

# Finiteness, phantom maps, completion, and the Segal conjecture

Cary Malkiewich

August 2013

The purpose of these notes is to make explicit some facts known by the cognoscenti of homotopy theory, which lead up to and illuminate the consequences of the Segal conjecture. To avoid confusion, we will proceed in logical order, starting with the simple foundational results on finiteness.

## 1 Finiteness

### 1.1 Finiteness of $[X, Y]$

We begin with a basic

**Question.** If  $X$  and  $Y$  are based CW-complexes, when is  $[X, Y]$  finite?

Of course, by  $[X, Y]$  we mean the set of based homotopy classes of based maps. In most of our examples  $Y$  will be a double loop space, so that  $[X, Y]$  is an abelian group. In this situation we may ask the related

**Question.** When is  $[X, Y]$  finitely generated?

One might naively hope that  $[X, Y]$  is finite if  $X$  and  $Y$  are finite complexes. But a counterexample is

$$[S^1, S^1] \cong \mathbb{Z}$$

OK, well  $\mathbb{Z}$  is at least finitely generated, so maybe  $[X, Y]$  is a finitely generated abelian group if  $X$  and  $Y$  are finite. But

$$[S^2, S^1 \vee S^2] \cong \bigoplus_{\infty} \mathbb{Z}$$

is not finitely generated. In this last example, one can still at least say that  $\pi_2(S^1 \vee S^2)$  is finitely generated as a module over  $\mathbb{Z}[\pi_1(S^1 \vee S^2)]$ . But the same cannot be said for  $\pi_3(S^1 \vee S^2)$ .

The most natural examples where  $[X, Y]$  is finite or finitely generated seem to arise when  $X$  is a finite complex, but  $Y$  is constructed in such a way that it receives only finitely many maps from finite complexes. It is not difficult to formalize this:

**Proposition 1.1.** *If  $X$  is a finite complex with cells in dimensions  $i_1, \dots, i_k$ , and  $Y$  is a (possibly infinite) CW complex with  $\pi_{i_1}(Y), \dots, \pi_{i_k}(Y)$  all finite, then  $[X, Y]$  is finite.*

**Proposition 1.2.** *If  $Y$  is instead a double loop space and  $\pi_{i_1}(Y), \dots, \pi_{i_k}(Y)$  are all finitely generated, then  $[X, Y]$  is finitely generated.*

Both facts are proven by simple induction on the skeleton of  $X$ .

So finiteness or finite generation boils down to the same condition on the homotopy groups of  $Y$ . Unfortunately, the above counterexamples show that it is not enough to assume that  $Y$  is finite, or of finite type (finitely many cells in each dimension). In some sense the fundamental group is the problem:

**Proposition 1.3.** *If  $Y$  is a simply-connected CW complex of finite type then  $\pi_n(Y)$  is a finitely generated abelian group for all  $n \geq 2$ .*

This follows from Serre's Hurewicz theorem in homotopy mod  $\mathcal{C}$ , where  $\mathcal{C}$  is finitely generated abelian groups.

If  $\pi_1(Y)$  is not 0, then we ought to think of  $\pi_n(Y)$  as a module over the group ring  $\mathbb{Z}[\pi_1(Y)]$ , and not just as an abelian group. Still, it is not true in general that  $\pi_n$  is finitely generated as a  $\mathbb{Z}[\pi_1]$ -module. One must pick a substitute, and the following classical results provide two choices for such a substitute:

**Proposition 1.4** (Wall, [Wal65]). *The CW complex  $Y$  is equivalent to one of finite type iff  $\pi_1(Y)$  is finitely presented, and for each finite skeleton  $Y^{(n-1)}$  with  $n \geq 2$  the relative homotopy groups  $\pi_n(Y, Y^{(n-1)})$  are finitely generated  $\mathbb{Z}[\pi_1(Y)]$ -modules.*

**Proposition 1.5** (Wall, [Wal65]). *If  $\mathbb{Z}[\pi_1(Y)]$  is Noetherian, then  $\pi_1(Y)$  is finitely presented. In this case,  $Y$  is equivalent to a CW complex of finite type iff  $H_n(\tilde{Y})$  is a finitely generated  $\mathbb{Z}[\pi_1(Y)]$ -module for  $n \geq 2$ . Here  $\tilde{Y}$  denotes the universal cover.*

## 1.2 Finiteness and Spanier-Whitehead Duality

Recall that we say that  $X$  is a *finite spectrum* if it is equivalent to a CW spectrum with finitely many stable cells.

