THE TRANSFER ON THE n-FOLD COVER OF THE CIRCLE

CARY MALKIEWICH

The purpose of this note is to give details about how to calculate the Becker-Gottlieb transfer [BG75] for the n-fold covering $S^1 \xrightarrow{\cdot n} S^1$. It was written in November of 2014, but I added references and put it on my webpage in December of 2018.

First we have to talk about choices of coordinates. In the stable homotopy category, any diagram manifesting A as a retract of X gives canonical isomorphisms in the homotopy category

$$A \vee F \xrightarrow{\sim} X \xrightarrow{\sim} A \times C$$

where F is the fiber of $X \longrightarrow A$ and C is the cofiber of $A \longrightarrow X$. We stress that we get not just an abstract isomorphism $A \oplus C \cong X$ in the stable homotopy category, but a *particular* isomorphism induced by the retract diagram for A and X. Any other isomorphism $A \oplus C \cong X$ may be easily compared against our canonical one, by composing them to get a self-map of $A \oplus C$. The two isomorphisms agree iff this self-map is the identity, and this is easily checked in practice by calculating separately the maps

$$A \longrightarrow A$$
$$C \longrightarrow C$$
$$A \longrightarrow C$$
$$C \longrightarrow A$$

and checking that the first two are the identity and the second two are zero.

Now consider the suspension spectrum of the circle, $\Sigma^\infty_+ S^1$. Choosing a basepoint for the circle gives a retract diagram

$$\Sigma^\infty S^0 \longrightarrow \Sigma^\infty_+ S^1 \longrightarrow \Sigma^\infty S^0$$

In the stable category, this leads to a canonical splitting of $\Sigma^\infty_+ S^1$ into a wedge of $\Sigma^\infty S^0$ and another summand. The second summand may be canonically identified with the cofiber of the first map above, or the fiber of the second map. The cofiber is simple enough, it is just the suspension spectrum of the based circle $\Sigma^\infty S^1$.

We want to describe a different way of getting the same isomorphism. We will map $\Sigma^\infty S^0 \longrightarrow \Sigma^\infty_+ S^1$ as before, by inclusion of the basepoint of S^1. For the second summand, we truncate $\Sigma^\infty S^1$ at spectrum level 1, and then map in by the circle transfer:

$$S^2 \longrightarrow \Sigma_+ S^1$$

We check that this transfer, when pushed forward to $\Sigma^\infty S^1 \times \Sigma^\infty S^0$, gives a map

$$\Sigma^\infty S^1 \longrightarrow \Sigma^\infty S^1 \times \Sigma^\infty S^0$$

which is up to homotopy the identity on the first factor and zero on the second factor. Therefore it defines a splitting $\Sigma^\infty S^0 \vee \Sigma^\infty S^1 \xrightarrow{\sim} \Sigma^\infty_+ S^1$ which is, in the homotopy category, the inverse of the above splitting.

This is the splitting we'll use for our calculation. Actually, we are lucky and it doesn't even matter, because any other splitting will give the same answer. To see this, abbreviate and let

$$\mathbb{S}^0 = \Sigma^\infty S^0, \qquad \mathbb{S}^1 = \Sigma^\infty S^1.$$

Then the automorphism group of the spectrum we are studying is

$$Aut(\mathbb{S}^0 \vee \mathbb{S}^1) \cong (\mathbb{Z}/2)^3$$

where one $\mathbb{Z}/2$ flips the sign of the $\mathbb{S}^0$, one flips the sign of the $\mathbb{S}^1$, and the last one acts by the matrix

	$\mathbb{S}^0$	$\mathbb{S}^1$
$\mathbb{S}^0$	$1 \in \pi_0^S$	$\eta \in \pi_1^S$
$\mathbb{S}^1$	$0 \in \pi_{-1}^S$	$1 \in \pi_0^S$

where $\eta \in \pi_1^S$ is the Hopf map and $2\eta = 0$. So there are really 8 choices of splitting in the stable category

$$\Sigma^\infty_+ S^1 \cong \mathbb{S}^0 \vee \mathbb{S}^1.$$

However, if we restrict to self-maps of $\Sigma^\infty_+ S^1$, the choice of splitting doesn't matter. Conjugating a self-map by any of the above three generators does not change a matrix with entries in the above groups. This is admittedly a bit of a fluke – it depends crucially on the fact that $2\eta = 0$. So even for $\Sigma^\infty_+ S^3$, the choice of splitting would matter.

