RESEARCH STATEMENT

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Stable homotopy theory and algebraic K-theory are deep, rich subjects, that draw a lot of their original motivation from the study of high-dimensional manifolds and fixed-point theory. My research continues this long tradition by finding new relationships between

- trace methods in algebraic K-theory and fixed-point theory,
- equivariant structures on algebraic K-theory and equivariant manifolds,
- and most recently, combinatorial forms of algebraic K-theory and scissors congruence.

To do this research I draw on my experience with equivariant spectra, parametrized spectra, K-theory, and hands-on geometric techniques for computations, from the previous work [Mal17b, Mal17c, Mal17a, KM18, MM19, LM19, LM18, DMP$^+$19, MP18, MM20a, Mal19, MM20b, CLM$^+$20, KM20, Mal22, KMR22, BGM$^+$23, GIMM23].

0.1. **The higher form of Hilbert's Third Problem.** I would like to start with the project that began in the spring of 2022 and resulted in the paper [Mal22]. It solves the higher version of Hilbert's Third Problem for each of the one-dimensional geometries. This is a fundamental advance in our understanding of scissors congruence, and the fact that it is even possible has come as a surprise to those in the area.

The scissors congruence problem asks, given two polytopes P and Q in a neat geometry (Euclidean, spherical, or hyperbolic), can P be cut into finitely many pieces and rearranged to form Q? The problem is effectively solved for low-dimensional geometries (up to dimension three or four, depending on the geometry). However, it is open in all geometries above dimension four. This problem is often called Hilbert's Third Problem because Hilbert asked specifically about the case of three-dimensional Euclidean geometry. See e.g. [Dup01, Sah79] for an overview of the progress made on this problem in the second half of the 20th century.

The higher scissors congruence problem, or higher Hilbert's Third Problem, asks about the *space* formed by taking a point for every polytope, a path for every scissors congruence between polytopes, a 2-simplex for every pair of composable paths, and so on. Group-completing this space gives the scissors congruence K-theory spectrum $K(\mathcal{P})$, first described by Zakharevich [Zak17]. We think of this object as a space whose connected components (π_0) describe the solution to the classical scissors congruence problem. The higher groups describe something beyond the polytopes up to scissors congruence – they describe *the scissors congruences themselves.*

My contribution to the area is to show that $K(\mathcal{P})$ is a particularly nice kind of spectrum called a Thom spectrum [Mal22]. Because of this new presentation, we can make computations that were not possible before. For instance, we now know that the spectrum is rational for every Euclidean geometry. We also know *all* of its higher K-groups when the geometry is one-dimensional. This is what solves the higher version of Hilbert's Third Problem in dimension one.

Several of the intermediate results leading to this are joint work with Bohmann, Gerhardt, Merling, and Zakharevich. In particular, one of the steps involves showing that homotopy orbits

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commute with a certain form of K-theory of multicategories [BGM$^+$23]. This on its own is an important advancement in K-theory, and will give us new ways to categorify the famous Farrell-Jones conjecture.

This breakthrough is already leading to a larger effort to understand and compute the higher K-groups of scissors congruence. Work in progress with Zakharevich defines the Dehn invariant on the higher scissors congruence groups, and proves a higher Dehn-Sydler theorem for spherical geometries (the statement that volume and Dehn invariants separate scissors congruence classes). I also have a large amount of ongoing work aimed at calculating and better understanding the trace invariants on the higher K-groups of geometries of dimension two and three. My student Ezekiel Lemann has succeeded in giving a combinatorial description of the higher K-groups of one-dimensional geometries, and is currently working on generalizations of my result to other geometries. The trace methods I developed with my co-authors have also produced many new nonzero classes in the higher K-groups, and it will take time to sort out all of the new information we have and to present it systematically to the rest of the community. I am excited to report more on these efforts in the coming years.

0.2. **Equivariant manifolds.** The surgery theory program to classify high-dimensional manifolds was one of the greatest successes of topology in the 20th century. When we access the space of diffeomorphisms, algebraic K-theory plays a critical role, in the form of Waldhausen's functor $A(X)$. The variant of this program for G-manifolds, when G is a finite group, is more difficult and the picture is far from complete.

In the paper [MM19], Mona Merling and I develop the equivariant version of Waldhausen's $A(X)$. The desired properties of $A_G(X)$ were discussed by Goodwillie, Rognes, Waldhausen, and others in the 1990s, but the necessary technology to build equivariant algebraic K-theory out of Waldhausen categories with appropriate G-actions has only recently appeared.

Our next two goals are to prove the equivariant analogs of Waldhausen's celebrated parametrized h-cobordism theorem and splitting theorem. Together, these theorems give a splitting $A(X) \simeq \Sigma^\infty_+ X \times Wh^{\mathrm{DIFF}}(X)$, where the second spectrum is a delooping of the stabilized space $\mathcal{H}^\infty(X)$ of h-cobordisms on X.

We achieved the category-theoretic part of this program in the paper [MM20b], building on the results from [BO15, BD17]. The next step is the manifold part, which is joint work with Tom Goodwillie, Kiyoshi Igusa, and Mona Merling. The main result is that the stable space of smooth equivariant h-cobordisms has a splitting

$$\mathcal{H}^\infty_G(X)^G \simeq \prod_{(H)\leq G} \mathcal{H}^\infty(X^H_{hWH}). \tag{0.2.1}$$

This in turn separates into two parts – proving a similar result unstably for *isovariant* h-cobordisms, and then using functoriality of the h-cobordism space to pass to the stable space, where isovariant and equivariant h-cobordisms coincide. The functoriality part of this program is carried out in [GIMM23] – we give a new presentation of the h-cobordism space as a functor up to coherent homotopy, using several new ideas to streamline earlier treatments, which makes it tractable to generalize the result to manifolds with a G-action. Even in the non-equivariant case, this is an important development for the literature on the space of h-cobordisms.

