

DUALITY AND LINEAR APPROXIMATIONS IN HOCHSCHILD HOMOLOGY,  
*K*-THEORY, AND STRING TOPOLOGY

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DOCTOR OF PHILOSOPHY

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

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# Abstract

This thesis encompasses at least three separate but related projects. The first project from a chronological point of view is our treatment of twisted Poincaré duality for manifolds with coefficients in spectra. This is not really a new result, but our treatment is conceptually useful for the second project, and so we include it as an appendix.

The second project investigates a certain map from the stabilization of the gauge group of the principal bundle  $\Omega M \rightarrow * \rightarrow M$  to the string topology spectrum  $LM^{-TM}$ . This map is a linear approximation in the sense of Goodwillie and Weiss’s embedding calculus. We describe the rest of the tower, in the process extending embedding calculus from manifolds to CW complexes.

The third project is an ongoing and open-ended exploration of contravariant forms of algebraic  $K$ -theory of spaces. We begin with a splitting of  $THH(DX)$  when  $X$  is a reduced suspension. The resulting calculation gives evidence that the known equivalence  $D(THH(DX)) \simeq \Sigma_+^\infty LX$  is actually an equivariant equivalence, meaning that it preserves the  $C_n$ -fixed points. We prove this for general finite simply-connected  $X$ , making use of a new approach to  $THH$  via the norm construction of Angeltveit, Blumberg, et al. In the process of simplifying our work, we prove a new rigidity result for the geometric fixed points of orthogonal  $G$ -spectra.

Next we apply our splitting result to calculate  $TC(DS^1)$  explicitly. This object has striking parallels with  $TC(S^1)$ , and the calculation allows us to rule out a certain kind of “dual” Novikov conjecture for  $K(DS^1)$ .

Finally, in a somewhat different direction we investigate a variant of  $THH(DBG)$  when  $G$  is a finite group. We prove that certain linear approximations into and out of this object compose to give an equivalence after  $p$ -completion, and this suggests some interesting behavior of the contravariant  $K$ -theory of  $BG$ . In future work we intend to study this behavior further and connect it to equivariant forms of algebraic  $K$ -theory of spaces.

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# Chapter 1

## Introduction

### 1.1 Gauge groups, string topology, and homotopy calculus

Let  $M$  be a closed manifold, not necessarily oriented, and let  $LM = \text{Map}(S^1, M)$  be its free loop space. The evaluation map  $LM \rightarrow M$  is a fiber bundle whose fibers are monoids, so when we add a disjoint copy of  $M$  to  $LM$  and take a fiberwise suspension spectrum, we get a parametrized ring spectrum denoted  $\Sigma_{M,+}^{\infty} LM$ . Taking sections over  $M$  gives an ordinary ring spectrum  $\Gamma_M(\Sigma_{M,+}^{\infty} LM)$ .

It is by now well-known that one may formulate Poincaré duality for a manifold  $M$  with coefficients given by a bundle of spectra over  $M$  ([CJ13], [MS06]). It is difficult to find an explicit and conceptually simple treatment of this result in the literature, so we attempt to give such a treatment in Appendix A. At any rate, the Poincaré duality isomorphism gives an equivalence of ring spectra

$$LM^{-TM} \simeq \Gamma_M(\Sigma_{M,+}^{\infty} LM)$$

where  $LM^{-TM}$  is the Thom spectrum of the virtual bundle  $-TM$ , pulled back from  $M$  to  $LM$ . The spectrum  $LM^{-TM}$  has an intersection product, first given by Cohen and Jones [CJ02]. When  $M$  is oriented, this product descends to the product on  $H_*(LM)$  originally described by Chas and Sullivan [CS99]. The term *string topology* refers to the study of this product on  $H_*(LM)$  and its many generalizations.

The benefit of the above Poincaré duality equivalence is that it makes the string topology spectrum  $LM^{-TM}$  look like the kind of “linear approximation” that arises in Goodwillie

calculus. In fact, we may define a string topology functor

$$S(X) = \Gamma_X(f^*\Sigma_{M,+}^\infty LM)$$

for every space  $X \xrightarrow{f} M$  over  $M$ . This functor is *linear* in the sense that it preserves homotopy equivalences, takes  $\emptyset$  to  $*$ , and takes homotopy pushout squares to homotopy pullback squares. Following the philosophy of Goodwillie calculus, is natural to ask whether there are interesting functors  $G(X)$  for which  $S$  is the universal linear approximation of  $G$ . In [CJ13], Cohen and Jones show that there is such a functor:

$$\begin{aligned} G(X) &= \Sigma^\infty \Gamma_X(X \amalg f^*LM) \\ G(M) &= \Sigma_+^\infty \Gamma_M(LM) \end{aligned}$$

They call this the *gauge group functor*, since the group  $\Gamma_M(LM) \cong \Omega_{\text{id}}\text{haut}(M)$  is equivalent to the gauge group of the principal bundle  $\mathcal{P} \rightarrow M$  whose total space is contractible.

We can take this analysis further: if excisive functors are like affine-linear functions, then there is a notion of *n-excisive functor* analogous to *n*th degree polynomial functions. One may approximate a homotopy functor  $F$  by a tower

$$F(X) \longrightarrow \dots \longrightarrow P_n F(X) \longrightarrow P_{n-1} F(X) \longrightarrow \dots \longrightarrow P_2 F(X) \longrightarrow P_1 F(X)$$

in which  $P_n F$  is the universal *n*-excisive approximation to  $F$ . Goodwillie and Weiss have shown in [Wei99] and [GW99] that such a tower exists when we consider our functors to be defined on open subsets of a fixed manifold  $M$ . In fact, their calculus holds more generally for the category of finite CW complexes over any fixed base space  $B$ . The first-order approximation  $P_1 F$  has already been studied in the special case that  $F(\emptyset) \simeq *$ , and in that case, the natural transformation  $F \rightarrow P_1 F$  is called a *coassembly map*. We therefore call the more general *n*th order approximation  $F \rightarrow P_n F$  a *higher coassembly map*.

In chapter 2 below, we give an explicit general construction of  $P_n F$ , and identify the homotopy type of the layers of the tower for  $G(X)$  in terms of Thom spectra of bundles over configuration spaces.

## 1.2 Algebraic $K$ -theory of spaces and $THH$

Algebraic  $K$ -theory provides a rich set of invariants for a wide range of mathematical objects. One particularly interesting form of algebraic  $K$ -theory is Waldhausen’s functor

$$A(X) \simeq K(\Sigma_+^\infty \Omega X)$$

This functor enjoys many striking relationships with the spaces of pseudoisotopies and diffeomorphisms of  $X$  when  $X$  is either a PL or smooth manifold [Wal85]. It is also equipped with a natural map to the suspension spectrum of the free loop space [Goo90]

$$A(X) \longrightarrow \Sigma_+^\infty LX$$

which has served as a fruitful starting point for applications of homotopy calculus to  $A(X)$ .

Waldhausen’s functor  $A(X)$  has a natural contravariant analogue sometimes denoted  $\forall(X)$ . Following [BM11b] we might call it the “geometric Swan theory” of the ring  $\Sigma_+^\infty \Omega X$ . Surprisingly little is known about  $\forall(X)$ , in contrast to  $A(X)$ , though Blumberg and Mandell have shown that when  $X$  is finite and simply connected,

$$\forall(X) \simeq K(DX)$$

where  $DX$  is simply the Spanier-Whitehead dual of  $X$  [BM11b]. In this paper we will write  $DX_+$  instead of  $DX$  in order to differentiate from the based dual.

There are a few reasons one might wish to understand  $\forall(X)$  better. The space is a universal receptacle for any invariant of finite fibrations over  $X$  which splits cofiber sequences. There are natural pairings between  $\forall(X)$  and  $A(X)$  which allow us to relate classes in  $\forall(X)$  with transfer maps in  $A(X)$  [Wil00]. In a different direction, Morava has built a conjectural theory of motives based on  $K(DX)$  [Mor09]. Finally, Barwick has recently demonstrated the existence of a genuine  $G$ -spectrum structure on  $K(\mathbb{S})$  in which the genuine  $H$ -fixed points are equivalent to  $\forall(BH)$  [Bar14].

We are therefore motivated to better understand  $\forall(X)$  in general, and so we begin with the topological Hochschild homology  $THH(DX_+)$  in the case where  $X$  is finite and simply-connected. Even before Blumberg and Mandell’s work, this has been considered an interesting object of study. In the course of some string-topology calculations, Cohen

observed that its functional dual is equivalent to the free loop space

$$THH(\Sigma_+^\infty \Omega X) \simeq \Sigma_+^\infty LX \xrightarrow{\sim} D(THH(DX_+)) \quad (1.1)$$

and Campbell extended this from  $(\Sigma_+^\infty \Omega X, DX_+)$  to other pairs of Koszul-dual ring spectra [Cam14]. Forthcoming work of Ayala and Francis describes a general form of Poincaré/Koszul duality for manifolds and ring spectra, which returns this result when the manifold is the circle  $S^1$  [AF]. Finally, one may abuse notation and let  $LM$  denote the space of smooth loops in  $M$ ; then there are analytic lines of argument which establish the plausibility of  $THH(DM_+)$  as a model for the Thom spectrum  $LM^{-T(LM)}$  when  $LM$  is regarded as an infinite-dimensional manifold.

The previous work on  $THH(DX_+)$  leaves open the question of how the natural circle actions and equivariant structure on  $THH(DX_+)$  and  $\Sigma_+^\infty LX$  are related under the map (1.1). In Chapter 3 we give a definitive answer:

**Theorem 1.2.1.** *The map (1.1) can be made  $S^1$ -equivariant in such a way that it induces an equivalence of fixed point spectra*

$$\begin{aligned} \Phi^{C_n} D(THH(DX_+)) &\simeq \Phi^{C_n} \Sigma_+^\infty LX \\ [D(THH(DX_+))]^{C_n} &\simeq [\Sigma_+^\infty LX]^{C_n} \end{aligned}$$

for all finite subgroups  $C_n \leq S^1$ .

In fact we show that the functional dual  $D(THH(R))$  can be given a natural *pre-cyclotomic* structure, and when  $R = DX_+$  with  $X$  finite and simply-connected it has a *cyclotomic* structure. This is the kind of structure that is known to exist already on  $THH(R)$  and  $\Sigma_+^\infty LX$ , and which makes  $TC(R)$  and  $TC(X)$  so computable.

In the process, we prove a rigidity result for smash powers and geometric fixed points of orthogonal spectra that appears to be new and of independent interest. It gives in particular a rigidity theorem for the Hill-Hopkins-Ravenel norm diagonal map

$$X \xrightarrow{\Delta} \Phi^G X^{\wedge G}$$

Our work on  $D(THH(DX_+))$  builds on very recent work of Angeltveit, Blumberg, Gerhardt, Hill, Lawson, and Mandell [ABG<sup>+</sup>12], [ABG<sup>+</sup>14], which starts from the observation that the Hill-Hopkins-Ravenel multiplicative norm [HHR09] has the equivariant homotopy

type one would need, if one wanted the cyclic bar construction in orthogonal spectra to have the same equivariant homotopy type as Bökstedt’s construction of topological Hochschild homology [Bök85]. However in contrast to Bökstedt’s construction, the cyclic bar construction is relatively simple on the point-set level. This leads to simplifications in the theory of  $THH$ , as well as new results, including those outlined above.

After establishing this equivariant equivalence, we demonstrate a splitting on  $THH(DX_+)$  when  $X$  is a reduced suspension:

**Theorem 1.2.2.** *There is a natural equivalence of genuine  $S^1$ -spectra*

$$THH(D_+\Sigma X) \simeq \mathbb{S} \vee \Sigma^{-1} \left( \bigvee_{n=1}^{\infty} D(X^{\wedge n}) \wedge_{C_n} S_+^1 \right)$$

This splitting does not preserve the  $E_\infty$  ring structure, but it implies a collapsing result which makes that structure trivial to compute on the level of homology. Dualizing and applying Thm 1.2.1, we recover a classical stable splitting of the free loop space of a suspension:

$$\Sigma^\infty L\Sigma X \simeq \bigvee_{n=1}^{\infty} \Sigma^\infty S_+^1 \wedge_{C_n} X^{\wedge n}$$

This splitting was observed by Goodwillie and proven in [Coh87]; it follows from the combinatorial model for  $L(\Sigma X)$  described in [CC87] and generalized in [BM88]. Our work easily implies that this splitting is equivariant, in the sense that it preserves genuine fixed points for all finite subgroups  $C_n \leq S^1$ .

Future work will explore the relationship between the  $TC$  of  $DX_+$  and  $\Sigma_+^\infty \Omega X$ . To lay the groundwork for this, we focus on  $THH(DS_+^1)$ , and calculate  $TC$ :

**Theorem 1.2.3.** *There is a splitting of  $TC$  of the dual of the circle*

$$TC(D_+S^1)_p^\wedge \simeq \mathbb{S} \vee \Sigma \mathbb{C}P_{-1}^\infty \vee \bigvee_{n \in \mathbb{N}} X$$

in which  $X$  is defined to be the homotopy fiber of

$$\bigvee_{k=1}^{\infty} \Sigma_+^\infty BC_{p^k} \longrightarrow \Sigma^{-1} \Sigma_+^\infty S^1$$

This is surprisingly close to the  $TC$  of the ring  $\Sigma_+^\infty \Omega S^1 \simeq \Sigma_+^\infty \mathbb{Z}$ . Using the same

spectrum  $X$  as above, we have

$$TC(\Sigma_+^\infty \Omega S^1)_p^\wedge \simeq S_+^1 \wedge (\mathbb{S} \vee \Sigma \mathbb{C}\mathbb{P}_{-1}^\infty) \vee \bigvee_{n \in \mathbb{Z} \setminus \{0\}} \Sigma X$$

In a different direction, we also use this calculation to get explicit dimension counts on the rational homotopy groups and conclude that the coassembly map

$$K(DS^1) \longrightarrow \text{Map}(S^1, K(\mathbb{S}))$$

is not rationally split surjective. However  $K(DS^1) \not\cong \vee(S^1)$ , and so this does not completely count out the possibility of a “dual”  $A$ -theory Novikov conjecture.

In the general case where  $X$  is not simply-connected, we can still recognize  $\vee(X)$  as the  $K$ -theory of the Waldhausen category  $\mathcal{P}'(\Sigma_+^\infty \Omega X)$  consisting of left  $\Sigma_+^\infty \Omega X$ -modules whose underlying spectrum is finite or dualizable. The Waldhausen category  $\mathcal{P}'(\Sigma_+^\infty \Omega X)$  may be enriched in orthogonal spectra in a natural way, and so it has a topological Hochschild homology, which by [BM11c] and [ABG<sup>+</sup>12] receives a trace map from  $\vee(X)$ . We therefore investigate the  $THH$  of this category when  $X = BG$ ,  $G$  a finite group:

**Theorem 1.2.4.** *The composite of the assembly and coassembly maps*

$$BG_+ \wedge THH(*) \xrightarrow{\alpha} THH(\Sigma_+^\infty G) \longrightarrow THH(\mathcal{P}'(\Sigma_+^\infty G)) \xrightarrow{c\alpha} F(BG_+, THH(*))$$

*is up to homotopy the transfer  $BG_+ \longrightarrow F(BG_+, \mathbb{S})$  along the bundle over  $BG \times BG$  with fiber  $G$  and monodromy given by left and right multiplication of  $G$  on itself. There is a similar composite*

$$BWH_+ \wedge THH(*) \xrightarrow{\alpha} THH(\Sigma_+^\infty WH) \longrightarrow THH(\mathcal{P}'(\Sigma_+^\infty G)) \xrightarrow{c\alpha} F(BG_+, THH(*))$$

*which is up to homotopy the transfer  $BWH_+ \longrightarrow F(BG_+, \mathbb{S})$  along the bundle over  $BG \times BWH$  with fiber  $G/H$  and monodromy given by the left action of  $G$  and the right action of  $WH \cong \text{Aut}_G(G/H)^{\text{op}}$ .*

**Corollary 1.2.5.** *If  $G$  is a finite  $p$ -group the coassembly map*

$$THH(\mathcal{P}'(\Sigma_+^\infty G)) \xrightarrow{c\alpha} F(BG_+, THH(*))$$

is split surjective after  $p$ -completion, as a map of coarse  $S^1$ -spectra.

Unfortunately this coassembly map does not appear to be surjective on the fixed points under the circle action. In some sense this is because the spectrum on the right is not really cyclotomic but only pre-cyclotomic. One may approximate it in a universal way by a cyclotomic spectrum, and we conjecture that the resulting map is then indeed split surjective on all fixed points and on  $TC$ .