Anyway, with that discussion of coordinates out of the way, we can now compute the Becker-Gottlieb transfer [BG75] for the covering

$$S^1 \xrightarrow{\cdot n} S^1, \quad n > 0$$

This will be a stable map from $\Sigma^\infty_+ S^1$ to itself. Rewriting this spectrum as $\mathbb{S}^1 \vee \mathbb{S}^0$, we see that it suffices to fill out the matrix

	$\mathbb{S}^0$	$\mathbb{S}^1$
$\mathbb{S}^0$	$? \in \pi_0^S$	$? \in \pi_1^S$
$\mathbb{S}^1$	$? \in \pi_{-1}^S$	$? \in \pi_0^S$

The answer is

$$\begin{pmatrix} n & (n-1)\eta \\ 0 & 1 \end{pmatrix}$$

where $\eta \in \pi_1^S$ is the Hopf map. Remember that $2\eta = 0$, so the upper-right entry is 0 or η depending on the parity of n.

Let's explain how we got these entries. The lower-left is easy, since $\pi_{-1}^S = 0$. To get the rest, we embed $S^1 \times D^2$ into $\mathbb{R}^3$ in the usual way, and embed another copy of S^1 inside of that by winding around n times. Quotienting the complement of $S^1 \times e^2$ to a point gives the domain $S^2 \wedge S^1_+$;

quotienting the complement of a tubular neighborhood of the windier S^1 gives the target $S^2 \wedge S^1_+$. To get the top-degree map

$$S^3 \longrightarrow S^2 \wedge S^1_+ \overset{\text{collapse}}{\longrightarrow} S^2 \wedge S^1_+ \overset{\text{collapse}}{\longrightarrow} S^3$$

we count degree. But everything is a collapse map, so the degree is 1; this gives the lower-right entry in the matrix. To get the bottom-degree map

$$S^2 \hookrightarrow S^2 \wedge S^1_+ \overset{\text{collapse}}{\longrightarrow} S^2 \wedge S^1_+ \overset{\text{project}}{\longrightarrow} S^2$$

we inspect geometrically and conclude that the composite is degree n; this gives the upper-left entry.

The last entry is tricky. We will use the Pontryagin-Thom correspondence between the stable 1-stem and framed 1-manifolds up to framed bordism. We inspect the map

$$S^3 \longrightarrow S^2 \wedge S^1_+ \overset{\text{collapse}}{\longrightarrow} S^2 \wedge S^1_+ \overset{\text{project}}{\longrightarrow} S^2$$

The first inclusion picks out only the S^3 and not the S^2, because if we follow it by projection to S^2 right away, we get the stable map represented by the circle with the trivial framing, which is 0. If we instead apply our transfer, the resulting map is represented by a circle winding around n times, with the "trivial" framing. We unwind this circle in $(n-1)$ steps to get the trivial circle. Each step involves removing a crossing in the knot diagram by flipping a loop; this adds 1 to the framing number. Therefore our map is represented by a trivial circle with framing $(n-1)$; this gives the upper-right entry of the matrix.

I'm not claiming this is an original result; the most it might be is an original proof of a known result. The calculation appears in a few places in the literature, but sometimes incorrectly. The correct answer appears in [Hes96], proof of Lemma 1.5.1. It's also used in Lemma 3.15 of [CDD11].

References

[BG75] J. C. Becker and D. H. Gottlieb, *The transfer map and fiber bundles*, Topology **14** (1975), no. 1, 1–12.

[CDD11] Gunnar Carlsson, Christopher L Douglas, and Bjørn Ian Dundas, *Higher topological cyclic homology and the Segal conjecture for tori*, Advances in Mathematics **226** (2011), no. 2, 1823–1874.

[Hes96] Lars Hesselholt, *On the p-typical curves in quillen's k-theory*, Acta Mathematica **177** (1996), no. 1, 1–53.

Department of Mathematics
Binghamton University
PO Box 6000
Binghamton, New York 13902-6000
malkiewich@math.binghamton.edu