**Proposition 1.6.** *A spectrum  $X$  is finite iff both of these conditions hold:*

- $X$  is bounded below, i.e. there is some integer  $N \in \mathbb{Z}$  such that  $\pi_n X = 0$  for all  $n \leq N$ .

- $X$  has finitely generated homology groups, which are nonzero in only finitely many degrees.

*Proof.* This follows from an easy generalization of the usual Hurewicz theorem to the stable setting.  $\square$

From this proposition it follows that a retract of a finite spectrum is finite. The homotopy groups of a finite spectrum are also finitely generated in each degree, but they may be nonzero in infinitely many degrees.

Every spectrum has a *functional dual*

$$DX := F(X, \mathbb{S})$$

which is characterized by a natural isomorphism

$$[Y, F(X, \mathbb{S})] \cong [Y \wedge X, \mathbb{S}]$$

If  $X$  is finite then we will call this the *Spanier-Whitehead dual* of  $X$ . In this case,  $DX$  is also finite and there is a natural equivalence

$$X \simeq D(DX)$$

Recall that if  $E$  is any spectrum, the  $E$ -homology of  $X$  is defined as

$$E_n(X) = [\mathbb{S}^n, X \wedge E]$$

We will define the  $E$ -cohomology of  $X$  as

$$E^n(X) = [\Sigma^n X, E]$$

WARNING: We have given our cohomology groups the *homological grading*. The more common convention of *cohomological grading* puts the  $\Sigma^n$  on the  $E$ . Of course, to go from one to the other, we negate  $n$ .

We are going to characterize how the homology and cohomology of  $DX$  is related to that of  $X$ . The last fact we need is the following: there is a natural map

$$F(X, Z) \wedge Y \longrightarrow F(X, Z \wedge Y)$$

and this map is an equivalence of spectra if  $X$  is finite or if  $Y$  is finite. The proof is simple: both sides are *excisive* in  $X$  and  $Y$ , meaning they preserve homotopy pushout/pullback squares of spectra. Since the map is obviously an equivalence if  $X = \mathbb{S}$ , a simple induction shows it's an equivalence for all finite  $X$ . Similarly, the map is obviously an equivalence when  $Y = \mathbb{S}$  and  $X$  is anything, so it's also an equivalence when  $Y$  is finite.

From the above facts, it quickly follows that if  $X$  is a finite spectrum, there are natural isomorphisms

$$E_n(DX) \cong E^n(X), \quad E^n(DX) \cong E_n(X) \quad \text{if } X \text{ is finite}$$

In particular, the first one follows from

$$[\mathbb{S}^n, DX \wedge E] \cong [\mathbb{S}^n, F(X, E)] \cong [\mathbb{S}^n \wedge X, E]$$

and the second follows from dualizing  $X$  twice. If  $X$  is not finite, then we cannot expect an analogue of the above isomorphisms, because the main step  $DX \wedge E \simeq F(X, E)$  fails for infinite  $X$ . If on the other hand  $E$  is finite, then the main step works again, so we still get one of the above two isomorphisms:

$$E_n(DX) \cong E^n(X) \quad \text{if } E \text{ is finite}$$

We remark that ordinary homology/cohomology with  $\mathbb{Z}/p$  coefficients is represented by the Eilenberg-MacLane spectrum  $H\mathbb{Z}/p$ , which is *not* finite! (One may check that the cohomology of  $H\mathbb{Z}/p$  is not concentrated in finitely many degrees.) The conclusion is that, in general,