The author has felt that this result is an indication of interesting behavior for some form of equivariant algebraic  $K$ -theory lying above these  $THH$  functors. A promising possibility is given in recent work of Barwick [Bar14], which provides a genuinely  $G$ -equivariant form of  $K(\mathbb{S})$  whose  $G$ -fixed points are  $\forall(BG)$ . It appears that the coassembly map we have studied lies below the map from this spectrum's fixed points to the homotopy fixed points

$$K(\mathbb{S})^G \longrightarrow K(\mathbb{S})^{hG}$$

In 1991 Waldhausen conjectured the existence of an equivariant form of  $K(\mathbb{S}) = A(*)$  for which the above map is an equivalence up to completion. This is a version of the Atiyah-Segal completion theorem, but for algebraic  $K$ -theory instead of topological  $K$ -theory. The work done so far on the  $THH$  level has consequences related to this conjecture, and the author intends to pursue this direction further.

### 1.3 Background on $K$ -theory and the cyclotomic trace

Though the bulk of our work is done on the level of  $THH$ , it is useful to know how such constructions relate back to interesting forms of  $K$ -theory, and we outline that here.

Most forms of algebraic  $K$ -theory seem to be special cases of Waldhausen  $K$ -theory, which is defined for any category  $\mathbf{C}$  equipped with a notion of *cofibrations* and *weak equivalences* satisfying the axioms found in [Wal85]. If  $\mathbf{M}$  is a model category, then the subcategory of cofibrant objects always forms such a “Waldhausen category.” We typically restrict to a subcategory obeying some finiteness condition, since otherwise the  $K$ -theory of  $\mathbf{C}$  would be trivial.

Given a Waldhausen category  $\mathbf{C}$ , one defines its 0th  $K$ -theory by taking the free abelian group on the weak equivalence classes of objects of  $\mathbf{C}$  and imposing the relation that  $b = a + c$

for every cofiber sequence  $a \rightarrow b \rightarrow c$ :

$$K_0(\mathbf{C}) = \mathbb{Z}\langle \text{ob}\mathbf{C} \rangle / \langle [b] = [a] + [c] : a \rightarrow b \rightarrow c \rangle$$

This generalizes via Waldhausen’s  $\mathcal{S}$ -construction to a sequence of groups  $K_0, K_1, \dots$ , which are the homotopy groups of a connective spectrum  $K(\mathbf{C})$ . The well-known Eilenberg Swindle guarantees that when the category  $\mathbf{C}$  contains arbitrary coproducts, this spectrum is contractible. However if  $\mathbf{C}$  consists only of “finite” objects then this spectrum often contains deep and interesting invariants of the category  $\mathbf{C}$ .

To give a specific example, let  $\mathcal{R}_{\text{fin}}(X)$  denote the category of retractive spaces  $Y$  over  $X$  for which the inclusion  $X \rightarrow Y$  is a classical cofibration, and up to homotopy equivalence it is obtained by attaching finitely many cells to  $X$ . The cofibrations and weak equivalences are defined as the maps which on the total space  $Y$  are classical cofibrations or homotopy equivalences, respectively. The  $K$ -theory of  $\mathcal{R}_{\text{fin}}(X)$  is then by definition Waldhausen’s functor  $A(X)$ , which has proven useful in the study of high-dimensional manifolds [Wal85].

Calculations in  $K$ -theory are often quite difficult, but calculus of functors has proven to be a surprisingly effective technique for understanding and calculating  $K$ -theory. Goodwillie constructed in [Goo90] a natural map from  $A(X)$  into the suspension spectrum of the free loop space

$$A(X) \longrightarrow \Sigma_+^\infty LX$$

Bökstedt, Hsiang, and Madsen showed in [BHM93] that this map lifts to genuine fixed points of the  $S^1$ -spectrum  $\Sigma_+^\infty LX$  under any finite cyclic subgroup  $C_n \leq S^1$ , and that these lifts can be made in a compatible way, resulting in a map

$$A(X) \longrightarrow TC(X) := \text{holim}_{F,R} (\Sigma_+^\infty LX)^{C_n}$$

This natural transformation of functors is an equivalence on first derivatives in the sense of Goodwillie calculus [Goo91], allowing one to use the very computable  $TC(X)$  to deduce a great deal about  $A(X)$  in the case where  $X$  is simply-connected. This analysis was successfully carried out in [BCC<sup>+</sup>96].

Now in the construction of Waldhausen’s  $A$ -theory of spaces, one may subtly change the definition by insisting that we study retractive spaces over  $X$  with finite homotopy *fiber*, instead of finite homotopy *cofiber*. To be specific, let  $\mathcal{R}^{\text{fin}}(X)$  be the category of retractive

spaces over  $X$  for which the projection  $Y \rightarrow X$  is a fibration, and for every  $x \in X$  the fiber  $Y_x$  is in  $\mathcal{R}_{\text{fin}}(*)$ . Define the cofibrations and weak equivalences by requiring that the map is a cofibration resp. weak equivalence in  $\mathcal{R}_{\text{fin}}(*)$  when restricted to the fiber over  $x$ , for every  $x \in X$ . The  $K$ -theory of  $\mathcal{R}^{\text{fin}}(X)$  is then (somewhat unceremoniously) called “upside-down  $A$ -theory”  $\forall(X)$ . Surprisingly, though the definition is “dual” to  $A(X)$ , relatively little is known about  $\forall(X)$ .

To apply modern methods it will be easiest to think of Waldhausen  $K$ -theory as coming from highly-structured ring spectra, as opposed to spaces. To be specific, let  $R$  be an orthogonal ring spectrum. Then the category of left  $R$ -modules has a model structure described in ([MMSS01], Thm 12.1(i)) where the weak equivalences and fibrations are given by forgetting the  $R$ -action. Restrict to the subcategory  $\mathcal{P}(R)$  consisting of cofibrant modules which are *perfect*, meaning that they lie in the smallest thick subcategory containing the ring  $R$  itself. These are equivalently the modules which are dualizable in a certain bicategorical sense, or the modules which are equivalent to retracts of finite cell complexes built out of the cells  $R \wedge F_n D_+^k$ . Then  $\mathcal{P}(R)$  is easily checked to be closed under homotopy pushouts, so it defines a Waldhausen category. We denote its Waldhausen  $K$ -theory by  $K(R)$ , and call it the  $K$ -theory of the ring spectrum  $R$  itself.

Furthermore, if  $R$  is an orthogonal ring spectrum then one may define its *topological Hochschild homology* by a cyclic bar construction in the category of orthogonal spectra

$$THH(R) := |N_{\bullet}^{\text{cyc}} R|, \quad N_k^{\text{cyc}} R = R^{\wedge(k+1)}$$

If the unit  $\mathbb{S} \rightarrow R$  is a cofibration then this construction is appropriately derived. In the stable homotopy category this receives a map from  $K(R)$ , which lifts to the genuine fixed points in a compatible way, yielding

$$K(R) \longrightarrow TC(R) := \text{holim}_{F,R} THH(R)^{C_n}$$

This map is often referred to as the *cyclotomic trace*.

It is well-known that when  $X$  is connected, the classical definition of  $A(X)$  above may be re-expressed as the  $K$ -theory of the ring spectrum  $\Sigma_+^{\infty} \Omega X$

$$K(\Sigma_+^{\infty} \Omega X) \simeq A(X)$$

when  $\Omega X$  is any topological monoid whose classifying space is equivalent to  $X$ . Furthermore the topological Hochschild homology of this ring is

$$THH(\Sigma_+^\infty \Omega X) \simeq \Sigma_+^\infty LX$$

and the trace analysis described above for  $A(X)$  coincides with the cyclotomic trace

$$K(\Sigma_+^\infty \Omega X) \longrightarrow TC(\Sigma_+^\infty \Omega X)$$

In the general case where  $X$  is not simply-connected,  $\forall(X)$  does not appear to be the  $K$ -theory of a ring. However one may take the Waldhausen category  $\mathcal{P}'(R)$  of all cofibrant left  $R$ -modules whose *underlying spectrum* is finite or dualizable. Then when  $R = \Sigma_+^\infty \Omega X$ ,  $K$ -theory  $K(\mathcal{P}'(\Sigma_+^\infty \Omega X))$  is equivalent to  $\forall(X)$ . Moreover, the derived mapping spectra between objects of  $\mathcal{P}'(R)$  give an enrichment in orthogonal spectra, and one may use a many-object form of the cyclic bar construction to define  $THH(\mathcal{P}'(R))$ . By [BM11c], which builds on earlier work in [DM96], the Dennis trace generalizes to a map

$$\forall(X) \simeq K(\mathcal{P}'(R)) \longrightarrow TC(\mathcal{P}'(R)) \longrightarrow THH(\mathcal{P}'(R))$$

There is no known analogue of the Dundas-McCarthy theorems telling us how close  $\forall \longrightarrow TC$  is to being an equivalence, but the simple existence of these maps has already turned out to be useful. In particular, they allow us to draw conclusions about  $\forall(BG)$  and its linear approximation from similar statements on the level of  $THH$ .

## Chapter 2

# A tower from gauge groups to string topology

This chapter represents work done in 2012 on an application of calculus of functors to string topology. In it we develop a variant of calculus of functors, and use it to relate the gauge group  $\mathcal{G}(\mathcal{P})$  of a principal bundle  $\mathcal{P}$  over  $M$  to the Thom ring spectrum  $(\mathcal{P}^{\text{Ad}})^{-TM}$ . If  $\mathcal{P}$  has contractible total space, the resulting Thom ring spectrum is  $LM^{-TM}$ , which plays a central role in string topology. Cohen and Jones have recently observed that, in a certain sense,  $(\mathcal{P}^{\text{Ad}})^{-TM}$  is the linear approximation of  $\mathcal{G}(\mathcal{P})$ . We prove an extension of that relationship by demonstrating the existence of higher-order approximations and calculating them explicitly. This also generalizes calculations done by Arone in [Aro99].

## 2.1 Introduction

If  $M$  is a closed oriented manifold and  $LM = \text{Map}(S^1, M)$  is its free loop space, then the homology  $H_*(LM)$  has a *loop product* first described by Chas and Sullivan [CS99]. This loop product is homotopy invariant [CKS08] and has been calculated in a number of examples [CJY04]. In [FT04], Félix and Thomas studied the loop product by defining a multiplication-preserving map

$$H_*(\Omega_{\text{id}}\text{haut}(M); \mathbb{Q}) \longrightarrow H_{*+\dim M}(LM; \mathbb{Q}) \quad (2.1)$$

where  $\text{haut}(M)$  is the space of self-homotopy equivalences of  $M$ , and the loops are based at the identity map of  $M$ .

In [CJ02], Cohen and Jones described a ring spectrum  $LM^{-TM}$  whose homology is  $H_*(LM)$  but with a grading shift. The multiplication on  $LM^{-TM}$  gives the loop product on  $H_*(LM)$ , and the map of Félix and Thomas (2.1) comes from a map of ring spectra

$$\Sigma_+^\infty \Omega_{\text{id}}\text{haut}M \longrightarrow LM^{-TM} \quad (2.2)$$

by taking rational homology groups [CJ13]. In the forthcoming paper [CJ13], Cohen and Jones extend this map of ring spectra to a *natural transformation of functors*

$$F \longrightarrow L \quad (2.3)$$

$$F, L : \mathcal{R}_M^{\text{op}} \longrightarrow \mathcal{S}p$$

$$F(M \amalg M) = \Sigma_+^\infty \Omega_{\text{id}}\text{haut}M$$

$$L(M \amalg M) \simeq LM^{-TM}$$

Here  $\mathcal{R}_M$  is the category of retractive spaces over  $M$  and  $\mathcal{S}p$  is the category of spectra. We will give these functors explicitly in section 2.3. Both  $F$  and  $L$  are required to be *homotopy functors*, meaning that they send equivalences of spaces to equivalences of spectra. Cohen and Jones show that  $L$  is the universal approximation of  $F$  by an *excisive* homotopy functor,

i.e. one that takes each homotopy pushout square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

to a homotopy pullback square

$$\begin{array}{ccc} L(A) & \longleftarrow & L(B) \\ \uparrow & & \uparrow \\ L(C) & \longleftarrow & L(D) \end{array}$$

Such an  $L$  takes finite sums of spaces to finite products of spectra. This type of analysis is similar in spirit to Goodwillie’s homotopy calculus of functors ([Goo90], [Goo03]), though it is different in substance because the functors  $F$  and  $L$  are contravariant. Instead, it is more similar to Weiss’s embedding calculus ([Wei99], [GW99]), though again it is different because  $F$  is defined on all spaces and not just manifolds and embeddings.

Of course, in homotopy calculus one approximates a functor  $F$  by an  $n$ -excisive functor  $P_n F$  for each integer  $n \geq 0$ . These fit into a tower

$$F \longrightarrow \dots \longrightarrow P_n F \longrightarrow \dots \longrightarrow P_2 F \longrightarrow P_1 F \longrightarrow P_0 F$$

and one extracts information about  $F$  from the layers

$$D_n F := \text{hofib}(P_n F \longrightarrow P_{n-1} F)$$

The map of functors (2.3) described by Cohen and Jones is only the first level of this tower:

$$F \longrightarrow P_1 F$$

The main goal of this paper is to extend their construction by building the rest of the tower. In order to do this we must also develop a variant of homotopy calculus for contravariant functors from spaces to spectra.

In Definition 2.2.2 we define  $n$ -excisive contravariant functors. Our main results on  $n$ -excisive functors are Theorems 2.7.1 and 2.8.8, which imply

**Theorem 2.1.1.** *Let  $F$  be a contravariant homotopy functor from  $\mathcal{C}$  to  $\mathcal{D}$ , where one of*

the following holds:

- $\mathcal{C}$  is the category of unbased finite CW complexes over  $M$ , and  $\mathcal{D}$  is the category of based spaces or spectra.
- $\mathcal{C}$  is the category of based finite CW complexes, and  $\mathcal{D}$  is the category of based spaces or spectra.
- $\mathcal{C}$  is the category of finite retractive CW complexes over  $M$ , and  $\mathcal{D}$  is the category of spectra.

Then there exists a universal  $n$ -excisive approximation to  $F$ , called  $P_n F$ , and the natural transformation  $F(X) \rightarrow P_n F(X)$  is an equivalence when  $X$  is a disjoint union of the initial object and  $i$  discrete points,  $0 \leq i \leq n$ .

**Remark.** It has been pointed out to the author that the procedure found in [dBW12] can be adapted to generalize embedding calculus from the category of manifolds to a broader category of topological spaces. This should give a result similar to Thm. 2.1.1.

In section 2.3 we explicitly define the functors of Cohen and Jones that extend the map of ring spectra (2.2), and in section 2.4.2 we explicitly calculate the tower that extends the map of Cohen and Jones. We explain in Proposition 2.2.5 why the above universal theorem is needed to conclude that our tower is correct. Along the way to proving Theorems 2.7.1 and 2.8.8, we prove a splitting result on homotopy limits in Proposition 2.6.7 that is reminiscent of a result of Dwyer and Kan ([DK80]) on mapping spaces of diagrams. This all implies the main result of the paper:

**Theorem 2.1.2.** *There is a tower of homotopy functors*

$$F \longrightarrow \dots \longrightarrow P_n F \longrightarrow \dots \longrightarrow P_2 F \longrightarrow P_1 F \longrightarrow P_0 F$$

from finite retractive CW complexes over  $M$  into spectra such that

1.  $P_n F$  is the universal  $n$ -excisive approximation of  $F$ .
2. The map  $F \rightarrow P_1 F$  is the map (2.3) of Cohen and Jones.
3.  $F(M \amalg M) \cong \Sigma_+^\infty \Omega_{\text{id}} \text{haut} M$ .
4.  $P_1 F(M \amalg M) \simeq LM^{-TM}$ .

5.  $P_0F(M \amalg M) = *$ .

6. If  $X$  is any finite retractive CW complex over  $M$ , the maps

$$F(X) \longrightarrow P_nF(X) \longrightarrow P_{n-1}F(X)$$

are maps of ring spectra.

7. For all  $n \geq 1$ ,  $D_nF(M \amalg M)$  is equivalent to the Thom spectrum

$$\mathcal{C}(LM; n)^{-TC(M; n)}$$

Here  $C(M; n)$  is the space of unordered configurations of  $n$  points in  $M$ , and  $\mathcal{C}(LM; n)$  is the space of unordered collections of  $n$  free loops in  $M$  with distinct basepoints.

