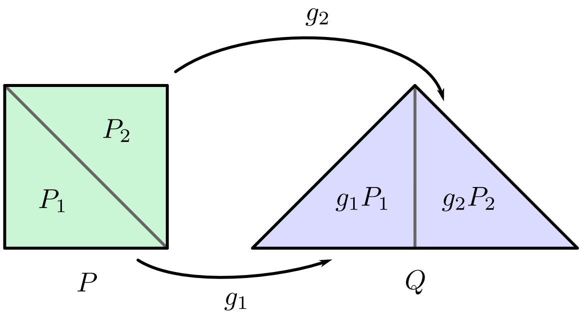
Algebraic Structures in Topology II

San Juan, Puerto Rico

joint work with Bohmann, Gerhardt, Merling, and Zakharevich,

and also with Kupers, Lemann, Miller, and Sroka

Scissors congruence: polytopes up to “cut and paste” operations



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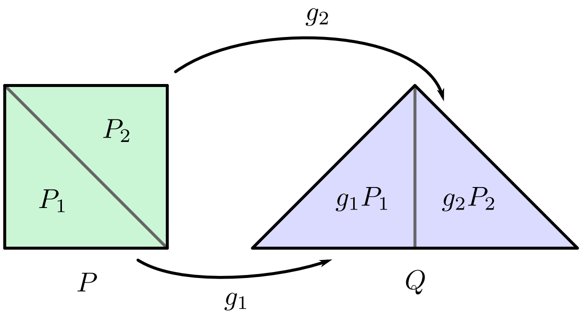


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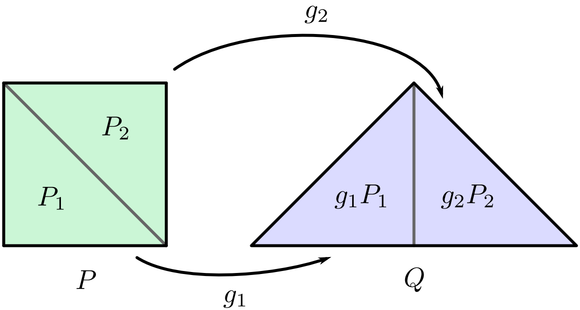


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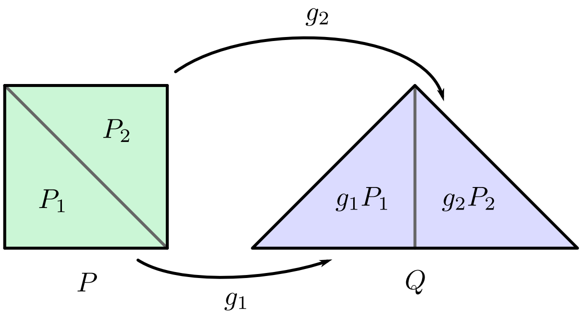


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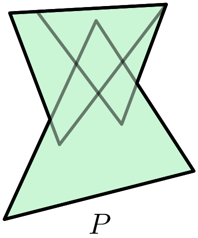

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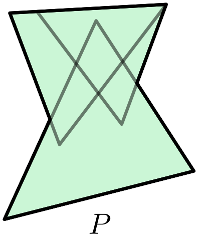


(simplices are $n$-dimensional and geodesic)

(convex hulls of $n + 1$ points in general position)

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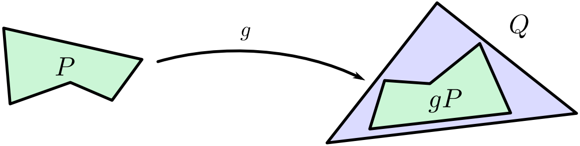

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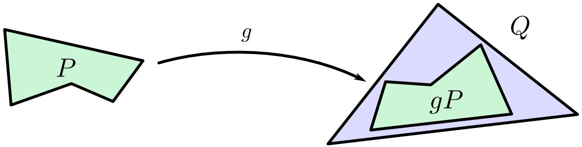
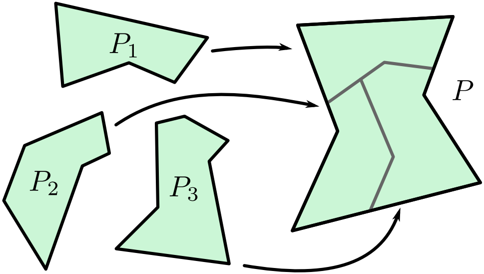



Note $P \cong Q$ iff they are congruent.