$$H_n(DX) \not\cong H^n(X)$$

Even with finite coefficients we get in general

$$H_n(DX; \mathbb{Z}/p) \not\cong H^n(X; \mathbb{Z}/p)$$

On the other hand, stable homotopy/cohomotopy is represented by  $\mathbb{S}$ , and stable homotopy/cohomotopy with finite coefficients is represented by the Moore spectrum  $M(\mathbb{Z}/p)$ , which is the cofiber

$$\mathbb{S} \xrightarrow{p} \mathbb{S} \longrightarrow M(\mathbb{Z}/p)$$

Both of these spectra are finite. Therefore, for all spectra  $X$ , we have natural isomorphisms

$$\begin{aligned} \pi_n(DX) &\cong \pi^n(X) \\ \pi_n(DX; \mathbb{Z}/p) &\cong \pi^n(X; \mathbb{Z}/p) \end{aligned}$$

## 2 Phantom Maps

Now suppose that  $X$  is a (possibly infinite) CW complex. Then  $X$  can be expressed as the colimit of its finite subcomplexes  $\{X_\alpha\}$  under the inclusion maps. This colimit system is both *filtered* and *homotopically correct*, so we may say that  $X$  is the filtered homotopy colimit of its finite subcomplexes.

If we inspect the space  $\text{Map}_*(X, Y)$  of all continuous maps from  $X$  to  $Y$  in the usual compactly generated compact-open topology, we get the isomorphisms

$$\begin{aligned} \text{Map}_*(X, Y) &\cong \text{Map}_*(\text{colim } X_\alpha, Y) \cong \lim \text{Map}_*(X_\alpha, Y) \\ &\simeq \text{Map}_*(\text{hocolim } X_\alpha, Y) \cong \text{holim } \text{Map}_*(X_\alpha, Y) \end{aligned}$$

and all is right in the world.

Suppose instead we try to understand only the *homotopy classes* of maps

$$[X, Y] = \pi_0(\text{Map}_*(X, Y))$$

Clearly we can restrict a map  $X \rightarrow Y$  to a finite subcomplex  $X_\alpha$ , and this respects inclusion of subcomplexes. Therefore we get a map

$$[X, Y] \rightarrow \lim_{\alpha} [X_\alpha, Y]$$

Another simple induction shows that this map is surjective; the essential point is that if I can modify a map by a homotopy and then extend it to a new cell, then I could have extended the original map directly to the new cell. So the above map is surjective. However, it is not always injective. Its kernel is, by definition, the set of *phantom maps* up to homotopy.

**Definition 2.1.** A map  $f : X \rightarrow Y$  is a *phantom map* if its restriction to each finite subcomplex  $X_\alpha \rightarrow Y$  is nullhomotopic. Equivalently,  $f$  is a phantom map if  $K \rightarrow X \rightarrow Y$  is nullhomotopic whenever  $K$  is a finite CW complex and  $K \rightarrow X$  is any map.

The virtue of the second condition is that it does not depend on the cell structure of  $X$ . Note that the existence of phantom maps does not contradict our above statements about mapping spaces. Rather, the phantom maps arise because  $\pi_0$  is a crude construction that does not commute with inverse limits or homotopy inverse limits.

We should give an example to convince the reader that phantom maps do in fact exist. Consider the space  $X = K(\mathbb{Q}, 1)$ . We may construct this space as a mapping telescope of copies of  $S^1$ , where the  $n$ th map  $S^1 \rightarrow S^1$  winds around  $n$  times. Applying  $H_1(-; \mathbb{Z})$  to the levels of this mapping telescope gives the system

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \dots$$

The colimit of this system is  $H_1(X; \mathbb{Z}) \cong \mathbb{Q}$ , and all other homology is trivial. By the universal coefficient theorem, the cohomology is therefore trivial except for  $H^2(X; \mathbb{Z}) \cong \text{Ext}(\mathbb{Q}, \mathbb{Z})$ . Note that when we take  $H^2$  of the levels of our mapping telescope we get zero, and the inverse limit of that is zero. But  $\text{Ext}(\mathbb{Q}, \mathbb{Z}) \neq 0$ , so we have found a nontrivial cohomology group

$$H^2(K(\mathbb{Q}, 1); \mathbb{Z}) \cong [K(\mathbb{Q}, 1), K(\mathbb{Z}, 2)]$$

which consists entirely of phantom maps. So, even in the very familiar case of ordinary cohomology with  $\mathbb{Z}$  coefficients, phantom maps exist.