At the level of π_0, the above results recover the smooth equivariant s-cobordism theorem of Browder-Quinn and Rothenberg [BQ75, Rot78]. Above this level, they allow us to calculate rational stable diffeomorphism groups of G-manifolds, starting with discs, spheres, and tori.

0.3. **Fixed-point theory and its generalizations.** Fixed point theory asks whether the fixed points can be removed from a map $f\colon X \to X$ by a homotopy, if X is a finite CW complex. The fiberwise variant considers fiber bundles $E \to B$ with fiber X, and fiberwise maps $f\colon E \to E$. See e.g. [Dol76]. The primary algebraic obstruction for these problems is the Lefschetz number $L(f)$, but it has a refinement called the Reidemeister trace $R(f)$ that is often a complete obstruction to removing fixed points.

Work of Geoghegan and Nicas in the 1990s on fiberwise Reidemeister traces [GN94a] provided strong hints that in the fiberwise case, topological Hochschild homology (THH) is the correct home for the Reidemeister trace $R(f)$. This idea was later realized through work of Klein, Williams, Ponto, Campbell, Lind, and myself [KW07, Pon10, LM19, CP19]. The central concept, from Ponto, is the notion of a "non-commutative" or "bicategorical" trace, generalizing the trace in a symmetric monoidal category that was used by Dold to form the fiberwise Lefschetz number. See Figure 0.3.1. This trace is carried out in a bicategory of parametrized spectra, and my book [Mal19] gives a precise and updated treatment of this category so that future work can proceed on firm footing.

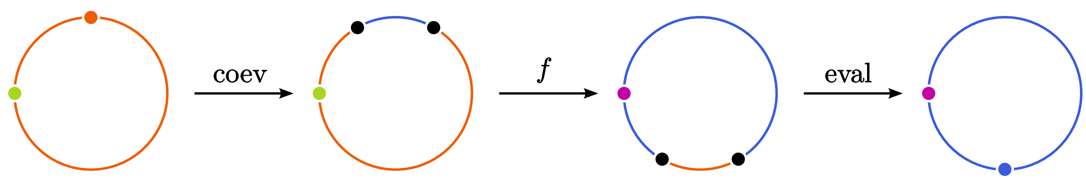

FIGURE 0.3.1. The "non-commutative" or "bicategorical" trace.

In [MP18] Kate Ponto and I consider the problem of removing n-periodic points from a family of maps $f\colon E \to E$ over B. We show that a certain invariant called the "Fuller trace" is a generalization of all previously known periodic-point invariants. The Fuller trace is essentially a Reidemeister trace carried out in genuinely equivariant spectra, so this is a surprising application of equivariant stable homotopy theory to a problem in classical topology.

I have constructed a counterexample showing that the Fuller trace is in fact strictly stronger, so that equivariant homotopy theory actually produces a better invariant than other methods. However, some work on the fiberwise, equivariant version of the Pontryagin-Thom isomorphism will be needed to make the calculation precise. After this, the next step is to show that our Fuller trace is a complete obstruction in families, by generalizing an argument by Jezierski [Jez01]. My student Lucas Williams is working on the equivariant fiberwise version of the Pontryagin-Thom isomorphism, which will give us the tools we need to finish this goal.

I have also used the Fuller trace to describe an invariant for periodic orbits of flows, in other words periodic solutions to ordinary differential equations. This generalizes a construction by Geoghegan and Nicas [GN94b] and recovers a construction of Michael Crabb. I was able to reinterpret it using trace methods in algebraic K-theory, showing that it defines a class in topological Frobenius homology (TF). This suggests that there are further connections between trace methods and the dynamics of ordinary differential equations, that have yet to be developed.

0.4. **Generalized characteristic polynomials.** The recent joint project [CLM+20] with Campbell, Lind, Ponto, and Zakharevich constructs a very general trace map from K-theory of endomorphisms to topological restriction homology (TR). For a commutative ring, on π_0 this map coincides with the characteristic polynomial from [Alm74]. In general, each endomorphism of an R-module goes to a class in TR that is described as a Fuller trace, just like in the work on periodic-point theory above. In other words:

The Fuller trace is a topological, non-commutative generalization of the characteristic polynomial.

This result fits the previous work with Ponto on periodic points into a larger K-theoretic context. It has other upshots as well. One is that, as in [LM19], these Fuller traces lie underneath wrong-way maps in algebraic K-theory.

0.5. **Transfers in algebraic K-theory and Hochschild homology.** John Lind and I began importing the Reidemeister trace from fixed-point theory into algebraic K-theory in [LM19]. It gives us a formula for "wrong-way" maps on THH lying underneath K-theory. This is a powerful technique. In that paper, it allowed us to prove that the Becker-Gottlieb transfer of [BG75] is a retract of the A-theory transfer of [Wil00], generalizing earlier results from [DJ12] and elsewhere. It also allowed us to do homology calculations of the transfer for the Hopf bundle.

With a tractable amount of additional work, this technique should allow us to compute K-theory transfer maps rationally. Through the connection between K-theory and differential topology, see e.g. [DWW03], this can be used to identify exotic families of smooth structures on manifold bundles. My student Shuchen Mu has been doing calculations of these that start with the case of spherical fibrations. Another upshot of the planned work in this area is additional calculations of the fiberwise Reidemeister trace over a base of dimension greater than one, and a new technique that uses the main result of [LM19] to compute the involution on the free loop space LX rationally. This latter calculation can be used to get rational stable diffeomorphism groups of various manifolds, see e.g. [WW14, BFJ17].

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