Cohen and Jones have also observed that this linearization phenomenon is more general. Consider any principal bundle

$$G \longrightarrow \mathcal{P} \longrightarrow M$$

The *gauge group*  $\mathcal{G}(\mathcal{P})$  is defined to be the space of automorphisms of  $\mathcal{P}$  as a principal bundle. It is a classical fact that there is an associated *adjoint bundle*  $\mathcal{P}^{\text{Ad}}$ , and that the gauge group  $\mathcal{G}(\mathcal{P})$  may be identified with the space of sections  $\Gamma_M(\mathcal{P}^{\text{Ad}})$ .

Gruher and Salvatore show in [GS08] that one may construct a Thom ring spectrum  $(\mathcal{P}^{\text{Ad}})^{-TM}$  out of the total space  $\mathcal{P}^{\text{Ad}}$  of the adjoint bundle. The multiplication on this ring spectrum gives a product on the homology  $H_*(\mathcal{P}^{\text{Ad}})$ . When the total space of  $\mathcal{P}$  is contractible, the adjoint bundle  $\mathcal{P}^{\text{Ad}}$  is equivalent to the free loop space  $LM$ , and the Gruher-Salvatore product on  $H_*(\mathcal{P}^{\text{Ad}})$  agrees with the Chas-Sullivan loop product on  $H_*(LM)$ .

In [CJ13], Cohen and Jones show that the map (2.2) of ring spectra generalizes to a map of ring spectra

$$\Sigma_+^\infty \mathcal{G}(\mathcal{P}) \longrightarrow (\mathcal{P}^{\text{Ad}})^{-TM} \tag{2.4}$$

Taking homology groups, they get a multiplication-preserving map

$$H_*(\mathcal{G}(\mathcal{P})) \longrightarrow H_{*+\dim M}(\mathcal{P}^{\text{Ad}}) \tag{2.5}$$

which generalizes the map (2.1) studied by Félix and Thomas. Cohen and Jones extend this generalized map of ring spectra to a map of functors  $F \longrightarrow L$  and show that  $L$  is the

universal approximation of  $F$  by an excisive functor. We extend their generalized result here as well:

**Theorem 2.1.3.** *There is a tower of homotopy functors*

$$F \longrightarrow \dots \longrightarrow P_n F \longrightarrow \dots \longrightarrow P_2 F \longrightarrow P_1 F \longrightarrow P_0 F$$

from finite retractive CW complexes over  $M$  into spectra such that

1.  $P_n F$  is the universal  $n$ -excisive approximation of  $F$ .
2. The map  $F \longrightarrow P_1 F$  is the generalized map of Cohen and Jones.
3.  $F(M \amalg M) \cong \Sigma_+^\infty \mathcal{G}(\mathcal{P})$ .
4.  $P_1 F(M \amalg M) \simeq (\mathcal{P}^{\text{Ad}})^{-TM}$ .
5.  $P_0 F(M \amalg M) = *$ .
6. If  $X$  is any finite retractive CW complex over  $M$ , the maps

$$F(X) \longrightarrow P_n F(X) \longrightarrow P_{n-1} F(X)$$

are maps of ring spectra.

7. For all  $n \geq 1$ ,  $D_n F(M \amalg M)$  is equivalent to the Thom spectrum

$$\mathcal{C}(\mathcal{P}^{\text{Ad}}; n)^{-TC(M;n)}$$

where  $\mathcal{C}(\mathcal{P}^{\text{Ad}}; n)$  is the space of unordered configurations of  $n$  points in the total space  $\mathcal{P}^{\text{Ad}}$  which have distinct images in  $M$ .

The outline of the chapter is as follows. In Section 2, we define  $n$ -excisive functors and give criteria for recognizing the universal  $n$ -excisive approximation  $P_n F$  of a given functor  $F$ . In Section 3, we give a detailed construction of a tower which generalizes the above two examples. In Section 4, we specialize to the above two examples and do some computations. Sections 5-8 supply a missing ingredient from the previous sections: a functorial construction of  $P_n F$  for general  $F$ . This material may be of independent interest in the general study of calculus of functors.

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## 2.2 Excisive Functors

Fix an unbased space  $B$ . Like every space that follows, we assume it is compactly generated and weak Hausdorff.

**Definition 2.2.1.** • Let  $\mathcal{U}_B$  be the category of spaces over  $B$ . The objects are spaces  $X$  equipped with maps  $X \rightarrow B$ . Define two subcategories

$$\mathcal{U}_{B,n} \subset \mathcal{U}_{B,\text{fin}} \subset \mathcal{U}_B$$

as follows. The subcategory  $\mathcal{U}_{B,\text{fin}}$  consists of all finite CW complexes over  $B$ . The subcategory  $\mathcal{U}_{B,n}$  consists of discrete spaces with at most  $n$  points over  $B$ . For simplicity, we may as well assume that the finite CW complexes are embedded in  $B \times \mathbb{R}^\infty$ , and that  $\mathcal{U}_{B,n}$  has only one space with  $i$  points for each  $0 \leq i \leq n$ .

• Let  $\mathcal{R}_B$  be the category of spaces containing  $B$  as a retract. As before, define two subcategories

$$\mathcal{R}_{B,n} \subset \mathcal{R}_{B,\text{fin}} \subset \mathcal{R}_B$$

where  $\mathcal{R}_{B,\text{fin}}$  consists of spaces  $X$  for which  $(X, B)$  is a finite relative CW complex, and  $\mathcal{R}_{B,n}$  consists of spaces of the form  $\underline{i} \amalg B$ ,  $\underline{i} = \{1, \dots, i\}$ , for  $0 \leq i \leq n$ .

Of course, if  $B = *$  then  $\mathcal{U}_B$  and  $\mathcal{R}_B$  are the familiar categories of unbased spaces  $\mathcal{U}$  and based spaces  $\mathcal{T}$ , respectively. The following definition should be seen as an analogue of Goodwillie's notion of  $n$ -excisive for covariant functors [Goo91]:

**Definition 2.2.2.** A contravariant functor  $\mathcal{R}_B^{\text{op}} \xrightarrow{F} \mathcal{T}$  is  $n$ -excisive if

- $F$  is a *homotopy functor*, meaning weak equivalences  $X \xrightarrow{\sim} Y$  of spaces containing  $B$  as a retract are sent to weak equivalences  $F(Y) \xrightarrow{\sim} F(X)$  of based spaces.
- $F$  takes *strongly co-Cartesian cubes* (equivalently, *pushout cubes*) of dimension at least  $n + 1$  to *Cartesian cubes* (see [Goo91]). When  $n = 1$ , this means that  $F$  takes homotopy pushout squares to homotopy pullback squares.

- $F$  takes filtered homotopy colimits to homotopy limits. In particular,  $F$  is determined up to equivalence by its behavior on relative finite CW complexes  $B \hookrightarrow X$ .

This definition is easily modified to suit many cases. When restricting to finite CW complexes  $(\mathcal{R}_{B,\text{fin}})$ , we drop the last condition. If  $\mathcal{S}p$  denotes a suitable model category of spectra, for example the category of prespectra described in [MMSS01], then a contravariant functor  $\mathcal{R}_B^{\text{op}} \xrightarrow{F} \mathcal{S}p$  is *n-excise* if it satisfies the above properties, with “equivalence of based spaces” replaced by “stable equivalence of spectra.” For most models of spectra, we are allowed to post-compose  $F$  with a fibrant replacement functor, and conclude that  $F$  is an *n-excise* functor to spectra iff each level  $F_j$  is an *n-excise* functor to based spaces. It is also straightforward to define *n-excise* for functors from unbased spaces  $\mathcal{U}_B$  or unbased finite spaces  $\mathcal{U}_{B,\text{fin}}$  to either spaces  $\mathcal{T}$  or spectra  $\mathcal{S}p$ .

If  $F$  is a contravariant *n-excise* functor, then  $F$  is completely determined by its values on the discrete spaces with at most  $n$  points:

**Proposition 2.2.3.** • *Let  $F$  and  $G$  be  $n$ -excise functors  $\mathcal{R}_B^{\text{op}} \rightarrow \mathcal{T}$ , and  $F \xrightarrow{\eta} G$  a natural transformation. If  $\eta$  is an equivalence when restricted to the subcategory  $\mathcal{R}_{B,n}^{\text{op}}$ , then  $\eta$  is also an equivalence on the rest of  $\mathcal{R}_B^{\text{op}}$ .*

- *If  $F$  and  $G$  are  $n$ -excise functors  $\mathcal{U}_B^{\text{op}} \rightarrow \mathcal{T}$ , and  $F \xrightarrow{\eta} G$  is an equivalence on  $\mathcal{U}_{B,n}^{\text{op}}$ , then  $\eta$  is also an equivalence on  $\mathcal{U}_B^{\text{op}}$ .*
- *The obvious analogues hold when the source of  $F$  and  $G$  is instead the category of finite CW complexes  $\mathcal{U}_{B,\text{fin}}$  or  $\mathcal{R}_{B,\text{fin}}$ , or when the target is spectra  $\mathcal{S}p$  instead of spaces  $\mathcal{T}$ .*

*Proof.* We will only prove the first statement, by induction on the dimension of the relative CW complex  $B \rightarrow X$ . The key fact is that a map of Cartesian cubes is an equivalence on the initial vertex if it is an equivalence on all the others.

We may construct a pushout  $(n+1)$ -cube whose final vertex is

$$\underline{n+1} \amalg B$$

but all other vertices are  $\underline{k} \amalg B$  with  $k \leq n$ . Applying  $F$  and  $G$  to this pushout cube gives two Cartesian cubes, and  $\eta$  gives a map between the two Cartesian cubes. This map is an equivalence on every vertex but the initial one, so it is an equivalence on the initial vertex as well:

$$\eta : F(\underline{n+1} \amalg B) \xrightarrow{\sim} G(\underline{n+1} \amalg B)$$

Similarly we may show that  $\eta$  is an equivalence on all spaces of the form  $\underline{k} \amalg B$ . Therefore  $\eta$  is an equivalence on all finite 0-dimensional complexes. For higher dimensional complexes, we need an additional definition.

For each subset  $T \subset \{1, 2, \dots, n\}$ , define the *layer cake space*  $L_T^d$  to be the subspace of the closed  $d$ -dimensional unit cube  $[0, 1]^d$  consisting of those points whose final coordinate is in the set

$$\left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\right\} \cup \{t : \lceil nt \rceil \in T\}$$

So  $L_{\{1, \dots, n\}}^d$  is the entire cube, while  $L_\emptyset^d$  is homotopy equivalent to  $n + 1$  copies of  $D^{p-1}$  glued along their boundaries. Intuitively,  $L_T^d$  consists of all the frosting in a layer cake, together with a selection of layers given by  $T$ . If  $T$  is a proper subset, then  $L_T^d$  is homotopy equivalent rel  $\partial I^d$  to  $\partial I^d$  with some  $(d - 1)$ -cells attached.

Now assume that  $\eta$  is an equivalence on all finite  $(d - 1)$ -dimensional complexes. Let  $X$  be a  $d$ -dimensional finite complex, with top-dimensional attaching maps  $\{\partial I^d \xrightarrow{\varphi_\alpha} X^{(d-1)}\}_{\alpha \in A}$ . Form an  $(n + 1)$ -dimensional pushout cube with the following description. The initial vertex is  $\coprod_A L_\emptyset^d$ , a disjoint union of one empty layer cake for each  $d$ -cell of  $X$ . Next, let  $n$  of the  $n + 1$  adjacent vertices be  $\coprod_A L_{\{i\}}^d$  as  $i$  ranges over  $\{1, \dots, n\}$ . Finally, let the last adjacent vertex be the pushout of  $X^{(d-1)}$  and  $\coprod_A L_\emptyset^d$  along  $\coprod_A \partial I^d$ . Then the final vertex of our pushout cube is homeomorphic to  $X$ , while every vertex other than the final one is homotopy equivalent to a  $(d - 1)$ -dimensional cell complex. After applying  $F$  and  $G$ ,  $\eta$  gives us a map between two Cartesian cubes, and the map is an equivalence on every vertex but the initial one. So  $F(X) \xrightarrow{\eta} G(X)$  is an equivalence as well, completing the induction.

If the source category of  $F$  and  $G$  has infinite CW complexes, we express each CW complex as a filtered homotopy colimit of its finite subcomplexes and invoke the colimit axiom. To move to all spaces, we recall that  $F$  and  $G$  preserve weak equivalences, and that every space over  $B$  has a functorial CW approximation.  $\square$

Now suppose  $F$  is a contravariant homotopy functor. We want to define a “best possible” approximation of  $F$  by an  $n$ -excisive functor. By this we mean an  $n$ -excisive functor  $P_n F$  with the same source and target as  $F$ , and a natural transformation  $F \longrightarrow P_n F$  that is

universal among all maps  $F \rightarrow P$  from  $F$  into an  $n$ -excisive functor  $P$ :

$$\begin{array}{ccc}
 F & \longrightarrow & P \\
 \downarrow & \nearrow & \\
 P_n F & & 
 \end{array}$$

We will relax this condition to take place in the *homotopy category* of functors. Following [Goo03], we get this homotopy category by formally inverting the *equivalences* of functors.

**Definition 2.2.4.** An *equivalence* of functors is a natural transformation  $F \rightarrow G$  that yields equivalences  $F(X) \xrightarrow{\sim} G(X)$  for all spaces  $X$ .

Unfortunately, this homotopy category of functors has significant set-theory issues. First of all, the category of all functors from spaces to spaces is not really a category in the usual sense. This is because when we choose two functors  $F$  and  $G$ , the collection of all natural transformations  $F \rightarrow G$  forms a proper class. In other words, the category of functors has large hom-sets. The homotopy category of functors has even larger hom-sets [Goo03].

One way of resolving this issue is to restrict to *small* functors as defined in [CD06]. The small functors form a model category, so their homotopy category has small hom-sets.

We will use a different fix, since we are ultimately interested in a result about compact manifolds. We will restrict our attention to functors defined on finite CW complexes ( $\mathcal{U}_{B,\text{fin}}$  or  $\mathcal{R}_{B,\text{fin}}$ ) instead of all spaces ( $\mathcal{U}_B$  or  $\mathcal{R}_B$ ). Finite CW complexes over  $B$  can always be embedded into  $B \times \mathbb{R}^\infty$ , so we can easily make  $\mathcal{U}_{B,\text{fin}}$  and  $\mathcal{R}_{B,\text{fin}}$  into small categories. Then the category of functors from  $\mathcal{U}_{B,\text{fin}}$  or  $\mathcal{R}_{B,\text{fin}}$  into spaces or spectra has the projective model structure, as discussed below in section 2.3.1. Therefore our homotopy category of functors has small hom-sets.

Now that we are on solid footing, we can return to the problem of finding a universal  $n$ -excisive approximation  $P_n F$  to the homotopy functor  $F$ . It turns out that  $P_n F$  will actually *agree* with  $F$  on the spaces with at most  $n$  points. This is similar to embedding calculus ([Wei99], [GW99]) but quite different from the case for covariant functors ([Goo03]). Extending the calculus analogy, we are calculating not a Taylor series but a polynomial interpolation: we sample our functor  $F$  at  $(n + 1)$  particular homotopy types  $\underline{0}, \dots, \underline{n}$  and then we build the unique degree  $n$  polynomial  $P_n F$  that has the same values on those  $(n + 1)$  homotopy types.

So let  $F$  be any contravariant homotopy functor from finite CW complexes ( $\mathcal{R}_{B,\text{fin}}$  or

$\mathcal{U}_{B,\text{fin}}$ ) to either based spaces or spectra. (There is one exception to this setup, as explained in section 2.7.) In sections 2.5, 2.7, and 2.8.2 below we will define a functor  $P_n F$  with the same source and target as  $F$ , and a natural transformation  $p_n F : F \rightarrow P_n F$ , both functorial in  $F$ . Then we will show two things:

- $P_n F$  is  $n$ -excisive.
- $F \rightarrow P_n F$  is an equivalence on  $\mathcal{R}_{B,n}^{\text{op}}$  (based case) or  $\mathcal{U}_{B,n}^{\text{op}}$  (unbased case).