Adams showed us in [Ada92] that we may “define away” phantom maps by changing our definition of homotopy.

**Definition 2.2.** Two maps  $f, g : X \rightarrow Y$  are *finite-homotopic* if  $f \circ h$  is homotopic to  $g \circ h$  for any map  $h : K \rightarrow X$  from a finite complex  $K$  into  $X$ . Let  $[X, Y]_f$  denote maps up to finite-homotopy.

Clearly there is a natural surjective map  $[X, Y] \rightarrow [X, Y]_f$ , and if  $X$  is a finite complex this map is an isomorphism.

**Proposition 2.3** (Adams). *For a fixed CW complex  $Y$ ,  $[-, Y]_f$  defines an extraordinary cohomology theory on connected CW complexes. Moreover if  $X$  is any CW complex then the map discussed above*

$$[X, Y]_f \xrightarrow{\cong} \lim_{\alpha} [X_{\alpha}, Y]$$

*is an isomorphism.*

It’s worth discussing what this result tells us about cohomology. Brown’s classical representability theorem starts with a cohomology theory that is defined on all pointed connected complexes, and proves that it is representable. However, intuitively, we would like cohomology theories to be determined by their behavior on finite connected complexes. Adams’ theorem tells us that when a cohomology theory is only defined on finite connected complexes, we really have two ways of extending it to infinite connected complexes. To start, any cohomology theory  $h^0, h^1, h^2, \dots$  (with a suspension isomorphism) defined on finite connected complexes is represented by an  $\Omega$ -spectrum  $Y_0, Y_1, \dots$ . So if  $X$  is a finite complex, then  $h^n(X)$  naturally isomorphic to  $[X, Y_n]$ . But when  $X$  is infinite, we may extend this theory by defining  $h^n(X)$  to be either  $[X, Y_n]$  or  $[X, Y_n]_f$ . The first construction is much better in general, because it is a true cohomology theory, and always satisfies excision. The second construction has the appealing property that it sends filtered homotopy colimits of finite complexes to inverse limits, but the price we pay for this is that it only satisfies excision for  $X = A \cup B$  if  $A \cap B$  is a finite complex.

In  $[X, Y]_f$  the phantom maps do not appear, pretty much by definition. In fact, there is a short exact sequence of sets

$$\{\text{phantom maps up to homotopy}\} \rightarrow [X, Y] \rightarrow [X, Y]_f$$

by which we mean that the first map is injective, the second map is surjective, and kernel of the second map is the image of the first. If  $Y$  is a double loop space then this is a short exact sequence of abelian groups, and it is naturally isomorphic to the  $\lim^1$ -short exact sequence

$$0 \rightarrow \lim_{\alpha}^1 [X_{\alpha}, Y] \rightarrow [X, Y] \rightarrow \lim_{\alpha} [X_{\alpha}, Y] \rightarrow 0$$

This gives us an algebraic way to handle phantom maps and characterize the difference between our two methods for extending cohomology theories from finite complexes to infinite ones.