A **cover** of $P$ is $\{P_i \xrightarrow{g_i} P\}_{i \in I}$, $I$ finite,

$P = \bigcup_{i \in I} g_i P_i$, interiors disjoint.



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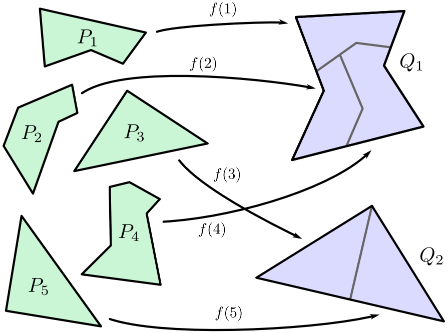

**Def.** (Zakharevich) The **category of covers** $\mathcal{W}(\mathcal{X})$:

Objects: finite tuples $\{P_i\}_{i\in I}$

Morphisms: covers as from the previous slide.

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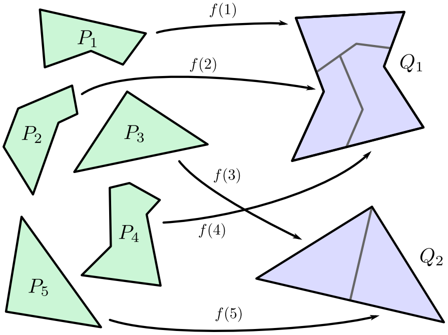

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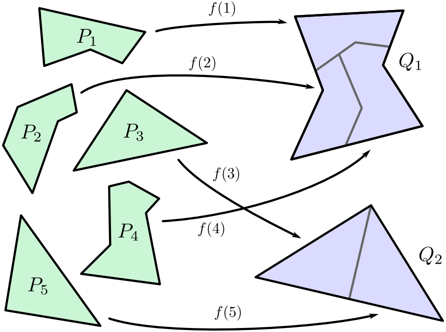


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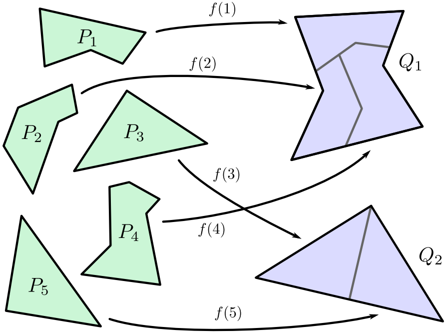


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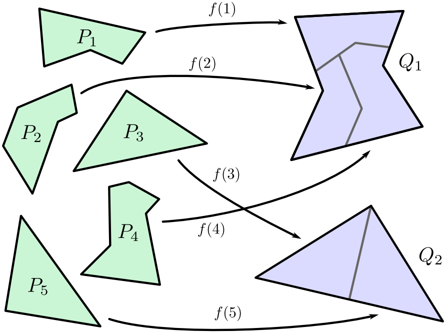


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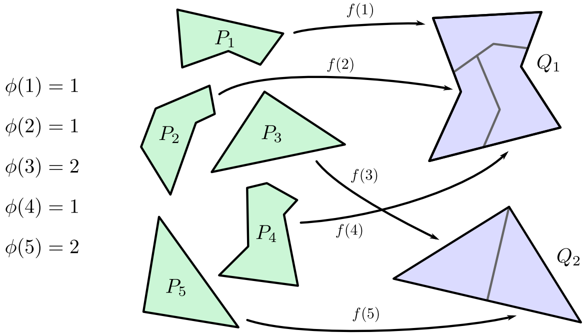


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Singletons $\{P\}$ and $\{Q\}$ are in same component of $\mathcal{W}(\mathcal{X})$ iff $P$ and $Q$ are scissors congruent.