**Proposition 2.2.5.** *If  $F \rightarrow P_n F$  is a functorial construction satisfying the above properties, then  $P_n F$  is universal among all  $n$ -excisive  $P$  with natural transformations  $F \rightarrow P$  in the homotopy category of functors.*

*Proof.* Easy adaptation of ([Goo03], 1.8). □

**Corollary 2.2.6** (Recognition Principle for  $P_n F$ ). *Given that such a construction  $P_n$  exists, if  $P$  is any  $n$ -excisive functor with a map  $F \rightarrow P$  that is an equivalence on  $\mathcal{R}_{B,n}^{\text{op}}$  or  $\mathcal{U}_{B,n}^{\text{op}}$ , then  $P$  is canonically equivalent to  $P_n F$ .*

*Proof.* By the universal property of  $P_n F$  there exists a unique map  $P_n F \rightarrow P$ , but this is a map of  $n$ -excisive functors and an equivalence on  $\mathcal{R}_{B,n}^{\text{op}}$  or  $\mathcal{U}_{B,n}^{\text{op}}$ , so it's an equivalence of functors. □

**Remark.** The above recognition argument applies equally well to the case of covariant functors, which is alarming, because in that case  $F \rightarrow P_n F$  is usually not an equivalence on the spaces with at most  $n$  points. The only possible conclusion is that, for covariant functors, there is no construction  $P_n$  satisfying the above properties.

We will delay the construction of  $P_n F$  to section 2.5. In the next section, we will apply this recognition theorem in a particular example.

### 2.3 The Tower of Approximations of a Mapping Space

Now we will compute the tower of universal  $n$ -excisive approximations of the functor

$$F(X) = \Sigma^\infty \text{Map}_B(X, E)$$

from retractive spaces over  $B$  to spectra. The map of Cohen and Jones described in the introduction is the special case  $X = M \amalg M$ ,  $B = M$ , and  $E = LM \amalg M$ . Our results in

this section are proven using techniques from model categories, so we will fix some notation for this following [MMSS01] and [MS06].

**Definition 2.3.1.** Let  $X$  and  $Y$  be unbased spaces over  $B$ , or retractive spaces over  $B$ .

- A  $q$ -cofibration  $X \rightarrow Y$  is a retract of a relative cell complex.
- A  $q$ -fibration  $X \rightarrow Y$  is a Serre fibration.
- An  $h$ -cofibration  $X \rightarrow Y$  is a map of spaces satisfying the homotopy extension property.
- An  $h$ -fibration  $X \rightarrow Y$  is a Hurewicz fibration.

We should also be precise about our definition of  $F(X) = \Sigma^\infty \text{Map}_B(X, E)$ .

**Definition 2.3.2.** • If  $E$  is a retractive space over  $B$ , let  $\Sigma_B E$  denote the *fiberwise reduced suspension* of  $E$ .

- An *ex-fibration* is a retractive space  $E$  over  $B$  such that  $E \rightarrow B$  is a Hurewicz fibration, and  $B \rightarrow E$  is well-behaved in a sense described in ([MS06], 8.2). For our purposes, the most important property of an ex-fibration  $E$  is that the fiberwise reduced suspension  $\Sigma_B E$  is again an ex-fibration.
- If  $X$  is a  $q$ -cofibrant retractive space over  $B$  and  $E$  is an ex-fibration over  $B$ , let  $\text{Map}_B(X, E)$  denote the space of maps  $X \rightarrow E$  respecting the maps into and out of  $B$ . If  $B$  is compact or  $X$  is finite CW then this space is well-based. If not, grow a whisker so that  $\Sigma^\infty$  will preserve equivalences.
- Similarly, let  $\text{Map}_B(X, \Sigma_B^\infty E)$  denote a spectrum whose  $k$ th level is fiberwise maps from  $X$  into  $\Sigma_B^k E$ .

Now we will build up to the definition of the functors that approximate  $F$ . Let  $\mathbf{M}_n$  be the category whose objects are the finite unbased sets  $\underline{0} = \emptyset, \underline{1} = *, \underline{2} = \{1, 2\}, \dots, \underline{n} = \{1, \dots, n\}$  and whose morphisms are the *surjective* maps. For any space  $X$ , we can form a diagram of unbased spaces indexed by the opposite category  $\mathbf{M}_n^{\text{op}}$ :

$$\underline{i} \mapsto X^i$$

The morphisms in this diagram are clear when we think of  $X^i$  as  $\text{Map}(\underline{i}, X)$ . So algebraically, this diagram is the functor represented by  $X$ . Geometrically, this is a diagram of generalized diagonal maps: each map  $\underline{i} \rightarrow \underline{i-1}$  results in a map  $X^{i-1} \rightarrow X^i$  whose image consists of points in which a particular pair of coordinates is repeated. The union of all such images the *fat diagonal*, which we will denote

$$\Delta \subset X^i$$

**Definition 2.3.3.** Let  $X$  be a  $q$ -cofibrant retractive space over  $B$  and let  $E$  be an ex-fibration over  $B$ .

- Let  $X \overline{\wedge} X$  denote the *external smash product* of  $X$  with itself; this is a retractive space over  $B \times B$  whose fiber over  $(b_1, b_2)$  is  $X_{b_1} \wedge X_{b_2}$ . More generally, if  $Y$  is a retractive space over  $C$  then  $X \overline{\wedge} Y$  is a retractive space over  $B \times C$  whose fiber over  $(b, c)$  is  $X_b \wedge Y_c$ .
- Let  $X^{\overline{\wedge} n}$  denote the  $n$ -fold iterated external smash product. It is a retractive space over  $B^n$ .
- Define

$$\text{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})}(X^{\overline{\wedge} i}, \Sigma_{B^i}^\infty E^{\overline{\wedge} i})$$

to be the spectrum whose  $k$ th level is collections of maps of retractive spaces

$$\begin{array}{ccccccc} \Sigma^{k*} & \Sigma_B^k E & \dots & \Sigma_{B^n}^k E^{\overline{\wedge} n} \\ \uparrow f_0 & \uparrow f_1 & & \uparrow f_n \\ * & X & \dots & X^{\overline{\wedge} n} \end{array}$$

such that each surjective map  $\underline{i} \leftarrow \underline{j}$  gives a commuting square

$$\begin{array}{ccc} \Sigma_{B^i}^k E^{\overline{\wedge} i} & \longrightarrow & \Sigma_{B^j}^k E^{\overline{\wedge} j} \\ \uparrow f_i & & \uparrow f_j \\ X^{\overline{\wedge} i} & \longrightarrow & X^{\overline{\wedge} j} \end{array}$$

**Remark.** Note that the collection of maps  $(f_0, f_1, \dots, f_n)$  is completely determined by the last map  $f_n$ , which must be  $\Sigma_n$ -equivariant. When  $n \geq 3$ , not every  $\Sigma_n$ -equivariant map

arises this way.

**Remark.** One might expect  $S^0 \rightarrow \Sigma^k S^0$  in the place of  $* \rightarrow \Sigma^k *$ , since an empty smash product is  $S^0$ . This answer would give the approximation to the functor  $F \vee \mathbb{S}$  instead of  $F$ . A similar phenomenon happens in Cor. 2.8.6 below.

Now we have defined a tower of functors

$$\begin{array}{rcl}
 F(X) & = & \Sigma^\infty \text{Map}_B(X, E) \\
 \downarrow & & \\
 \vdots & & \vdots \\
 P_n F(X) & = & \text{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})}(X^{\bar{\wedge} i}, \Sigma_{B^i}^\infty E^{\bar{\wedge} i}) \\
 \vdots & & \vdots \\
 \downarrow & & \\
 P_2 F(X) & = & \text{Map}_{B \times B}(X \bar{\wedge} X, \Sigma_{B \times B}^\infty E \bar{\wedge} E)^{\Sigma_2} \\
 \downarrow & & \\
 P_1 F(X) & = & \text{Map}_B(X, \Sigma_B^\infty E) \\
 \downarrow & & \\
 P_0 F(X) & = & *
 \end{array}$$

on the category of finite retractive CW complexes over  $B$ . We will justify the notation with Theorem 2.3.13, which shows that  $P_n F(X)$  is the universal  $n$ -excisive approximation to  $F(X)$ . This is a generalization of an observation made by Greg Arone about the tower in [Aro99].

**Remark.** It is more natural to examine the functor  $X \mapsto \text{Map}_B(X, E)$  first, before applying  $\Sigma^\infty$  to it. But this functor is already 1-excisive, so it does not give an interesting tower. It is also natural to consider

$$\hat{F}(X) = \Sigma^\infty \text{Map}_B(X, E)$$

for unbased  $X$  over  $B$ , without a basepoint section. But  $\hat{F}(X) = F(X \amalg B)$ , so  $P_n \hat{F}(X) = P_n F(X \amalg B)$  by comparing universal properties. Therefore the case of  $F$  on  $\mathcal{R}_B^{\text{op}}$  is more general than the case of  $\hat{F}$  on  $\mathcal{U}_B^{\text{op}}$ . This is true in general when the desired functor on  $\mathcal{U}_B^{\text{op}}$  extends to a functor on  $\mathcal{R}_B^{\text{op}}$ .

### 2.3.1 Cell Complexes of Diagrams

Many of the proofs that follow rely on the same basic idea: we start with a diagram of spaces or spectra that is built inductively out of cells, and we define maps of diagrams one cell at a time. In doing so, we are using the following standard facts. First, both spaces and spectra have compactly generated model structures [MMSS01]. Therefore the category of diagrams indexed by  $\mathbf{I}$  can be endowed with the *projective model structure*. The weak equivalences  $F \rightarrow G$  of diagrams are the maps that give objectwise equivalences  $F(i) \xrightarrow{\sim} G(i)$ , and the fibrations are the objectwise ( $q$ -)fibrations. The projective model structure is again compactly generated.

To understand the cofibrant diagrams, define a functor that takes a based space (or spectrum)  $X$  and produces the diagram

$$F_i(X)(j) = \mathbf{I}(i, j)_+ \wedge X$$

A map of diagrams  $F_i(X) \rightarrow G$  is the same as a map of spaces (or spectra)  $X \rightarrow G(i)$ . This property is clearly useful for defining maps of diagrams one cell at a time. We can define a *diagram cell* by applying  $F_i$  to the maps  $S_+^{n-1} \rightarrow D_+^n$ , and then define a *diagram CW complex* as an appropriate iterated pushout of diagram cells. Every diagram CW complex is cofibrant in the projective model structure. More generally, if we weaken the definition from relative CW complexes to retracts of relative cell complexes, we get all of the cofibrations in the projective model structure.

We will now apply these ideas and check that a certain diagram is cofibrant in the projective model structure. Recall that  $\mathbf{M}_n$  is the category with one object  $\underline{i} = \{1, \dots, i\}$  for each integer  $0 \leq i \leq n$ , with maps  $\underline{i} \rightarrow \underline{j}$  the surjective maps of sets. The maps are not required to preserve ordering, so in particular  $\mathbf{M}_n(\underline{i}, \underline{i}) \cong \Sigma_i$ , the symmetric group on  $i$  letters.

**Proposition 2.3.4.** *If  $X$  is a based CW complex, then  $\{X^{\wedge i}\}_{i=0}^n$  is a CW complex of  $\mathbf{M}_n^{\text{op}}$  diagrams. Similarly for Cartesian products  $\{X^i\}$ . If  $X$  is  $q$ -cofibrant then  $\{X^{\wedge i}\}$  and  $\{X^i\}$  are cofibrant diagrams.*

*Proof.* It suffices to put a new cell complex structure on  $X^{\wedge n}$  so that the fat diagonal is a subcomplex, and every cell outside of the fat diagonal is permuted freely by the  $\Sigma_n$ -action. We will do the case where  $X$  has a single cell, since the general case follows easily.

The product  $\prod^n D^m \cong \prod^n [0, 1]^m$  may be identified with the space of all  $n \times m$  matrices, with real entries between 0 and 1. The  $\Sigma_n$  action permutes the rows. Divide this space into open simplices as follows. We define a simplex of dimension  $d$  for each partition of the  $nm$  entries of the matrix into  $d$  nonempty equivalence classes, along with a choice of total ordering on the equivalence classes. This simplex corresponds to the subspace of matrices for which the equivalent entries have the same value, and the values are ordered according to the chosen total ordering.

The closures of these simplices give a triangulation of the cube  $\prod^n D^m \cong \prod^n [0, 1]^m$ . Each generalized diagonal in  $(D^m)^n$  is defined by setting an equivalence relation on the rows of the matrix, and requiring that equivalent rows have the same values. This is clearly an intersection of conditions we used to define the simplices above, so each generalized diagonal is a union of simplices. In addition, the simplices off the fat diagonal are freely permuted by the  $\Sigma_n$  action. This finishes the proof.

For the last statement in the proposition, a general  $q$ -cofibration is a retract of a relative cell complex, but retracts of maps of spaces clearly give retracts of maps of diagrams, so we are done.  $\square$

**Proposition 2.3.5.** *If  $X$  is a based CW complex and  $A$  is a subcomplex then  $\{A^{\wedge i}\} \rightarrow \{X^{\wedge i}\}$  is a relative CW complex of  $\mathbf{M}_n^{\text{op}}$  diagrams. If  $* \rightarrow A \rightarrow X$  are  $q$ -cofibrations then  $\{A^{\wedge i}\} \rightarrow \{X^{\wedge i}\}$  is a cofibration of  $\mathbf{M}_n^{\text{op}}$  diagrams.*

*Proof.* Each cell of  $X^{\wedge i}$  lying outside  $A^{\wedge i}$  is a product of cells in  $X$ , at least one of which is not a cell in  $A$ . As above, we subdivide each of these cells so that  $\Delta \cup A^{\wedge i}$  is a subcomplex when  $\Delta$  is any of the generalized diagonals. Off the fat diagonal, the  $\Sigma_i$  action still freely permutes the cells. This gives the recipe for building the map of diagrams  $\{A^{\wedge i}\} \rightarrow \{X^{\wedge i}\}$  out of free cells of diagrams.

Suppose that  $* \rightarrow A \rightarrow X$  are  $q$ -cofibrations, and we want to show that  $\{A^{\wedge i}\} \rightarrow \{X^{\wedge i}\}$  is a cofibration of diagrams. Then without loss of generality we can replace  $A \rightarrow X$  by a relative cell complex  $A \rightarrow X'$ . Then we can replace  $* \rightarrow A$  by a relative cell complex  $* \rightarrow A'$ , and we get the sequence of relative cell complexes  $* \rightarrow A' \rightarrow X' \cup_A A'$  containing  $* \rightarrow A \rightarrow X$  as a retract. Then we apply the same argument as above.  $\square$

**Proposition 2.3.6.** *If  $X$  is a retractive CW complex over  $B$  then  $\{B^i\} \rightarrow \{X^{\bar{\wedge} i}\}$  is a relative CW complex of  $\mathbf{M}_n^{\text{op}}$  diagrams. If  $B \rightarrow A \rightarrow X$  are  $q$ -cofibrations over  $B$  then  $\{A^{\bar{\wedge} i}\} \rightarrow \{X^{\bar{\wedge} i}\}$  is a cofibration of  $\mathbf{M}_n^{\text{op}}$  diagrams.*

*Proof.* We must verify that  $B^i \hookrightarrow X^{\wedge i}$  is a relative cell complex with one cell for each  $i$ -tuple of relative cells of  $B \hookrightarrow X$ . This is a straightforward adaptation of standard arguments, but it is worth pointing out that these arguments derail if we don't work in the category of compactly generated weak Hausdorff spaces. Once we are assured that everything is a cell complex, the rest of the proof follows as above.  $\square$

Recall that an *acyclic cofibration* (of spaces, spectra, or diagrams) is a map that is both a ( $q$ -)cofibration and a weak equivalence. An *acyclic cell* of spaces is a map  $D^n \times \{0\} \hookrightarrow D^n \times I$  for some  $n \geq 0$ . Every acyclic cofibration of spaces is a retract of a cell complex built out of these acyclic cells [MMSS01]. Similarly, an acyclic cell of diagrams is what we get by applying  $F_i$  to the map  $D^n \times \{0\} \hookrightarrow D^n \times I$ , and every acyclic cofibration of diagrams is a retract of a relative complex built out of these acyclic cells. With this language, we now give the following result:

**Corollary 2.3.7.** *If  $* \rightarrow A \rightarrow X$  are  $q$ -cofibrations and  $A \rightarrow X$  is acyclic then  $\{A^{\wedge i}\} \rightarrow \{X^{\wedge i}\}$  is an acyclic cofibration of  $\mathbf{M}_n^{\text{op}}$  diagrams. Similarly for Cartesian products.*

*Proof.* Since  $A$  and  $X$  are  $q$ -cofibrant they are well-based (i.e.  $* \rightarrow A$  is an  $h$ -cofibration). Therefore since  $A \rightarrow X$  is a weak equivalence,  $A^{\wedge i} \rightarrow X^{\wedge i}$  is a weak equivalence as well.  $\square$

**Corollary 2.3.8.** *Each acyclic cofibration  $A \rightarrow X$  of  $q$ -cofibrant retractive spaces over  $B$  induces a acyclic cofibration of  $\mathbf{M}_n^{\text{op}}$  diagrams  $\{A^{\wedge i}\} \rightarrow \{X^{\wedge i}\}$ .*

*Proof.* Again, we just need to show that  $A^{\wedge i} \rightarrow X^{\wedge i}$  is a weak equivalence of total spaces. The case where  $i = 2$  generalizes easily. Let  $H_A$  be the homotopy pushout of

$$\begin{array}{ccc} & B \times B & \\ & \uparrow & \\ (A \times B) \cup_{B \times B} (B \times A) & \longrightarrow & A \times A \end{array}$$

Then  $H_A$  is equivalent to the strict pushout  $A \overline{\wedge} A$ , because the bottom map is an  $h$ -cofibration. This gives a square

$$\begin{array}{ccc} H_A & \xrightarrow{\sim} & H_X \\ \downarrow \sim & & \downarrow \sim \\ A \overline{\wedge} A & \longrightarrow & X \overline{\wedge} X \end{array}$$

from which we see that the bottom map is an equivalence. For  $i > 2$  simply replace one of the two copies of  $A$  by the space  $A^{\overline{\wedge}(i-1)}$ .  $\square$

### 2.3.2 Proof that the Tower is Correct

**Proposition 2.3.9.**  $F(X) = \Sigma^\infty \text{Map}_B(X, E)$  as defined above in 2.3.3 takes weak equivalences between  $q$ -cofibrant retractive spaces over  $B$  to level equivalences of spectra. In particular,  $F$  is a homotopy functor on the relative CW complexes over  $B$ .