We will not stop to define  $\lim^1$  here, but we will remark that it vanishes if our inverse system of groups  $F : \mathbf{C} \rightarrow \mathbf{Ab}$  satisfies the *Mittag-Leffler condition*: for each  $i \in \mathbf{C}$ , there is some  $j \rightarrow i$  such that every  $k \rightarrow j \rightarrow i$  has

$$\text{image } F(k) = \text{image } F(j) \subset F(i)$$

One can verify Mittag-Leffler in particular examples, but we would also like a general statement. It would be enough if every decreasing nested (transfinite) sequence of subgroups of  $F(i)$  stabilizes. This happens rather trivially if  $F(i)$  is a finite abelian group. Therefore, if each  $[X_\alpha, Y]$  is finite, we get an isomorphism

$$[X, Y] \xrightarrow{\cong} \lim_{\alpha} [X_\alpha, Y]$$

and our two notions of cohomology for infinite  $X$  actually agree. From the last section, we know this will happen if  $Y$  has finite homotopy groups.

The above analysis applies to cohomology with  $\mathbb{Z}/p$  coefficients, in other words,  $Y = \Omega^\infty(H\mathbb{Z}/p)$ . So the ordinary cohomology of an infinite complex  $X$  is calculated as the inverse limit over the finite subcomplexes of  $X$ , provided we take  $\mathbb{Z}/p$  coefficients. This is yet another reason why working mod  $p$  makes computations easier. As a warning, this does NOT imply that any other filtration of  $X$  gives an inverse limit system that calculates  $H^*(X; \mathbb{Z}/p)$ . If a filtration contains some infinite subcomplexes, then Mittag-Leffler is not automatically satisfied, so there may be a  $\lim^1$ -term. A similar warning applies to spectra: the  $\mathbb{Z}/p$ -cohomology of a spectrum is often not the inverse limit of the  $\mathbb{Z}/p$ -cohomology of the levels of the spectrum.

Let's give another example, relevant to  $p$ -completion of Spanier-Whitehead duals. Define the Moore spectrum  $M(\mathbb{Z}/p)$  to be the cofiber

$$\mathbb{S} \xrightarrow{p} \mathbb{S} \rightarrow M(\mathbb{Z}/p)$$

and take  $Y = \Omega^\infty(M(\mathbb{Z}/p))$  to be its infinite loop space. Then the homotopy groups of  $Y$  are all finite. (We could show this with the Atiyah-Hirzebruch SS, which calculates  $\pi_*^S(M(\mathbb{Z}/p, n))$  from its homology). The maps  $[X, Y]$  may be called the *cohomotopy of  $X$  mod  $p$* :

$$\pi^0(X; \mathbb{Z}/p) := [X, \Omega^\infty(M(\mathbb{Z}/p))]$$

The above analysis shows that this may be calculated as an inverse limit over finite subcomplexes of  $X$ , with no  $\lim^1$  terms. This explains the cryptic phrase “ $\lim^1$ s vanish when working mod  $p$ .” (One should be careful: not every inverse limit system ever built involves maps out of finite complexes, so the above slogan is not true if interpreted too literally.)

### 3 $p$ -Completion and Spanier-Whitehead Duals

We will not construct explicit  $p$ -completions of spaces and spectra here, but we will recall their defining properties. The authoritative references are [BK72], [Bou75], and [Bou79]; a nice modern treatment is given in [MP11].

**Definition 3.1.** The Moore spectrum with  $\mathbb{Z}/p$  coefficients is the cofiber

$$\mathbb{S} \xrightarrow{p} \mathbb{S} \longrightarrow M(\mathbb{Z}/p)$$

If  $X$  is a spectrum, its homotopy with  $\mathbb{Z}/p$  coefficients is defined to be

$$\pi_*(X; \mathbb{Z}/p) := \pi_*(X \wedge M(\mathbb{Z}/p))$$

**Definition 3.2.** A map of spaces  $X \longrightarrow Y$  is a  $p$ -adic equivalence if it induces an isomorphism

$$H_*(X; \mathbb{Z}/p) \xrightarrow{\cong} H_*(Y; \mathbb{Z}/p)$$

A map of spectra  $X \longrightarrow Y$  is a  $p$ -adic equivalence if it induces an isomorphism

$$\pi_*(X; \mathbb{Z}/p) \xrightarrow{\cong} \pi_*(Y; \mathbb{Z}/p)$$

Using the fact that  $H\mathbb{Z}/p$  is a field spectrum (smashing with it gives a wedge of shifted copies of  $H\mathbb{Z}/p$ ), we can easily calculate

$$M(\mathbb{Z}/p) \wedge H\mathbb{Z}/p \simeq H\mathbb{Z}/p \vee \Sigma H\mathbb{Z}/p$$