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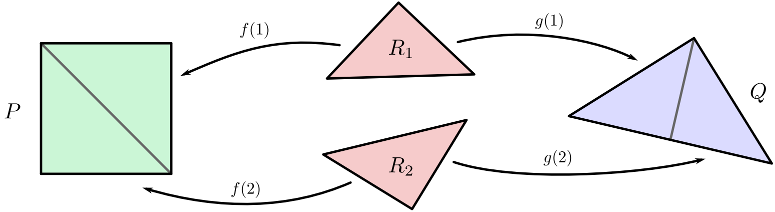

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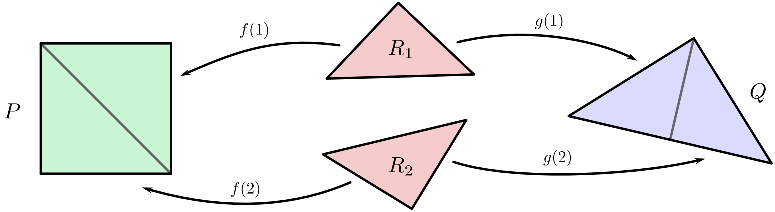


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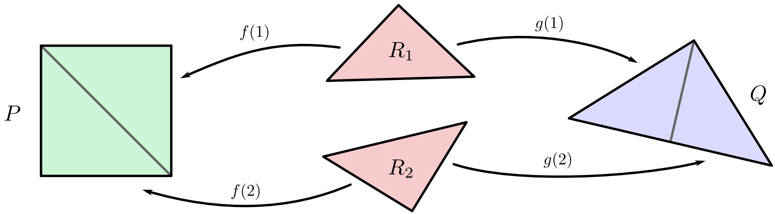


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These are the morphisms of a category $\mathcal{G}(\mathcal{X}) = \mathcal{W}(\mathcal{X})[\mathcal{W}(\mathcal{X})^{-1}]$.

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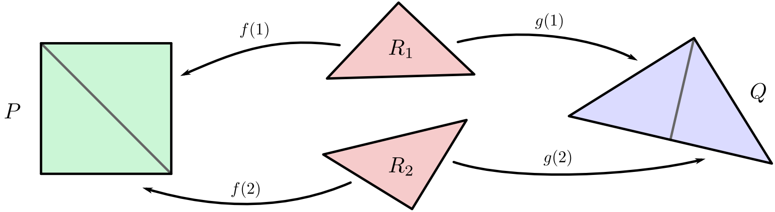


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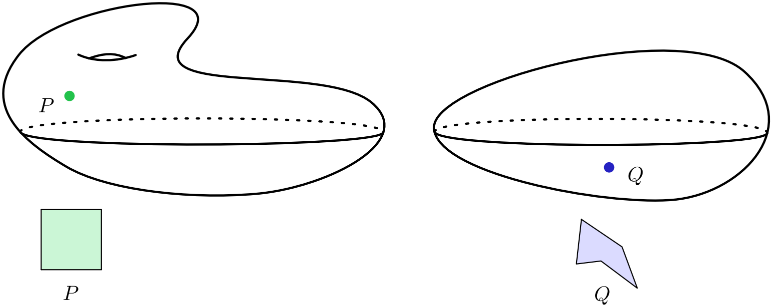

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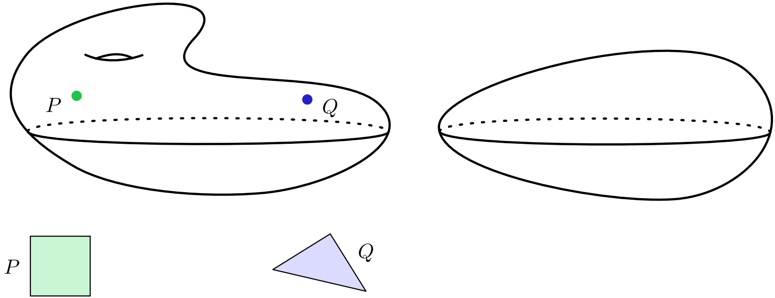