*Proof.* Since the spaces are modified to be well-based, it suffices to do this for the functor  $\text{Map}_B(X, E)$ . By Ken Brown’s lemma, it suffices to take an acyclic  $q$ -cofibration  $X \rightarrow Y$  and show that

$$\text{Map}_B(Y, E) \rightarrow \text{Map}_B(X, E)$$

is a weak equivalence. So take the square

$$\begin{array}{ccc} S_+^{n-1} & \longrightarrow & \text{Map}_B(Y, E) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ D_+^n & \longrightarrow & \text{Map}_B(X, E) \end{array}$$

where the  $+$  means disjoint basepoint and is there to remind us that the map must be an isomorphism on homotopy groups at all points. The right-hand vertical map is a weak equivalence (actually an acyclic fibration) if we can show the dotted diagonal map exists. This is equivalent to

$$\begin{array}{ccc} (S^{n-1} \times Y) \cup (D^n \times X) & \longrightarrow & E \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ D^n \times Y & \longrightarrow & B \end{array}$$

Since  $E \rightarrow B$  is a  $q$ -fibration, it suffices to show that the left-hand vertical map is an acyclic

$q$ -cofibration. This is the main axiom for checking that (compactly generated) spaces form a monoidal model category, and it follows from a similar condition on the generating maps  $S^{n-1} \rightarrow D^n$  and  $D^n \times \{0\} \rightarrow D^n \times I$  as in [Hov99], so we are done. Alternatively, the homotopy invariance of mapping spaces from cofibrant objects to fibrant objects could also be deduced from the results of Dwyer and Kan on hammock localization [DK80].  $\square$

**Proposition 2.3.10.**  $P_n F(X) = \text{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})}(X^{\wedge i}, \Sigma_{B^i}^\infty E^{\wedge i})$  as defined above in 2.3.3 takes weak equivalences between  $q$ -cofibrant retractive spaces over  $B$  to level equivalences of spectra.

*Proof.* Again, by Ken Brown's lemma it suffices to take an acyclic  $q$ -cofibration  $X \rightarrow Y$  and show that

$$\text{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})}(Y^{\wedge i}, \Sigma_{B^i}^\infty E^{\wedge i}) \rightarrow \text{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})}(X^{\wedge i}, \Sigma_{B^i}^\infty E^{\wedge i})$$

is a level equivalence of spectra. So take the square of spaces

$$\begin{array}{ccc} S_+^{n-1} & \longrightarrow & \text{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})}(Y^{\wedge i}, \Sigma_{B^i}^k E^{\wedge i}) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ D_+^n & \longrightarrow & \text{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})}(X^{\wedge i}, \Sigma_{B^i}^k E^{\wedge i}) \end{array}$$

and show the dotted diagonal map exists. This is equivalent to a lift in this square of diagrams indexed by  $\mathbf{M}_n^{\text{op}}$ :

$$\begin{array}{ccc} (S_+^{n-1} \wedge Y^{\wedge i}) \cup (D_+^n \wedge X^{\wedge i}) & \longrightarrow & \Sigma_{B^i}^k E^{\wedge i} \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ D_+^n \wedge Y^{\wedge i} & \longrightarrow & B^i \end{array}$$

Since we assumed that  $E \rightarrow B$  was an ex-fibration, the right-hand vertical map is an ex-fibration as well ([MS06], 8.2.4). Therefore it is also a  $q$ -fibration. So it suffices to show the left-hand vertical map is an acyclic cofibration of diagrams. Using ([MS06], 7.3.2), this reduces to showing that

$$X^{\wedge i} \rightarrow Y^{\wedge i}$$

is an acyclic cofibration of diagrams, but we already did that in Prop. 2.3.8 above.  $\square$

**Proposition 2.3.11.**  $F \rightarrow P_n F$  is an equivalence on the  $q$ -cofibrant space  $\underline{i} \amalg B$  when  $0 \leq i \leq n$ .

*Proof.* When  $X = \underline{i} \amalg B$ , the fat diagonal covers all of  $X^{\wedge j}$  for  $j > i$ . Therefore a natural transformation of  $\mathbf{M}_n^{\text{op}}$ -diagrams is determined by what it does on  $X^{\wedge i} = \underline{i}^i \amalg B$ . This is an  $i^i$ -tuple of points in various fibers of  $\Sigma_{B^i}^\infty E^{\wedge i}$ , with compatibility conditions. The compatibility conditions force us to have only one point for each nonempty subset of  $\underline{i}$ . Therefore the map  $F(\underline{i} \amalg B) \rightarrow P_n F(\underline{i} \amalg B)$  becomes

$$\Sigma^\infty(E_{b_1} \times \dots \times E_{b_i}) \rightarrow \prod_{S \subset \underline{i}, S \neq \emptyset} \Sigma^\infty \bigwedge_{s \in S} E_{b_s}$$

From Cor. 2.8.6 below, this map is always an equivalence.  $\square$

**Proposition 2.3.12.**  $P_n F$  is  $n$ -excisive.

*Proof.* Start with a strongly co-Cartesian cube indexed by the subsets of a fixed finite set  $S$ , with  $|S| \geq n + 1$ . This cube is equivalent to a cube of pushouts along relative CW complexes

$$A \rightarrow X_s \quad s \in S$$

of retractive spaces over  $B$ . Applying  $P_n F$  to this cube, we get a cube of spectra

$$T \rightsquigarrow P_n F\left(\bigcup_{s \in T} X_s\right)$$

Let's show that this cube is level Cartesian. Fix a nonnegative integer  $k$  and restrict attention to level  $k$  of the spectra in the cube. This turns out to be a *fibration cube* as defined in [Goo91]. To prove this, we have to construct this lift for any space  $K$ :

$$\begin{array}{ccc} K & \longrightarrow & \text{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})} \left( \left( \bigcup_{s \in S} X_s \right)^{\wedge i}, \Sigma_{B^i}^k E^{\wedge i} \right) \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ K \times I & \longrightarrow & \lim_{T \subsetneq S} \text{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})} \left( \left( \bigcup_{t \in T} X_t \right)^{\wedge i}, \Sigma_{B^i}^k E^{\wedge i} \right) \\ & & \parallel \\ & & \text{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})} \left( \text{colim}_{T \subsetneq S} \left( \bigcup_{t \in T} X_t \right)^{\wedge i}, \Sigma_{B^i}^k E^{\wedge i} \right) \end{array}$$

Rearranging gives

$$\begin{array}{ccc}
 K \times \left[ \left( \{0\} \times (\bigcup_{s \in S} X_s)^{\bar{\wedge} i} \right) \cup \left( I \times \operatorname{colim}_{T \subsetneq S} (\bigcup_{t \in T} X_t)^{\bar{\wedge} i} \right) \right] & \longrightarrow & \Sigma_{B^i}^k E^{\bar{\wedge} i} \\
 \downarrow & \dashrightarrow & \downarrow \\
 K \times I \times (\bigcup_{s \in S} X_s)^{\bar{\wedge} i} & \longrightarrow & B^i
 \end{array}$$

This is a square of maps of diagrams. The left and right vertical maps are vertexwise  $h$ -cofibrant and  $h$ -fibrant, respectively. Unfortunately, our model structure on diagrams is  $q$ -type, not  $h$ -type. Fortunately, we can define maps of  $\mathbf{M}_n^{\text{op}}$ -diagrams one level at a time, one cell at a time. So consider inductively the modified square

$$\begin{array}{ccc}
 K \times \left[ \left( \{0\} \times (\bigcup_{s \in S} X_s)^{\bar{\wedge} i} \right) \cup \left( I \times \left[ \Delta \cup \operatorname{colim}_{T \subsetneq S} (\bigcup_{t \in T} X_t)^{\bar{\wedge} i} \right] \right) \right] & \longrightarrow & \Sigma_{B^i}^k E^{\bar{\wedge} i} \\
 \downarrow & \dashrightarrow & \downarrow \\
 K \times I \times (\bigcup_{s \in S} X_s)^{\bar{\wedge} i} & \longrightarrow & B^i
 \end{array}$$

where  $\Delta \subset (\bigcup_{s \in S} X_s)^{\bar{\wedge} i}$  is the fat diagonal. From Prop. 2.3.6 above we know that  $(\bigcup_S X_s)^{\bar{\wedge} i}$  is built up from its fat diagonal by attaching free  $\Sigma_i$ -cells, so we can define the lift one free  $\Sigma_i$ -cell at a time. Each time, we get an acyclic  $h$ -cofibration on the left, and the map on the right is an  $h$ -fibration, so the lift exists. By construction, it's natural with respect to all the maps in  $\mathbf{M}_n^{\text{op}}$ .

Now that we have a fibration cube of spaces

$$(T \subset S) \mapsto \operatorname{Map} \left( \left( \bigcup_{t \in T} X_t \right)^{\bar{\wedge} i}, \Sigma_{B^i}^k E^{\bar{\wedge} i} \right)$$

we check that the map from the initial vertex into the ordinary limit of the rest is a weak equivalence:

$$\begin{aligned}
 \operatorname{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})} \left( (\bigcup_{s \in S} X_s)^{\bar{\wedge} i}, \Sigma_{B^i}^k E^{\bar{\wedge} i} \right) &\longrightarrow \lim_{T \subsetneq S} \operatorname{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})} \left( (\bigcup_{t \in T} X_t)^{\bar{\wedge} i}, \Sigma_{B^i}^k E^{\bar{\wedge} i} \right) \\
 &\parallel \\
 &\operatorname{Map}_{(\mathbf{M}_n^{\text{op}}, \{B^i\})} \left( \operatorname{colim}_{T \subsetneq S} \left[ (\bigcup_{t \in T} X_t)^{\bar{\wedge} i} \right], \Sigma_{B^i}^k E^{\bar{\wedge} i} \right)
 \end{aligned}$$

But since  $i \leq n < |S|$ , every choice of  $i$  points in  $\bigcup_S X_s$  lies in some  $\bigcup_T X_t$  for some proper subset  $T$  of  $S$ . Therefore this map is a homeomorphism.  $\square$

**Theorem 2.3.13.**  *$P_n F$  is the universal  $n$ -excisive approximation of  $F$ .*

*Proof.* This follows from Cor. 2.2.6 above and Thm. 2.8.8 below. Together, they tell us that the universal  $n$ -excisive approximation  $P_n F$  exists and is uniquely identified by the property that  $P_n F$  is  $n$ -excisive and  $F \rightarrow P_n F$  is an equivalence on the spaces with at most  $n$  points.  $\square$

### 2.3.3 The Layers

We would like to identify each *layer*  $D_n F$  of the tower, defined to be the homotopy fiber of

$$P_n F \longrightarrow P_{n-1} F$$

In fact, the natural map  $P_n F \rightarrow P_{n-1} F$  is a level fibration of spectra, so  $D_n F$  is equivalent to the ordinary fiber. This in turn consists of all collections of maps that are trivial on  $X^{\bar{\wedge} i}$  for  $i < n$  and that vanish on the fat diagonal of  $X^{\bar{\wedge} n}$ . This spectrum may be written

$$D_n F(X) \simeq \text{Map}_{B^n}(X^{\bar{\wedge} n}/_{B^n} \Delta, \Sigma_{B^n}^\infty E^{\bar{\wedge} n})^{\Sigma_n}$$

Here the decoration  $(-)^{\Sigma_n}$  means *strict* or *categorical* fixed points. These are not the *genuine* fixed points as in [MM02], as we are not taking a fibrant replacement in the model structure given by Mandell and May for orthogonal  $\Sigma_n$ -spectra. For concreteness, the above spectrum is given at level  $k$  by the  $\Sigma_n$ -equivariant maps

$$X^{\bar{\wedge} n}/_{B^n} \Delta \longrightarrow \Sigma_{B^n}^k E^{\bar{\wedge} n}$$

of retractive spaces over  $B^n$ .

**Proposition 2.3.14.**  *$D_n F(X)$  is an  $(m - d)n - 1$  connected spectrum, where  $d = \dim X$  and  $m$  is the connectivity of  $E \rightarrow B$  (and so  $m - 1$  is the connectivity of the fiber  $E_b$ ).*

*Proof.* If  $Y$  is a  $c$ -connected based  $\Sigma_n$ -space and  $X$  is a  $d$ -dimensional based free  $\Sigma_n$ -CW complex, then  $\text{Map}(X, Y)^{\Sigma_n}$  has connectivity at least  $c - d$ . We prove this by constructing an equivariant homotopy of  $S^k \wedge X \rightarrow Y$  to the constant map, one free  $\Sigma_n$ -cell at a time. A straightforward adaptation of this argument gives the above result.  $\square$

**Corollary 2.3.15.** *If  $m > d$  then the tower converges to its limit*

$$P_\infty F(X) = \text{Map}_{(\mathbf{M}_\infty^{\text{op}}, \{B^i\})}(X^{\wedge i}, \Sigma_{B^i}^\infty E^{\wedge i})$$

The tower may converge to  $F(X)$  itself when  $m > d$ . In the case where  $B = *$  this is shown to be true in [Aro99]. For general  $B$ , here is a partial result:

**Proposition 2.3.16.** *If  $m > d$  then the map  $F \rightarrow P_0 F$  is  $m - d$  connected and the map  $F \rightarrow P_1 F$  is  $2(m - d) - 1$  connected.*

*Proof.* First we replace every  $\Sigma^\infty$  with  $Q = \Omega^\infty \Sigma^\infty$ ; this doesn't change the homotopy groups under the assumption that  $m > d$  because our spectra are connective. The first result follows easily from the fact that if  $X$  is  $k$  connected then so is  $QX$ . The second follows from the fact that if  $X$  is well-based and  $k$  connected then  $X \rightarrow QX$  is  $2k + 1$  connected. (This in turn comes from the Freudenthal Suspension theorem.) We look at the diagram

$$\begin{array}{ccc} \text{Map}_B(X, E) & \longrightarrow & Q\text{Map}_B(X, E) \\ \downarrow & & \swarrow \\ \text{Map}_B(X, Q_B E) & & \end{array}$$

The vertical map is  $2m - d - 1$  connected and the horizontal map is  $2m - 2d - 1$  connected, so the diagonal is  $2m - 2d - 1$  connected.  $\square$

We will finish this section by specializing to the case where  $M$  is a closed manifold,  $B = M$ , and  $X = M \amalg M$ . Consider the following spaces:

- $\Delta \subset M^n$  is the fat diagonal.
- $F(M; n) \cong M^n - \Delta$  is the noncompact manifold of ordered  $n$ -tuples of distinct points in  $M$ .
- $\iota : M^n - \Delta \hookrightarrow M^n$  is the inclusion map.
- $C(M; n) \cong F(M; n)_{\Sigma_n}$  is the noncompact manifold of unordered  $n$ -tuples of distinct points in  $M$ .