Therefore every map of spectra which gives an isomorphism on  $\pi_*(-; \mathbb{Z}/p)$  also gives an isomorphism on  $H_*(-; \mathbb{Z}/p)$ . The converse is true if our spectra are connective, using the Adams spectral sequence. From the first section, we already know that homotopy with finite coefficients is much better behaved on non-connective spectra than homology with finite coefficients, so there is good reason to use the above definition of  $p$ -adic equivalence of spectra. On the other hand, one cannot define unstable homotopy with finite coefficients without running into serious technical problems. This explains why  $p$ -adic equivalence means something very different for spectra than it did for spaces.

Now we may define  $p$ -completions of spaces and spectra.

**Definition 3.3.** A space  $Z$  is  $p$ -complete if every  $p$ -adic equivalence  $X \longrightarrow Y$  of CW complexes gives an equivalence

$$[Y, Z] \xrightarrow{\cong} [X, Z]$$

A spectrum  $Z$  is  $p$ -complete if every  $p$ -adic equivalence  $X \longrightarrow Y$  of spectra gives an equivalence

$$[Y, Z] \xrightarrow{\cong} [X, Z]$$

where  $[-, -]$  denotes maps in the stable homotopy category.

**Theorem 3.4** (Bousfield). *Every space  $X$  has a  $p$ -completion  $X_p^\wedge$ , which is initial among  $p$ -complete spaces receiving maps from  $X$ , and final among spaces receiving a  $p$ -adic equivalence from  $X$ . In particular  $X_p^\wedge$  is  $p$ -complete and there is a  $p$ -adic equivalence*

$$X \longrightarrow X_p^\wedge$$

*The same is true for spectra.*

**Proposition 3.5.** *If  $X$  and  $Y$  are  $p$ -complete spaces or spectra and  $X \longrightarrow Y$  is a  $p$ -adic equivalence, then it is a weak equivalence as well. Therefore  $X \longrightarrow X_p^\wedge$  is a weak equivalence when  $X$  is  $p$ -complete, and  $X_p^\wedge \longrightarrow (X_p^\wedge)_p^\wedge$  is always a weak equivalence.*

**Proposition 3.6.** *If  $X$  and  $Y$  are spectra then*

$$F(X, Y) \longrightarrow F(X, Y_p^\wedge)$$

*is a  $p$ -completion.*

The last proposition is an excellent exercise that can be proven quite formally from the others, using the definition of completion and the adjunction between smash and mapping spectrum. At one point it uses the fact that  $M(\mathbb{Z}/p)$  is finite. Be warned that proof does not adapt to more general kinds of localization unless  $X$  is finite. In particular, the Segal conjecture will help us see that

$$F(\Sigma_+^\infty BC_p, \mathbb{S}) \longrightarrow F(\Sigma_+^\infty BC_p, \mathbb{S}_\mathbb{Q})$$

is not a rationalization. (The right-hand side is rational but on  $\pi_0^\mathbb{Q}$  the map is  $\mathbb{Q} \oplus \mathbb{Q}_p^\wedge \longrightarrow \mathbb{Q}$ .)