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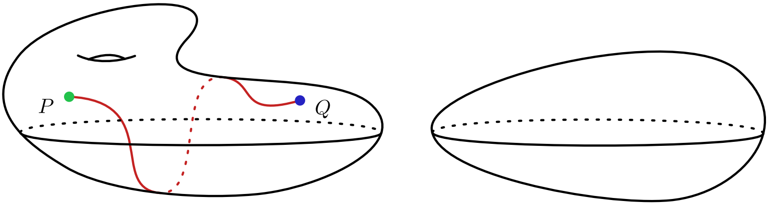


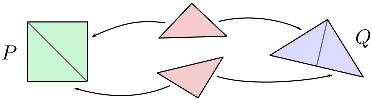

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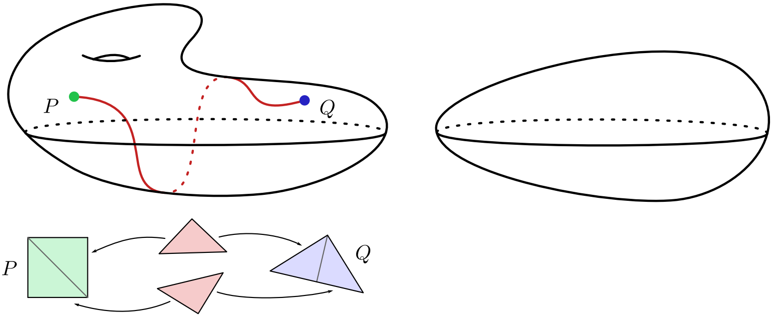


Scissors congruence *is* understanding the homotopy type of this space.

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"Stable" or "additive" scissors congruence is understanding the homotopy type of this space.

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**Theorem.** (Zylev, 1965) Group completion is injective:
$\pi_0 B\mathcal{G}(\mathcal{X}) \hookrightarrow \pi_0(B\mathcal{G}(\mathcal{X}))[\pi_0^{-1}] = K_0(\mathcal{X})$.

$K_0(\mathcal{X})$ = free abelian group on polytopes, up to s.c.

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**Example.** Two-dimensional hyperbolic or spherical geometry, $X = H^2$ or $S^2$.

Same! $K_0 = \mathbb{R}$, given by area.

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(Sydler 1965) $K_0(\mathcal{E}^3) \to \mathbb{R} \oplus (\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\pi\mathbb{Z}))$ is injective!

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Similar, but with $SU(2)$.

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Get the volume and two Dehn invariants, one for 3-dimensional faces, one for 1-dimensional faces.

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**Open question:** Are these jointly injective?

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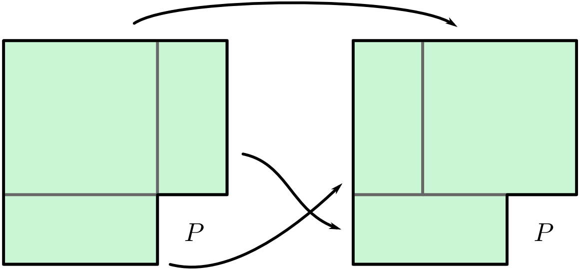

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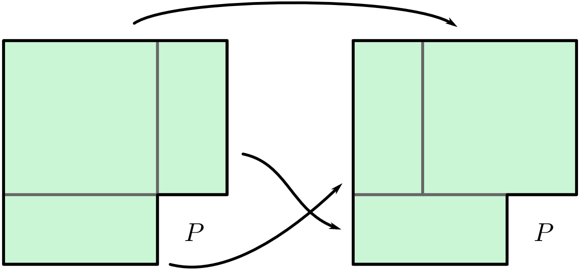


$$B\mathcal{G}(\mathcal{X}) = \coprod_{\text{s.c. classes}} B\mathrm{Aut}(P),$$

so it's about understanding this symmetry group.