Then the layers of the above tower can be rewritten

$$\begin{aligned}
 D_n F(M \amalg M) &= \Gamma_{(M^n, \Delta)}(\Sigma_{M^n}^\infty E^{\bar{\wedge} n})^{\Sigma_n} \\
 &\simeq \Gamma_{M^n - \Delta}^c \left( \Sigma_{M^n}^\infty E^{\bar{\wedge} n} \Big|_{M^n - \Delta} \right)^{\Sigma_n} \\
 &\cong \Gamma_{F(M; n)}^c \left( \Sigma_{F(M; n)}^\infty \iota^* E^{\bar{\wedge} n} \right)^{\Sigma_n} \\
 &\cong \Gamma_{C(M; n)}^c (\Sigma_{C(M; n)}^\infty (\iota^* E^{\bar{\wedge} n})_{\Sigma_n}) \\
 &\simeq ((\iota^* E^{\bar{\wedge} n})_{\Sigma_n})^{-T(C(M; n))}
 \end{aligned}$$

The last step is the application of Poincaré duality (see [MS06], [CK09]) to the noncompact manifold  $C(M; n)$  with twisted coefficients given by the bundle of spectra  $(\iota^* E^{\bar{\wedge} n})_{\Sigma_n}$ . Since the manifold in question is noncompact, Poincaré duality gives an equivalence between cohomology with compact supports and homology desuspended by the tangent bundle of  $C(M; n)$ . The result is the Thom spectrum

$$((\iota^* E^{\bar{\wedge} n})_{\Sigma_n})^{-T(C(M; n))}$$

We will see a few examples of this in the next section.

## 2.4 Examples and Calculations

**Example 2.4.1.** Taking  $B = *$  and  $E = Y$  for any based space  $Y$  gives

$$\begin{aligned}
 F(X) &= \Sigma^\infty \text{Map}_*(X, Y) \\
 \vdots &\quad \vdots \\
 P_n F(X) &= \text{Map}_{\mathbf{M}_n^{\text{op}}, *} (X^{\wedge n}, \Sigma^\infty Y^{\wedge n}) \\
 \vdots &\quad \vdots \\
 P_2 F(X) &= \text{Map}_*(X \wedge X, \Sigma^\infty Y \wedge Y)^{\Sigma_2} \\
 P_1 F(X) &= \text{Map}_*(X, \Sigma^\infty Y) \\
 P_0 F(X) &= *
 \end{aligned}$$

with  $n$ th layer

$$D_n F(X) = \text{Map}_*(X^{\wedge n} / \Delta, \Sigma^\infty Y^{\wedge n})^{\Sigma_n}$$

This coincides with Arone's tower in [Aro99], and therefore converges when the connectivity of  $Y$  is at least the dimension of  $X$ . It is curious that the Taylor tower in the  $Y$  variable should agree with the polynomial interpolation tower in the  $X$  variable. We also expect this to happen in the case  $B \neq *$ , though we do not prove this here.

**Example 2.4.2.** If  $X = S^1$  and  $Y$  is simply connected then the  $n$ th layer of the tower is

$$\mathrm{Map}_*(S^n/\Delta, \Sigma^\infty Y^{\wedge n})^{\Sigma_n} \cong \Omega^n \Sigma^\infty Y^{\wedge n}$$

If  $Y = \Sigma Z$  with  $Z$  connected, then the  $n$ th layer is

$$\Sigma^\infty Z^{\wedge n}$$

It is well known that the tower splits in this case ( [Aro99], [Böd87]):

$$\Sigma^\infty \Omega \Sigma Z \simeq \prod_{n=1}^{\infty} \Sigma^\infty Z^{\wedge n}$$

**Example 2.4.3.** If  $X$  is unbased we get the tower

$$\begin{aligned} F(X) &= \Sigma^\infty \mathrm{Map}(X, Y) \\ &\vdots \\ P_n F(X) &= \mathrm{Map}_{\mathbf{M}_n^{\mathrm{op}}}(X^i, \Sigma^\infty Y^{\wedge i}) \\ &\vdots \\ P_2 F(X) &= \mathrm{Map}(X \times X, \Sigma^\infty Y \wedge Y)^{\Sigma_2} \\ P_1 F(X) &= \mathrm{Map}(X, \Sigma^\infty Y) \\ P_0 F(X) &= * \end{aligned}$$

If  $Y = S^0$  and  $X$  is any finite unbased CW complex then the  $n$ th layer of this tower is

$$D_n F(X) \simeq \mathrm{Map}(X^n/\Delta, \mathbb{S})^{\Sigma_n} \cong D((X^n/\Delta)_{\Sigma_n}) \cong D(C(X; n))$$

where  $D$  denotes Spanier-Whitehead dual. If  $Y = S^m$  and  $m > \dim X$  then the tower

converges to  $\Sigma^\infty \text{Map}(X, S^m)$ , and the  $n$ th layer is

$$D_n F(X) \simeq \text{Map}(X^n / \Delta, \Sigma^{mn} \mathbb{S})^{\Sigma_n} \simeq \Sigma^{mn} D((X^n / \Delta)_{\Sigma_n}) \cong \Sigma^{mn} D(C(X; n))$$

### 2.4.1 Gauge Groups and Thom Spectra

Let  $B = M$  be a closed connected manifold, and let  $\mathcal{P} \rightarrow M$  be a  $G$ -principal bundle. The *gauge group*  $\mathcal{G}(\mathcal{P})$  is defined to be the space of automorphisms of  $\mathcal{P}$  as a principal bundle. Consider the quotient

$$\mathcal{P}^{\text{Ad}} = \mathcal{P} \times_G G^{\text{Ad}}$$

where  $G^{\text{Ad}}$  is the group  $G$  as a right  $G$ -space with the conjugation action. Then we may identify  $\mathcal{G}(\mathcal{P})$  with the space of sections  $\Gamma_M(\mathcal{P}^{\text{Ad}})$ . Taking  $E$  to be the ex-fibration  $(\mathcal{P}^{\text{Ad}}) \amalg M$  and  $X$  to be the retractive space  $M \amalg M$  gives the tower

$$\begin{aligned} F(M \amalg M) &= \Sigma^\infty \Gamma_M(\mathcal{P}^{\text{Ad}} \amalg M) && \cong \Sigma_+^\infty \mathcal{G}(\mathcal{P}) \\ &\vdots \\ P_n F(M \amalg M) &= \Gamma_{(\mathbf{M}_n^{\text{op}}, \{M^i\})}(\Sigma_{M^i}^\infty (\mathcal{P}^{\text{Ad}})^i \amalg M^i) \\ &\vdots \\ P_2 F(M \amalg M) &= \Gamma_{M \times M}(\Sigma_{M \times M}^\infty (\mathcal{P}^{\text{Ad}})^2 \amalg M^2)^{\Sigma_2} \\ P_1 F(M \amalg M) &= \Gamma_M(\Sigma_M^\infty \mathcal{P}^{\text{Ad}} \amalg M) && \simeq (\mathcal{P}^{\text{Ad}})^{-TM} \\ P_0 F(M \amalg M) &= * \end{aligned}$$

The description of the  $n$ th layer in section 2.3.3 above becomes

$$D_n F(M \amalg M) \simeq \mathcal{C}(\mathcal{P}^{\text{Ad}}; n)^{-T(C(M; n))}$$

Here  $\mathcal{C}(\mathcal{P}^{\text{Ad}}; n)$  is configurations of  $n$  points in  $\mathcal{P}^{\text{Ad}}$  with distinct images in  $C(M; n)$ . This relates the stable homotopy type of the gauge group  $\mathcal{G}(\mathcal{P})$  to Thom spectra of configuration spaces.

If we use orthogonal spectra instead of prespectra, we get a tower of strictly associative ring spectra. This proves Theorem 2.1.3 from the introduction. If  $G$  is replaced by a grouplike  $A_\infty$  space then we get a tower of  $A_\infty$  ring spectra.

By the Thom isomorphism, the homology of  $\mathcal{C}(\mathcal{P}^{\text{Ad}}; n)^{-TC(M; n)}$  is the same as the

homology of the base space  $\mathcal{C}(\mathcal{P}^{\text{Ad}}; n)$ , with coefficients twisted by the orientation bundle of  $C(M; n)$  pulled back to  $\mathcal{C}(\mathcal{P}^{\text{Ad}}; n)$ . We can calculate this homology using the zig-zag of homotopy pullback squares

$$\begin{array}{ccccc}
 & G^n & & G^n & & G^n \\
 & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{C}(\mathcal{P}^{\text{Ad}}; n) & \xleftarrow{/\Sigma_n} & \mathcal{F}(\mathcal{P}^{\text{Ad}}; n) & \longrightarrow & (\mathcal{P}^{\text{Ad}})^n \\
 & \downarrow & \downarrow & & \downarrow \\
 C(M; n) & \xleftarrow{/\Sigma_n} & F(M; n) & \longrightarrow & M^n
 \end{array}$$

where  $F(M; n) \cong M^n - \Delta$  is ordered configurations of  $n$  points in  $M$ . Note that the manifold  $F(M; n)$  is orientable iff  $M$  is orientable, while  $C(M; n)$  is orientable iff  $M$  is orientable and  $\dim M$  is even.

Another approach to understanding the homology of configuration spaces comes from the scanning map

$$\begin{aligned}
 C(M; n) &\longrightarrow \Gamma_M(S^{TM})_n \\
 \mathcal{C}(\mathcal{P}^{\text{Ad}}; n) &\longrightarrow \Gamma_M(S^{TM} \wedge_M (\mathcal{P}^{\text{Ad}} \amalg M))_n
 \end{aligned}$$

Here the subscript of  $n$  denotes sections that are degree  $n$  in the appropriate sense. If  $M$  is open, the scanning map gives an isomorphism on integral homology in a stable range [McD75]. If  $M$  is closed, it gives an isomorphism on rational homology in a stable range [Chu11].

### 2.4.2 String Topology

Now we will finally construct the tower we described in the introduction. We may start with the tower of section 2.3 and set  $B = M$ ,  $E = LM \amalg M$ , and  $X = M \amalg M$ . Or, we may take the tower from section 2.4.1 and set  $G \simeq \Omega M$  and  $\mathcal{P} \simeq *$ , so that  $\mathcal{P}^{\text{Ad}} \simeq LM$ . Either

construction gives the tower

$$\begin{aligned}
 F(M \amalg M) &= \Sigma^\infty \Gamma_M(LM \amalg M) && \simeq \Sigma_+^\infty \Omega_{\text{id}} \text{haut}(M) \\
 &\quad \vdots \\
 P_n F(M \amalg M) &= \Gamma_{(\mathbf{M}_n^{\text{op}}, \{M^i\})}(\Sigma_{M^i}^\infty LM^i \amalg M^i) \\
 &\quad \vdots \\
 P_2 F(M \amalg M) &= \Gamma_{M \times M}(\Sigma_{M \times M}^\infty LM^2 \amalg M^2)^{\Sigma_2} \\
 P_1 F(M \amalg M) &= \Gamma_M(\Sigma_M^\infty LM \amalg M) && \simeq LM^{-TM} \\
 P_0 F(M \amalg M) &= *
 \end{aligned}$$

The  $n$ th layer is

$$D_n F(M \amalg M) \simeq \mathcal{C}(LM; n)^{-TC(M; n)}$$

As before,  $\mathcal{C}(LM; n)$  is configurations of  $n$  unmarked free loops in  $M$  with distinct basepoints. This proves Theorem 2.1.2 from the introduction.

**Remark.** The connectivity of the  $n$ th layer  $\mathcal{C}(LM; n)^{-TC(M; n)}$  is negative, and decreases to  $-\infty$  as  $n \rightarrow \infty$ . Therefore the tower does not converge to  $F$ . We may phrase this another way: if the first layer is  $k$ -connected, then the  $n$ th layer is approximately  $nk$ -connected. This is actually consistent with other results from calculus of functors (cf. [Goo03] Thm. 1.13 and [Aro99] Thm. 2), the difference here being that  $k$  is negative. In the more general case of a principal bundle  $\mathcal{P}$  over  $M$ , it seems likely that the tower will converge to  $F$  when the dimension of  $M$  is at most the connectivity of  $G$ . (This is after factoring out the tower for  $F(X) = \mathbb{S}$  by taking a cofiber on each level.) We will not prove such a convergence result in this paper.

We conclude this section with a short homology calculation. We will skip over the first layer  $LM^{-TM}$ , since it can be calculated using methods from [CJY04]. Instead, taking  $M = S^n$ , we use standard Serre spectral sequence arguments to calculate the second layer in rational homology

$$H_*(\mathcal{C}(LS^n; 2)^{-TC(S^n; 2)}; \mathbb{Q})$$

If  $n$  is odd, then  $H_q(\mathcal{C}(LS^n; 2); \mathbb{Q})$  with twisted coefficients is

$$\left\{ \begin{array}{ll} \mathbb{Q} & q = n - 1, 2n - 2, 2n - 1, 3n - 2 \\ \mathbb{Q}^2 & q = 3n - 3, 4n - 4, 4n - 3, 5n - 4 \\ \vdots & \vdots \\ 0 & \text{otherwise} \end{array} \right.$$

and if  $n$  is even the answer (with untwisted coefficients) is

$$\left\{ \begin{array}{ll} \mathbb{Q} & q = n - 1, 3n - 3, 3n - 2, 4n - 4, \\ & 5n - 4, 6n - 6, 6n - 5, 8n - 7 \\ \mathbb{Q}^2 & q = 5n - 5, 7n - 7, 7n - 6, 8n - 8, 9n - 9, \\ & 9n - 8, 10n - 10, 10n - 9, 12n - 11 \\ \vdots & \vdots \\ 0 & \text{otherwise} \end{array} \right.$$

To get the homology of the spectrum  $\mathcal{C}(LS^n; 2)^{-TC(S^n; 2)}$  we subtract  $2n$  from each degree. This spectrum is a homotopy fiber of a map of rings, so its homology carries an associative multiplication with no unit. It is easy to check however that most of the products are zero, and when  $n$  is odd, all the products are zero.

## 2.5 First Construction of $P_n F$

We still need to add teeth to Cor. 2.2.6 by giving a functorial construction of  $P_n F$  for general  $F$  with the desired properties. We begin with a description of  $P_n F$  in the non-fiberwise case ( $B = *$ ) that the author learned from Greg Arone. Broadly, the construction in this section is the cellular approach to  $P_n F$ , whereas our second construction in section 2.7 is the simplicial approach.

Let  $F : \mathcal{T}_{\text{fin}}^{\text{op}} \rightarrow \mathcal{T}$  be a contravariant homotopy functor from finite based spaces to based spaces. We want to construct another functor  $P_n F$  that agrees with  $F$  on the spaces  $\mathcal{T}_n$  with at most  $n$  points. A reasonable guess is to take a Kan extension from  $\mathcal{T}_n$  back to all of  $\mathcal{T}_{\text{fin}}$ . In fact, if we assume in addition that  $F$  is *topological* (enriched over spaces) and that we take the homotopy right Kan extension over topological functors, then we get the

right answer.

We can give  $P_n F$  more explicitly as follows. Recall that  $\mathcal{T}_n \subset \mathcal{T}$  is the subcategory of finite based sets  $\underline{i}_+ = \{1, \dots, i\}_+$  with  $0 \leq i \leq n$  and based maps between them. For a fixed finite based space  $X$ , define two diagrams of unbased spaces over  $\mathcal{T}_n^{\text{op}}$ :

$$\underline{i}_+ \rightsquigarrow X^i = \text{Map}_*(\underline{i}_+, X)$$

$$\underline{i}_+ \rightsquigarrow F(\underline{i}_+)$$

Then consider the space of (unbased!) maps between these two diagrams

$$P_n F(X) \stackrel{?}{=} \text{Map}_{\mathcal{T}_n^{\text{op}}}(X^i, F(\underline{i}_+))$$

Note that since  $F$  is topological, there is a natural map from  $F(X)$  into this space. Furthermore, this map is a homeomorphism when  $X = \underline{i}_+$  for  $0 \leq i \leq n$ , since then the diagram  $X^i$  is generated freely by a single point at level  $\underline{i}_+$ , corresponding to the identity map of  $\underline{i}_+$ . This is good, but we have missed the mark a little bit because this construction is not  $n$ -excisive in general.