Let's put this all together for spectra. Every spectrum  $X$  has a  $p$ -completion  $X_p^\wedge$ , and a natural map

$$X \longrightarrow X_p^\wedge$$

which is an equivalence on  $\pi_*(-; \mathbb{Z}/p)$  and therefore an equivalence on  $H_*(-; \mathbb{Z}/p)$ . The functional dual  $DX$  has a  $p$ -completion as well, and the completion map is equivalent to

$$F(X, \mathbb{S}) \longrightarrow F(X, \mathbb{S}_p^\wedge)$$

which is also an equivalence on  $\pi_*(-; \mathbb{Z}/p)$ , and therefore an equivalence on  $H_*(-; \mathbb{Z}/p)$ . The results in the next section will be less confusing if we remember this:  $\pi_*(DX; \mathbb{Z}/p) \cong \pi^*(X; \mathbb{Z}/p)$  is understandable in terms of finite subcomplexes of  $X$ , but  $H_*(DX; \mathbb{Z}/p) \not\cong H^*(X; \mathbb{Z}/p)$  is not.

### Exercises.

- Prove  $B\mathbb{Z}/p^n$  is a  $p$ -complete space for every  $n \geq 1$ .

- If  $G$  is a finite  $p$ -group, prove  $BG = K(G, 1)$  is  $p$ -complete. If  $G$  is also abelian, prove that  $K(G, n)$  is  $p$ -complete, and that the Eilenberg-MacLane spectrum  $HG$  is  $p$ -complete.
- It turns out that if a spectrum  $X$  has all homotopy groups finite  $p$ -groups, then  $X$  is  $p$ -complete. Use this to prove that  $\Sigma^\infty B\mathbb{Z}/p^n$  is  $p$ -complete.
- Prove that  $\mathbb{S}_p^\wedge \simeq M(\hat{\mathbb{Z}}_p)$ , the Moore spectrum whose 0th homology is the  $p$ -adic integers  $\hat{\mathbb{Z}}_p$  and whose other homology vanishes.
- It also turns out that finite hocolims and arbitrary holims of  $p$ -complete spectra are  $p$ -complete. Use this to prove that  $D(\Sigma^\infty B\mathbb{Z}/p^n)$  is  $p$ -complete.

These and other useful basic properties are found in [Bou79].

## 4 The Segal Conjecture

Now we come to our main goal of understanding the statement of the Segal conjecture. Let  $C_p$  denote the cyclic group of order  $p$ , and let  $BC_p = K(C_p, 1)$  denote its classifying space. The Segal conjecture (which is not a conjecture because it was proven in the 1980s) establishes among other things an equivalence of spectra

$$D(\Sigma^\infty BC_p) \simeq (\Sigma_+^\infty BC_p)_p^\wedge \simeq \mathbb{S}_p^\wedge \vee \Sigma^\infty BC_p$$

Adding a disjoint basepoint gives the equivalence

$$D(\Sigma_+^\infty BC_p) \simeq \mathbb{S} \vee (\Sigma_+^\infty BC_p)_p^\wedge \simeq \mathbb{S} \vee \mathbb{S}_p^\wedge \vee \Sigma^\infty BC_p$$

Now the first equivalence above seems very wrong, since it implies that the  $\mathbb{Z}/p$ -homology of these two spectra agree:

$$D(\Sigma^\infty BC_p), \quad \Sigma_+^\infty BC_p$$

We know the  $\mathbb{Z}$ -homology of the right-hand side vanishes below degree 0, and starting in degree 0 it is

$$\mathbb{Z} \quad \mathbb{Z}/p \quad 0 \quad \mathbb{Z}/p \quad 0 \quad \mathbb{Z}/p \quad 0 \quad \dots$$

therefore the  $\mathbb{Z}/p$ -homology also vanishes below 0, and the rest is

$$\mathbb{Z}/p \quad \mathbb{Z}/p \quad \mathbb{Z}/p \quad \mathbb{Z}/p \quad \mathbb{Z}/p \quad \mathbb{Z}/p \quad \mathbb{Z}/p \quad \dots$$

We are forced to conclude that the  $\mathbb{Z}/p$ -homology of the left side is the same as this. So the dual of  $BC_p$  has homology concentrated in nonnegative degrees, even though the dual of any finite skeleton of  $BC_p$  has homology concentrated in negative degrees! We know now why this isn't a contradiction: the homology of  $D(\Sigma^\infty BC_p)$  is not necessarily the cohomology of  $\Sigma^\infty BC_p$ . In fact, this example establishes that there is no direct relationship between

the homology of  $D(\Sigma^\infty BC_p)$  and the inverse limit of the homology of  $D(\Sigma^\infty (BC_p)^{(k)})$  as  $k \rightarrow \infty$ .

