RESEARCH STATEMENT

CARY MALKIEWICH

For much of their history, stable homotopy theory and algebraic K-theory have drawn motivation from the study of high-dimensional manifolds, and from fixed-point theory. My current research continues this trend by relating

- fixed-point theory to trace methods in algebraic K-theory, and
- equivariant manifolds to equivariant structures on algebraic K-theory.

To do this research I draw on my experience with equivariant spectra, parametrized spectra, K-theory, and hands-on geometric techniques for computations, from the previous work [Mal17b, Mal17c, Mal17a, KM18, MM19, LM19, LM18, DMP$^+$19, MP18, MM20a, Mal19, MM20b, CLM$^+$20, KM20].

0.1. **Fixed-point theory and its generalizations.** Fixed point theory asks whether the fixed points can be removed from a map $f \colon X \to X$ by a homotopy, if X is a finite CW complex. The fiberwise variant considers fiber bundles $E \to B$ with fiber X, and fiberwise maps $f \colon E \to E$. See e.g. [Dol76]. The primary algebraic obstruction for these problems is the Lefschetz number $L(f)$, but it has a refinement called the Reidemeister trace $R(f)$ that is often a complete obstruction to removing fixed points.

Work of Geoghegan and Nicas in the 1990s on fiberwise Reidemeister traces [GN94] provided strong hints that in the fiberwise case, topological Hochschild homology (THH) is the correct home for the Reidemeister trace $R(f)$. This idea was later realized through work of Klein, Williams, Ponto, Campbell, Lind, and myself [KW07, Pon10, LM19, CP19]. The central concept, from Ponto, is the notion of a "non-commutative" or "bicategorical" trace, generalizing the trace in a symmetric monoidal category that was used by Dold to form the fiberwise Lefschetz number. See Figure 0.1.1. This trace is carried out in a bicategory of parametrized spectra, and my book [Mal19] gives a precise and updated treatment of this category so that future work can proceed on firm footing.

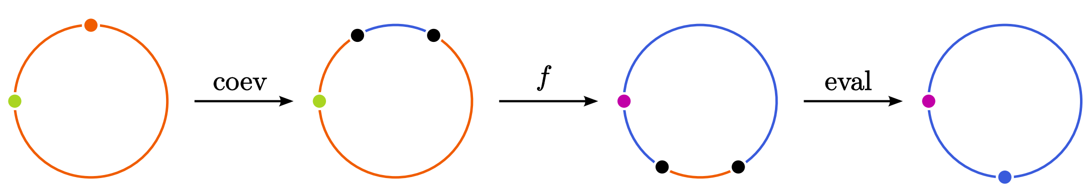

FIGURE 0.1.1. The "non-commutative" or "bicategorical" trace.

In [MP18] Kate Ponto and I consider the problem of removing n-periodic points from a family of maps $f \colon E \to E$ over B. We show that a certain invariant called the "Fuller trace" is a generalization of all previously known periodic-point invariants. The Fuller trace is essentially a

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Reidemeister trace carried out in genuinely equivariant spectra. I consider this to be a surprising application of equivariant stable homotopy theory to a problem in classical topology.

I have constructed a counterexample showing that the Fuller trace is in fact strictly stronger, so that equivariant homotopy theory actually produces a better invariant than other methods. However, some work on the fiberwise, equivariant version of the Pontryagin-Thom isomorphism will be needed to make the calculation precise. After this, the next step is to show that our TR invariant is a complete obstruction in families, by generalizing an argument by Jezierski [Jez01].

0.2. **Generalized characteristic polynomials.** Proceeding further, the recent joint project [CLM+20] with Campbell, Lind, Ponto, and Zakharevich constructs a very general trace map from K-theory of endomorphisms to topological restriction homology (TR). For a commutative ring, on π_0 this map coincides with the characteristic polynomial from [Alm74]. In general, each endomorphism of an R-module goes to a class in TR that is described as a Fuller trace, just like in the work on periodic-point theory above. In other words:

The Fuller trace is a topological, non-commutative generalization of the characteristic polynomial.