To fix this, we take the *derived* or *homotopically correct* mapping space of diagrams instead. We could do this by fixing a model structure on  $\mathcal{T}_n^{\text{op}}$  diagrams in which the weak equivalences are defined objectwise. Then we would replace  $\{X^i\}$  by a cofibrant diagram and  $\{F(\underline{i}_+)\}$  by a fibrant diagram. The space of maps between these replacements is by definition the homotopically correct mapping space.

More concretely, we can fatten up the diagram  $\{X^i\}$  to the diagram

$$\underline{i}_+ \rightsquigarrow \text{hocolim}_{\underline{j}_+ \in (\mathcal{T}_n^{\text{op}} \downarrow \underline{i}_+)} X^j$$

and leave  $\{F(\underline{i}_+)\}$  alone. Then the above conditions are satisfied for the projective model structure defined above in section 2.3.1. This standard thickening is sometimes called a *two-sided bar construction* [May75] [Shu06].

Equivalently, we can leave  $\{X^i\}$  alone and fatten up  $\{F(\underline{i}_+)\}$  to

$$\underline{i}_+ \rightsquigarrow \text{holim}_{\underline{j}_+ \in (\underline{i}_+ \downarrow \mathcal{T}_n^{\text{op}})} F(\underline{j}_+)$$

Then the above conditions would be satisfied for the injective model structure, if it existed. Note that the two spaces we get in either case are actually homeomorphic:

$$P_n F(X) = \text{Map}_{\mathcal{T}_n^{\text{op}}} \left[ \text{hocolim}_{\underline{j}_+ \in (\mathcal{T}_n^{\text{op}} \downarrow \underline{i}_+)} X^j, F(\underline{i}_+) \right] \cong \text{Map}_{\mathcal{T}_n^{\text{op}}} \left[ X^j, \text{holim}_{\underline{i}_+ \in (\underline{j}_+ \downarrow \mathcal{T}_n^{\text{op}})} F(\underline{i}_+) \right]$$

Take either of these as our definition of  $P_n F(X)$ . The natural map  $F(X) \rightarrow P_n F(X)$  can be seen by taking the previous case and observing in addition that there are always natural maps

$$\text{hocolim} \longrightarrow \text{colim} \quad \text{or} \quad \text{lim} \longrightarrow \text{holim}$$

Consider the second description

$$P_n F(X) \cong \text{Map}_{\mathcal{T}_n^{\text{op}}} \left[ X^j, \text{holim}_{\underline{i}_+ \in (\underline{j}_+ \downarrow \mathcal{T}_n^{\text{op}})} F(\underline{i}_+) \right]$$

When  $X = \underline{i}_+$  and  $i \leq n$ , we may evaluate our map of diagrams at the “identity” point of  $(\underline{i}_+)^i$ , giving a homeomorphism

$$P_n F(\underline{i}_+) \xrightarrow{\cong} \text{holim}_{\underline{j}_+ \in (\underline{i}_+ \downarrow \mathcal{T}_n^{\text{op}})} F(\underline{j}_+)$$

of spaces under  $F(\underline{i}_+)$ . But the map

$$F(\underline{i}_+) \xrightarrow{\sim} \text{holim}_{\underline{j}_+ \in (\underline{i}_+ \downarrow \mathcal{T}_n^{\text{op}})} F(\underline{j}_+)$$

is an equivalence too, and this forces  $F(\underline{i}_+) \rightarrow P_n F(\underline{i}_+)$  to be an equivalence.

Next, consider the first description

$$P_n F(X) = \text{Map}_{\mathcal{T}_n^{\text{op}}} \left[ \text{hocolim}_{\underline{j}_+ \in (\mathcal{T}_n^{\text{op}} \downarrow \underline{i}_+)} X^j, F(\underline{i}_+) \right]$$

Remembering that  $X$  is a CW complex, the diagram on the left is a CW complex of diagrams. It has one free cell of dimension  $d + m$  at the vertex  $\underline{i}_+$  for every choice of  $d$ -cell in  $X^j$  and choice of  $m$ -tuple of composable arrows

$$\underline{j}_+ = \underline{i}_{0+} \longrightarrow \underline{i}_{1+} \longrightarrow \dots \longrightarrow \underline{i}_{m+} = \underline{i}_+$$

Therefore the techniques of section 2.3.2 above tell us that  $P_n F$  is  $n$ -excisive. This completes the proof that  $n$ -excisive approximations exist for topological functors from based spaces to based spaces.

The assumption that  $F$  is topological is not much stronger than the assumption that  $F$  is a homotopy functor. To see this, first define  $\Delta_X^{\text{nd}}$  as below (2.6.4) as the category of nondegenerate simplices  $\Delta^p \rightarrow X$ . A map from  $\Delta^p \rightarrow X$  to  $\Delta^q \rightarrow X$  is a factorization  $\Delta^q \hookrightarrow \Delta^p \rightarrow X$ , where  $\Delta^q \hookrightarrow \Delta^p$  is a composition of inclusions of faces. The classifying space  $|\Delta_X^{\text{nd}}|$  is homeomorphic to the thin geometric realization of  $S.X$ .

Even when  $F$  is not topological, each map  $\Delta_+^p \wedge Y \rightarrow X$  gives a map

$$\Delta_+^p \wedge F(X) \xrightarrow{\sim} F(X) \rightarrow F(\Delta_+^p \wedge Y)$$

which assemble into a zig-zag

$$\begin{aligned} F(X) &\longrightarrow \operatorname{holim}_{\Delta_{\operatorname{Map}_*(Y,X)}^{\text{nd}}} F(\Delta_+^p \wedge Y) \xleftarrow{\sim} \operatorname{holim}_{\Delta_{\operatorname{Map}_*(Y,X)}^{\text{nd}}} F(Y) \\ &\cong \operatorname{Map}(|\Delta_{\operatorname{Map}_*(Y,X)}^{\text{nd}}|, F(Y)) \xleftarrow{\sim} \operatorname{Map}(\operatorname{Map}_*(Y, X), F(Y)) \end{aligned}$$

assuming  $\operatorname{Map}_*(Y, X)$  has the homotopy type of a CW complex. So we don't quite get a map from  $F(X)$  to the far right-hand side, but we get something close enough for the purposes of homotopy theory. In particular, setting  $Y = \underline{i}_+$  we get a natural zig-zag

$$F(X) \longrightarrow \dots \longrightarrow \operatorname{Map}(X^i, F(\underline{i}_+))$$

and therefore we get a natural zig-zag from  $F(X)$  to  $P_n F(X)$ , which gives a natural map  $F \rightarrow P_n F$  in the homotopy category of functors. This map is still an equivalence when  $X$  has at most  $n$  points, because in that case the homotopy limits become ordinary products and we can use the same argument as above.

So much for the assumption that  $F$  is topological. Once we have the case where  $F$  takes based spaces to spaces, we can also easily handle the case when  $F$  takes based spaces to spectra. Simply post-compose  $F$  with fibrant replacement of spectra, and work one level at a time. This works because every stable equivalence of fibrant spectra gives a weak equivalence of spaces on each level. The above constructions naturally commute with taking the based loop space  $\Omega$ , so they pass to a construction on spectra.

If  $F$  is defined on unbased spaces then we make the same construction, except that we replace  $\mathcal{T}_n$  with the category of finite unbased sets  $\mathcal{U}_n$ . Using Prop. 2.2.5 above, we have finished the proof of the following:

**Theorem 2.5.1.** *If  $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$  is a homotopy functor, where  $\mathcal{C} = \mathcal{U}_{\text{fin}}$  or  $\mathcal{T}_{\text{fin}}$  and  $\mathcal{D} = \mathcal{T}$  or  $\mathcal{S}p$ , then there is a universal  $n$ -excisive approximation  $P_n F$ , and  $F \rightarrow P_n F$  is an equivalence on spaces with at most  $n$  points.*

This result is a good first step, but we really want to know that approximations exist for functors defined on the categories  $\mathcal{U}_{B,\text{fin}}$  and  $\mathcal{R}_{B,\text{fin}}$  of fiberwise spaces. We will do this in section 2.7 by switching to a more simplicial construction

$$P_n F(X) = \text{holim}_{\Delta^p \times \underline{i} \rightarrow X} F(\underline{i} \times \Delta^p)$$

It is worth pointing out that our “cellular” approach here can also be modified to work, though there is a significant issue when dealing with functors

$$\mathcal{R}_{B,\text{fin}}^{\text{op}} \rightarrow \mathcal{T}$$

from retractive spaces to spaces. Curiously, our simplicial approach also runs into a similar problem, as discussed below in section 2.7.3.

### 2.5.1 Higher Brown Representability

Before we move on, we should point out that this first construction is better suited to proving a kind of Brown Representability for homogeneous  $n$ -excisive functors. Let  $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$  be a homotopy functor as in Thm. 2.5.1 above. Then  $F$  is  $n$ -reduced if  $P_{n-1} F \simeq *$ , or equivalently if  $F(X) \simeq *$  whenever  $X$  is a space with at most  $(n - 1)$  points. Note that

$$F_n(X) := \text{hofib}(F(X) \rightarrow P_{n-1} F(X))$$

is always  $n$ -reduced, and  $F_n(\underline{n})$  is the *cross effect*  $\text{cross}_n F(\underline{1}, \dots, \underline{1})$  defined below in section 2.8.

We say that  $F$  is *homogeneous  $n$ -excisive* if it is  $n$ -excisive and  $n$ -reduced. So  $D_n F(X) = \text{hofib}(P_n F(X) \rightarrow P_{n-1} F(X))$  is always homogeneous  $n$ -excisive. Homogeneous 1-excicive functors are a good notion of space-valued or spectrum-valued cohomology theories. From

numerous sources (e.g. [Cho07], [CK09], [MS06]) we expect such cohomology theories to be represented by spaces or spectra.

Examining the construction of  $P_n F$  in this section, we see that  $P_n F(X) \rightarrow P_{n-1} F(X)$  is a Serre fibration when  $X$  is a CW complex. Therefore the ordinary fiber is equivalent to  $D_n F$ . This can be rephrased as the following:

**Proposition 2.5.2.** • *If  $F : \mathcal{U}_{\text{fin}}^{\text{op}} \rightarrow \mathcal{T}$  is an  $n$ -reduced homotopy functor then there is a natural map*

$$F(X) \rightarrow D(X) := \text{Map}_*(X^n/\Delta, F(\underline{n}))^{\Sigma_n}$$

*in the homotopy category of functors on  $\mathcal{U}_{\text{fin}}^{\text{op}}$ . If  $F$  is homogeneous  $n$ -excisive then this map is an equivalence.*

• *If  $F : \mathcal{T}_{\text{fin}}^{\text{op}} \rightarrow \mathcal{T}$  then the same is true for*

$$F(X) \rightarrow D(X) := \text{Map}_*(X^{\wedge n}/\Delta, F(\underline{n}_+))^{\Sigma_n}$$

• *Analogous statements hold when the target of  $F$  is spectra.*

We will now strengthen this to an equivalence of homotopy categories. Let  $G$  be a finite group. Recall that the usual notion of  $G$ -equivalence of  $G$ -spaces is an equivariant map  $X \rightarrow Y$  which induces equivalences  $X^H \rightarrow Y^H$  for all subgroups  $H < G$ . We will call an equivariant map  $X \rightarrow Y$  a *naïve  $G$ -equivalence* if it is merely an equivalence when we forget the  $G$  action. It is well known that there are at least two cofibrantly generated model structures on  $G$ -spaces, one which gives the  $G$ -equivalences and one which gives the naïve  $G$ -equivalences.

Examining the behavior of  $D(X)$  on the spaces  $\underline{i}$  or  $\underline{i}_+$  for  $i \leq n$ , it is clear that the homotopy type of  $D(X)$  is determined by the *nonequivariant* or *naïve* homotopy type of  $F(\underline{n})$  or  $F(\underline{n}_+)$ . The following is then straightforward:

**Proposition 2.5.3.** *The above construction gives an equivalence between the homotopy category of homogeneous  $n$ -excisive functors to spaces and the naïve homotopy category of  $\Sigma_n$ -spaces. A similar statement holds for functors to spectra.*

## 2.6 Properties of Homotopy Limits

In order to carry out our second construction of  $P_n F$ , we need a small collection of facts about homotopy limits. This section is expository except for Prop. 2.6.7.

Let  $[n]$  denote the totally ordered set  $\{0, 1, \dots, n\}$  as a category. Let  $\Delta[p]$  denote the standard  $p$ -simplex as a simplicial set, and let  $\Delta^p = |\Delta[p]|$  denote its geometric realization. Let  $\mathbf{I}$  be any small category. Recall [BK87] that if  $A : \mathbf{I} \rightarrow \mathcal{T}$  is a diagram of based spaces, the homotopy limit is defined

$$\operatorname{holim}_{\mathbf{I}} A \subset \prod_{g: \Delta[n] \rightarrow \mathbf{I}} \operatorname{Map}_*(\Delta_+^n, A(g(n))),$$

as the subset of all collections of maps that agree in the obvious way with the face and degeneracy maps of the nerve  $\mathbf{N}\mathbf{I}$ . The following is perhaps the most standard result about homotopy limits, and we have already used it several times. It is included here for completeness.

**Proposition 2.6.1.** *If  $A, B : \mathbf{I} \rightarrow \mathcal{T}$  are two diagrams indexed by  $\mathbf{I}$ , and  $A \rightarrow B$  is a natural transformation that on each object  $i \in \mathbf{I}$  gives a weak equivalence  $A(i) \rightarrow B(i)$ , then it induces a weak equivalence*

$$\operatorname{holim}_{\mathbf{I}} A \rightarrow \operatorname{holim}_{\mathbf{I}} B$$

Recall that if  $\mathbf{I} \xrightarrow{\alpha} \mathbf{J}$  is a functor and  $A : \mathbf{J} \rightarrow \mathcal{T}$  is a diagram of spaces, then there is a naturally defined map

$$\operatorname{holim}_{\mathbf{J}} A \rightarrow \operatorname{holim}_{\mathbf{I}} (A \circ \alpha)$$

The functor  $\mathbf{I} \xrightarrow{\alpha} \mathbf{J}$  is *homotopy initial* (or *homotopy left cofinal*) if for each object  $j \in \mathbf{J}$  the overcategory  $(\alpha \downarrow j)$  has contractible nerve.

**Proposition 2.6.2.** *If  $\mathbf{I} \xrightarrow{\alpha} \mathbf{J}$  is homotopy initial and  $A : \mathbf{J} \rightarrow \mathcal{T}$  is a diagram of spaces, then*

$$\operatorname{holim}_{\mathbf{J}} A \rightarrow \operatorname{holim}_{\mathbf{I}} (A \circ \alpha)$$

*is an equivalence.*

Given a functor  $\alpha$ , we will use these results to determine whether  $\alpha$  is homotopy initial:

**Lemma 2.6.3.** • *Each adjunction of categories induces a homotopy equivalence on the nerves.*

- *If  $(\alpha \downarrow j)$  is related by a zig-zag of adjunctions to the one-point category  $*$ , then its nerve is contractible and therefore  $\alpha$  is homotopy initial.*
- *If  $(\alpha \downarrow j)$  has an initial or terminal object then  $\alpha$  is homotopy initial.*
- *If  $\alpha$  is a left adjoint then it is homotopy initial.*

We will frequently use this example of a homotopy initial functor:

**Definition 2.6.4.** • *If  $X$  is a space, let  $\Delta_X^{\text{nd}}$  denote the category of nondegenerate simplices  $\Delta^p \rightarrow X$ . A map from  $\Delta^p \rightarrow X$  to  $\Delta^q \rightarrow X$  is a factorization  $\Delta^q \hookrightarrow \Delta^p \rightarrow X$ , where  $\Delta^q \hookrightarrow \Delta^p$  is a composition of inclusions of faces. The classifying space of  $\Delta_X^{\text{nd}}$  is homeomorphic to the thin geometric realization of  $X^i$ :*

$$B\Delta_X^{\text{nd}} \cong |\text{sd}(S.(X^i))| \cong |S.(X^i)| \cong |(S.X)^i| \cong |S.X|^i$$

- *Let  $\Delta_X$  be the category of all (possibly degenerate) simplices in  $X$ , with face and degeneracy maps between them. Then the inclusion  $\Delta_X^{\text{nd}} \rightarrow \Delta_X$  is a left adjoint, therefore homotopy initial.*
- *If  $X$  is a simplicial set, there are obvious analogues of  $\Delta_X^{\text{nd}}$  and  $\Delta_X$ . As before, the inclusion  $\Delta_X^{\text{nd}} \hookrightarrow \Delta_X$  is a left adjoint, therefore homotopy initial.*

Next, we need a fact about iterated homotopy limits. We recall the colimit version first. If  $F : \mathbf{I} \rightarrow \mathbf{Cat}$  is a small diagram of small categories, the *Grothendieck construction* gives a larger category  $\mathbf{I} \int F$ , whose objects are pairs  $(i, x)$  of an object  $i \in \mathbf{I}$  and an object  $x \in F(i)$ . The maps  $(i, x) \rightarrow (j, y)$  are arrows  $i \xrightarrow{f} j$  in  $\mathbf{I}$ , and arrows  $F(f)(x) \rightarrow y$  in  $F(j)$ . *Thomason's Theorem* tells us that a homotopy colimit of a diagram  $A : \mathbf{I} \int F \rightarrow \mathcal{T}$  is expressed as an iterated homotopy colimit:

$$\text{hocolim}_{\mathbf{I} \int F} A \simeq \text{hocolim}_{i \in \mathbf{I}} \left( \text{hocolim}_{F(i)} A \right)$$

To formulate the result for homotopy limits, we again let  $F : \mathbf{I} \rightarrow \mathbf{Cat}$  be a small diagram of small categories. Then the *reverse Grothendieck construction* gives a larger

category  $\mathbf{I} \int^R F$ , whose objects are again pairs  $(i, x)$  of an object  $i \in \mathbf{I}$  and an object  $x \in F(i)$ . The maps  $(i, x) \rightarrow (j, y)$  are arrows  $j \xrightarrow{f} i$  in  $\mathbf{I}$ , and arrows  $x \rightarrow F(f)(y)$  in  $F(i)$ . Note that this is related to the original Grothendieck construction in that

$$\mathbf{I} \int^R F \cong (\mathbf{I} \int (\text{op} \circ F))^{\text{op}}$$

**Proposition 2.6.5** (Dual of Thomason’s Theorem). *For a diagram  $A : \mathbf{I} \int^R F \rightarrow \mathcal{T}$ , there is a natural weak equivalence*

$$\text{holim}_{\mathbf{I} \int^R F} A \xrightarrow{\sim} \text{holim}_{i \in \mathbf{I}^{\text{op}}} \left( \text{holim}_{F(i)} A \right)$$

We will not give a proof of this since Schlichtkrull gives an excellent one in [Sch09].