Let's stop dwelling on how this could be false and start thinking about why it might be true. Recall from our earlier discussion that an inverse limit of finite groups never has a  $\lim^1$ , and therefore a filtered colimit of finite complexes has stable cohomotopy  $\pi^n(-; \mathbb{Z}/p)$  given as an inverse limit. In this example, the skeleta  $(BC_p)^{(k)}$  are all finite, so we conclude that the cohomotopy of  $BC_p$  with  $\mathbb{Z}/p$  coefficients is nothing more than an inverse limit of the cohomotopy of the skeleta:

$$\pi^n(\Sigma^\infty BC_p; \mathbb{Z}/p) \cong \lim_k \pi^n(\Sigma^\infty (BC_p)^{(k)}; \mathbb{Z}/p)$$

Using the fact that stable homotopy of the dual is stable cohomotopy, we also get

$$\pi_n(D(\Sigma^\infty BC_p); \mathbb{Z}/p) \cong \lim_k \pi_n(D(\Sigma^\infty (BC_p)^{(k)}); \mathbb{Z}/p)$$

So despite our utter failure to connect the homology of  $D(\Sigma^\infty BC_p)$  to that of  $D(\Sigma^\infty (BC_p)^{(k)})$ , there is actually a direct relationship between their homotopy groups with finite coefficients. And we could even use the Atiyah-Hirzebruch spectral sequence to go from cohomology to cohomotopy:

$$E_2 = H^*(\Sigma^\infty BC_p; \pi^*(\mathbb{S}; \mathbb{Z}/p)) \Rightarrow \pi^*(\Sigma^\infty BC_p; \mathbb{Z}/p)$$

This is a second-quadrant spectral sequence when we use the homological grading for everything (or a fourth-quadrant spectral sequence with the cohomological grading). That makes it possible to have homotopy classes in any degree, so there is no contradiction.

It turns out to be easier to use the Adams spectral sequence to actually calculate this homotopy with finite coefficients and to get the above equivalence of spectra. This is still quite hard, but it is enough to compare the  $E^2$ -pages for the Adams spectral sequences of the two sides. This was successfully done by Lin for  $p = 2$  [Lin80] and Gunawardena for odd  $p$  [Gun80].

There is much more than can be said, but we have given everything one needs to get a basic grasp of the Segal conjecture, so we will stop here. The interested reader may find much more information in the references in [Car92], section 4(C).

## References

- [Ada92] JF Adams, *A variant of EH Brown's representability theorem*, The Selected Works of J. Frank Adams **2** (1992), 408.
- [BK72] A.K. Bousfield and D.M. Kan, *Homotopy limits, completions and localizations*, vol. 304, Springer, 1972.

- [Bou75] A.K. Bousfield, *The localization of spaces with respect to homology*, *Topology* **14** (1975), no. 2, 133–150.
- [Bou79] ———, *The localization of spectra with respect to homology*, *Topology* **18** (1979), no. 4, 257–281.
- [Car92] G. Carlsson, *A survey of equivariant stable homotopy theory*, *Topology* **31** (1992), no. 1, 1–27.
- [Gun80] J.H.C. Gunawardena, *Segal’s conjecture for cyclic groups of (odd) prime order*, JT Knight prize essay, Cambridge (1980).
- [Lin80] W.H. Lin, *On conjectures of mahowald, segal and sullivan*, *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 87, Cambridge Univ Press, 1980, pp. 449–458.
- [MP11] J.P. May and K. Ponto, *More concise algebraic topology: localization, completion, and model categories*, University of Chicago Press, 2011.
- [Wal65] C.T.C. Wall, *Finiteness conditions for cw-complexes*, *The Annals of Mathematics* **81** (1965), no. 1, 56–69.