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Restrict from all isometries to translations.

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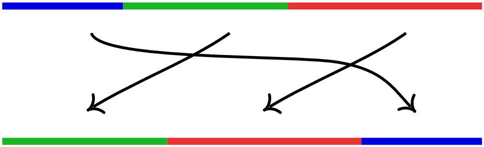

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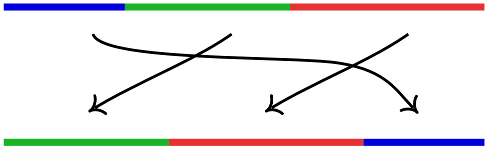

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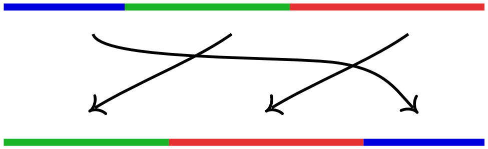

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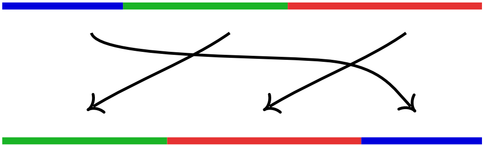

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Is there some relationship between $\mathrm{Aut}(P)$ for different $P \subseteq X$?

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For any nonempty polytopes $P, Q \subseteq X$,

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(Canonical! Always get the same isomorphism no matter how $P \to R \leftarrow Q$.)

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These dirty tricks give us an $\cong$ on homology in every degree. $\square$

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Higher additive invariants of scissors congruence!

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We get a surprising simplification though.

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Note $PT(X) \simeq ST(X)$ if $X$ is Euclidean or hyperbolic.

Can de-suspend by the tangent bundle of $X$:

$$\Sigma^{-TX} PT(X) = \frac{\Sigma_{+}^{-TX} \left( \operatorname*{hocolim}_{\emptyset \subsetneq U \subseteq X} U \right)}{\Sigma_{+}^{-TX} \left( \operatorname*{hocolim}_{\emptyset \subsetneq U \subsetneq X} U \right)}.$$

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$Pt(X) = H_n(PT(X)) =$ a free abelian group, the **polytope group**. For $E^n$ or $H^n$, also called $St(X)$, the **Steinberg module**.

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(Note $G$ is discrete here!)

**Corollary.** (Everyone)

$$H_*(\mathrm{Aut}(P)) \cong H_*(\Omega_0^\infty K(\mathcal{X})),$$

$$H_*(K(\mathcal{X})) \cong H_*(G; Pt(X)).$$

Can go from homology of the (big) group $G$ to homology of the (gigantic!) group $\mathrm{Aut}(P)$.

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Turns out to be rational, get

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We calculate

$$K_n(\mathcal{E}^2) \cong H_n(K(\mathcal{E}^2); \mathbb{Q}) \cong \bigoplus_{p+2q=n} H_p(O(2); \Lambda^{2q+2}(\mathbb{R}^2)^t),$$

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In particular, $\mathrm{Aut}(P)^{ab} = 0$.

**Example.** "Rectangle exchange transformations" in *n*-dimensional Euclidean space. Only allow boxes aligned with coordinate axes, and you can only translate them. (Cornulier, Lacourte 2022)

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In particular, $K_0 = \mathbb{R}^{\otimes n}$, $K_1 = H_1 = (\Lambda^2\mathbb{R} \otimes \mathbb{R}^{\otimes(n-1)})^{\oplus n}$.

**Example.** Thompson's group $V$: cut $[0,1]$ at points of the form $\frac{a}{2^k}$, allow translations and scalings by powers of 2.

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Can also do variants where the homology is not known yet, e.g. the "irrational slope Thompson's group" (Burillo, Nucinkis, Reeves 2022).

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Thank you!