This result fits the previous work with Ponto on periodic points into a larger K-theoretic context. It has other upshots as well. One is that, as in [LM19], these Fuller traces lie underneath wrong-way maps in algebraic K-theory.

0.3. **Transfers in algebraic K-theory and Hochschild homology.** John Lind and I began importing the Reidemeister trace from fixed-point theory into algebraic K-theory in [LM19]. It gives us a formula for "wrong-way" maps on THH lying underneath K-theory. This is a powerful technique. In that paper, it allowed us to prove that the Becker-Gottlieb transfer of [BG75] is a retract of the A-theory transfer of [Wil00], generalizing earlier results from [DJ12] and elsewhere. It also allowed us to do homology calculations of the transfer for the Hopf bundle.

With a tractable amount of additional work, this technique should allow us to compute K-theory transfer maps rationally. Through the connection between K-theory and differential topology, see e.g. [DWW03], this can be used to identify exotic families of smooth structures on manifold bundles. I plan to do calculations of these that start with the case of spherical fibrations. Another upshot of the planned work in this area is additional calculations of the fiberwise Reidemeister trace over a base of dimension greater than one, and a new technique that uses the main result of [LM19] to compute the involution on the free loop space LX rationally. This latter calculation can be used to get rational stable diffeomorphism groups of various manifolds, see e.g. [WW14, BFJ17].

0.4. **Equivariant manifolds.** The surgery theory program to classify high-dimensional manifolds was one of the greatest successes of topology in the 20th century. When we access the space of diffeomorphisms, algebraic K-theory plays a critical role, in the form of Waldhausen's functor $A(X)$. The variant of this program for G-manifolds, when G is a finite group, is more difficult and the picture is far from complete.

In the paper [MM19] Mona Merling and I develop the equivariant version of Waldhausen's $A(X)$. The desired properties of $A_G(X)$ were discussed by Goodwillie, Rognes, Waldhausen, and others in the 1990s, but the necessary technology to build equivariant algebraic K-theory out of Waldhausen categories with appropriate G-actions has only recently appeared.

Our next two goals are to prove the equivariant analogs of Waldhausen's celebrated parametrized h-cobordism theorem and splitting theorem. Together, these theorems give a splitting $A(X) \simeq \Sigma^\infty_+ X \times Wh^{\mathrm{DIFF}}(X)$, where the second spectrum is a delooping of the stabilized space $\mathcal{H}^\infty(X)$ of h-cobordisms on X.

We achieved the category-theoretic part of this program in the recent paper [MM20b], building on the results from [BO15, BD17]. The next step is the manifold part, which is joint work with Tom Goodwillie, Kiyoshi Igusa, and Mona Merling. The main result is that the stable space of smooth equivariant h-cobordisms has a splitting

$$\mathcal{H}^\infty_G(X)^G \simeq \prod_{(H)\leq G} \mathcal{H}^\infty(X^H_{hWH}). \tag{0.4.1}$$

This in turn separates into two parts – proving a similar result unstably for *isovariant* h-cobordisms, and then using functoriality of the h-cobordism space to pass to the stable space, where isovariant and equivariant h-cobordisms coincide. Both are currently in progress.

At the level of π_0 these results recover the smooth equivariant s-cobordism theorem of Browder-Quinn and Rothenberg [BQ75, Rot78]. Above this level, they allow us to calculate rational stable diffeomorphism groups of G-manifolds, starting with discs, spheres, and tori.

0.5. **Connections to cut-and-paste K-theory.** The projects I described above fall under the scope of the NSF grant DMS-2005524, "Algebraic K-Theory in Fixed-Point Theory and Smooth Manifolds." I have also recently started a collaboration with several mathematicians as part of the grant NSF DMS-2052923, "FRG: Collaborative Research: Trace Methods and Applications for Cut-and-Paste K-Theory." As a part of this grant I will be applying ideas from cut-and-paste K-theory to fixed-point theory and its generalizations. The first step is to identify trace maps that lie beneath cut-and-paste K-theory. This would land in a more general form of THH than that discussed in the first section. This project is still in its early stages, but I expect this to lead to a new, more refined Reidemeister trace $R(f)$ that detects the dynamical behavior of f up to homeomorphism, not just up to homotopy.

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