In this paper, we will come upon several homotopy limits that are indexed by forwards Grothendieck constructions  $\mathbf{I} \int F$  instead of reverse ones. Here we will demonstrate that such a homotopy limit splits, but the result is more complicated.

**Definition 2.6.6.** If  $\mathbf{I}$  is a small category, the *twisted arrow category*  $a\mathbf{I}$  has as its objects the arrows  $i \rightarrow j$  of  $\mathbf{I}$ . The morphisms from  $i \rightarrow j$  to  $k \rightarrow \ell$  are the factorizations of  $k \rightarrow \ell$  through  $i \rightarrow j$ :

$$\begin{array}{ccc} i & \longleftarrow & k \\ \downarrow & & \downarrow \\ j & \longrightarrow & \ell \end{array}$$

**Proposition 2.6.7.** *Given a diagram  $A : \mathbf{I} \int F \rightarrow \mathcal{T}$  there is a natural weak equivalence*

$$\text{holim}_{\mathbf{I} \int F} A \xrightarrow{\sim} \text{holim}_{(i \xrightarrow{f} j) \in a\mathbf{I}} \left( \text{holim}_{F(i)} A \circ F(f) \right)$$

**Remark.** This proposition is motivated by a result of Dwyer and Kan on function complexes [DK80]. Roughly, the left-hand side is the space of maps between two diagrams indexed by  $\mathbf{I}$ . The first diagram sends  $i$  to the nerve of  $F(i)$ , while the other sends  $i$  to  $A(i)$ . Mapping spaces of this form, if they are “homotopically correct,” are equivalent to a homotopy limit of mapping spaces  $\text{Map}(NF(i), A(j))$  over the twisted arrow category  $a\mathbf{I}$ ; this is roughly what we get on the right-hand side.

*Proof.* Recall that we already have a functor  $F : \mathbf{I} \rightarrow \mathbf{Cat}$ . Define another functor

$(a\mathbf{I})^{\text{op}} \rightarrow \mathbf{Cat}$  by taking  $i \rightarrow j$  to  $F(i)$ , and call this functor  $F$  by abuse of notation. Then we can build the reverse Grothendieck construction  $(a\mathbf{I})^{\text{op}} \int^R F$ .

The desired weak equivalence is the composite

$$\text{holim}_{\mathbf{I} \int F} A \xrightarrow{\sim} \text{holim}_{(a\mathbf{I})^{\text{op}} \int^R F} A \circ \alpha \xrightarrow{\sim} \text{holim}_{(i \xrightarrow{f} j) \in a\mathbf{I}} \left( \text{holim}_{F(i)} A \circ F(f) \right)$$

The second map is a weak equivalence by the dual of Thomason’s theorem, stated above.

The first map is induced by pullback along a functor

$$(a\mathbf{I})^{\text{op}} \int^R F \xrightarrow{\alpha} \mathbf{I} \int F$$

and it suffices to show that this functor is homotopy initial. Specifically,  $\alpha$  does the following to objects and morphisms:

$$\begin{array}{ccccc} \begin{array}{ccc} (i \xrightarrow{f} j, & & \\ \uparrow g & & \downarrow h \\ (i' \xrightarrow{f'} j', & & \end{array} & \begin{array}{ccc} x \in F(i) \xrightarrow{\alpha} (j, & & \\ \downarrow \varphi & & \downarrow h \\ F(g)(x') \in F(i) & & \\ \uparrow F(g) & & \\ x' \in F(i') \xrightarrow{\alpha} (j', & & \end{array} & \begin{array}{ccc} F(f)(x) \in F(j) & & \\ \downarrow F(h) & & \\ F(hf)(x) \in F(j') & & \\ \downarrow F(hf)\varphi & & \\ F(f')(x') \in F(j') & & \end{array} \end{array}$$

Fix an object  $(\ell, z \in F(\ell))$  in the target category  $\mathbf{I} \int F$ . We’ll show that the overcategory  $(\alpha \downarrow (\ell, z))$  is contractible. A typical map between objects of this overcategory is given by the data

$$\begin{array}{ccccc} \begin{array}{ccccc} i & \xrightarrow{f} & j & \xrightarrow{p} & \ell, \\ \uparrow g & & \downarrow h & & \parallel \\ i' & \xrightarrow{f'} & j' & \xrightarrow{p'} & \ell, \end{array} & \begin{array}{ccccc} x \in F(i) & \xrightarrow{F(pf)} & F(pf)(x) & \xrightarrow{\sigma} & z \\ \downarrow \varphi & & \downarrow F(pf)\varphi & & \parallel \\ F(g)(x') \in F(i) & & & & \\ \uparrow F(g) & \xrightarrow{F(pf)} & & & \\ x' \in F(i') & \xrightarrow{F(p'f')} & F(p'f')(x') & \xrightarrow{\sigma'} & z \end{array} \end{array}$$

where everything commutes. Let  $\mathbf{J}$  be the subcategory of  $(\alpha \downarrow (\ell, z))$  consisting of objects for which  $j = \ell$  and  $p$  is the identity. Then there is a projection  $P : (\alpha \downarrow (\ell, z)) \rightarrow \mathbf{J}$  which is left adjoint to the inclusion  $I : \mathbf{J} \rightarrow (\alpha \downarrow (\ell, z))$ . We can exhibit  $P$  and the natural

transformation from the identity to  $I \circ P$  in the following diagram:

$$\begin{array}{ccc}
 i \xrightarrow{f} j \xrightarrow{p} \ell, & x \in F(i) \xrightarrow{F(pf)} F(pf)(x) \xrightarrow{\sigma} z \\
 \parallel & \parallel \\
 i & x \in F(i) \\
 \parallel & \uparrow \text{id} \\
 i \xrightarrow{pf} \ell \xrightarrow{=} \ell, & x \in F(i) \xrightarrow{F(pf)} F(pf)(x) \xrightarrow{\sigma} z
 \end{array}$$

To check the adjunction, it suffices to check that a map from any object of  $(\alpha \downarrow (\ell, z))$  into an object of  $\mathbf{J}$  factors uniquely through this projection. Once this is checked, the next step is to show that  $\mathbf{J}$  has an initial subcategory  $\mathbf{K}$ . A typical object of  $\mathbf{K}$  is given in the first row below.

$$\begin{array}{ccc}
 \ell \xrightarrow{=} \ell \xrightarrow{=} \ell, & F(f)(x) \in F(\ell) \xrightarrow{\text{id}} F(f)(x) \xrightarrow{\sigma} z \\
 \uparrow f & \parallel \\
 i \xrightarrow{f} \ell \xrightarrow{=} \ell, & F(f)(x) \in F(\ell) \\
 & \uparrow F(f) \\
 & x \in F(i) \xrightarrow{F(f)} F(f)(x) \xrightarrow{\sigma} z
 \end{array}$$

The rest of the diagram justifies the claim that  $\mathbf{K}$  is initial. Finally,  $\mathbf{K}$  is isomorphic to the category of objects over  $z$  in  $F(\ell)$ , which has terminal object  $z$ . We have completed a zig-zag of adjunctions between  $(\alpha \downarrow (\ell, z))$  and  $*$ , so  $(\alpha \downarrow (\ell, z))$  is contractible. Therefore  $\alpha$  is homotopy initial and the equivalence is complete.

The equivalence is clearly natural in  $A$ , but it is also natural in  $F$  in the following sense. A map of diagrams of categories  $F \xrightarrow{\eta} G$  gives a map  $\mathbf{I} \int F \xrightarrow{\mathbf{I} \int \eta} \mathbf{I} \int G$ , so a diagram  $A : \mathbf{I} \int G \rightarrow \mathcal{T}$  can be pulled back to  $\mathbf{I} \int F$ . Our equivalence then fits into a commuting square:

$$\begin{array}{ccc}
 \text{holim}_{\mathbf{I} \int F} (\mathbf{I} \int \eta)^* A & \longrightarrow & \text{holim}_{(i \xrightarrow{f} j) \in a\mathbf{I}} \left( \text{holim}_{F(i)} ((\mathbf{I} \int \eta)^* A) \circ F(f) \right) \\
 \uparrow & & \uparrow \\
 \text{holim}_{\mathbf{I} \int G} A & \longrightarrow & \text{holim}_{(i \xrightarrow{f} j) \in a\mathbf{G}} \left( \text{holim}_{G(i)} A \circ G(f) \right)
 \end{array}$$

□

Lastly, we want a result on diagrams  $A : \mathbf{J} \rightarrow \mathcal{T}$  for which every arrow  $i \rightarrow j$  induces a weak equivalence  $A(i) \rightarrow A(j)$ . Call such a diagram *almost constant*. Of course, if  $A$  is a constant diagram sending everything to the space  $X$ , then its homotopy limit is

$$\operatorname{holim}_{\mathbf{J}} A = \operatorname{Map}(B\mathbf{J}, X)$$

where  $B\mathbf{J} = |N\mathbf{J}|$  is the classifying space of  $\mathbf{J}$ . If  $A$  is instead almost constant, then we get (see [CK09], [Dwy96])

**Proposition 2.6.8.** *If  $A : \mathbf{J} \rightarrow \mathcal{T}$  is almost constant, then there is a fibration  $E_A \rightarrow B\mathbf{J}$  and a natural weak equivalence*

$$\operatorname{holim}_{\mathbf{J}} A \simeq \Gamma_{B\mathbf{J}}(E_A)$$

Moreover, if  $\mathbf{I} \xrightarrow{\alpha} \mathbf{J}$  is a functor then there is a homotopy pullback square

$$\begin{array}{ccc} E_{A \circ \alpha} & \longrightarrow & E_A \\ \downarrow & & \downarrow \\ B\mathbf{I} & \longrightarrow & B\mathbf{J} \end{array}$$

**Corollary 2.6.9.** *If  $A : \mathbf{J} \rightarrow \mathcal{T}$  is almost constant, and  $\mathbf{I} \xrightarrow{\alpha} \mathbf{J}$  induces a weak equivalence  $B\mathbf{I} \rightarrow B\mathbf{J}$ , then the natural map*

$$\operatorname{holim}_{\mathbf{J}} A \rightarrow \operatorname{holim}_{\mathbf{I}} (A \circ \alpha)$$

*is a weak equivalence.*

## 2.7 Second Construction of $P_n F$ : The Higher Coassembly Map

Here we will describe how to construct  $P_n F(X)$  as a homotopy limit

$$P_n F(X) = \operatorname{holim}_{\Delta^p \times \underline{i} \rightarrow X} F(\underline{i} \times \Delta^p)$$

When  $n = 1$  and  $F$  is reduced, this construction is essentially the same as the *coassembly map* described in [CK09]. The coassembly map is formally dual to the *assembly map* ([WW93]) often found in treatments of algebraic K-theory.

We will prove that our construction of  $P_n F$  satisfies four properties:

1.  $P_n F$  is a homotopy functor.
2.  $P_n F$  takes pushout cubes whose dimension is at least  $n + 1$  to Cartesian cubes.
3. If  $X$  is a CW complex then  $P_n F(X) \longrightarrow \operatorname{holim}_{X' \subset X \text{ finite complex}} P_n F(X')$  is an equivalence.
4.  $F \longrightarrow P_n F$  is an equivalence on  $\mathcal{R}_{B,n}^{\text{op}}$  or  $\mathcal{U}_{B,n}^{\text{op}}$ .

For functors on finite CW complexes, conditions (1), (2), and (4) are enough to imply  $P_n F$  is the universal  $n$ -excisive approximation of  $F$ . Condition (3) is a bit weaker than the standard condition that filtered homotopy colimits go to homotopy limits; it is here because the technology we need for (2) happens to make (3) easy.

There are 8 different setups we might consider, based on whether  $B$  is a point or not, the spaces over  $B$  are fiberwise based (retractive) or unbased, and  $F$  goes into spaces or spectra. We will first handle all cases where the spaces over  $B$  are unbased. Then we'll handle all cases where  $B = *$  and the spaces over  $B$  are based. Together this gives an extension and a second proof of Theorem 2.5.1 above:

**Theorem 2.7.1.** *If  $F : \mathcal{C} \longrightarrow \mathcal{D}$  is a homotopy functor, where  $\mathcal{C} = \mathcal{U}_{B,\text{fin}}$  or  $\mathcal{T}_{\text{fin}}$  and  $\mathcal{D} = \mathcal{T}$  or  $Sp$ , then there is a universal  $n$ -excisive approximation  $P_n F$ , and  $F \longrightarrow P_n F$  is an equivalence on spaces with at most  $n$  points.*

Finally, in section 2.8.2 below we will do the case of functors from retractive spaces over  $B$  to spectra. We do not have a method that works for retractive spaces over  $B$  to spaces.

### 2.7.1 $P_n F$ for Unbased Spaces over $B$

Let  $\mathbf{C}_{B,n}$  denote a subcategory of simplicial sets over  $S.B$  consisting of objects of the form

$$\underline{i} \times \Delta[p], \quad p \geq 0 \text{ and } 0 \leq i \leq n.$$

Specifically, we take one such object for each choice of  $p$  and  $i$ , and each choice of map of simplicial sets  $\underline{i} \times \Delta[p] \longrightarrow S.B$ . We do *not* take the full subcategory on these objects. Each

map

$$\underline{j} \times \Delta[q] \longrightarrow \underline{i} \times \Delta[p]$$

must be a product of a single simplicial map  $\Delta[q] \longrightarrow \Delta[p]$  and a map of finite sets  $\underline{j} \longrightarrow \underline{i}$ . Intuitively,  $\mathbf{C}_{B,n}$  is a simplicial fattening of  $\mathcal{U}_{B,n}$ .

Now let  $F$  be any contravariant homotopy functor from unbased spaces over  $B$  to spaces or spectra. If  $F$  is a functor to spectra, compose it with fibrant replacement. This gives an equivalent functor that takes weak equivalences of spaces to level equivalences of spectra, and we can argue one level at a time. So now without loss of generality,  $F$  is a homotopy functor to based spaces.

If  $X$  is a simplicial set over  $S.B$ , define

$$P_n F(X) = \operatorname{holim}_{(\mathbf{C}_{B,n} \downarrow X)^{\text{op}}} F(\underline{i} \times \Delta^p)$$

Abusing notation, define  $P_n F$  on spaces as the composite

$$\mathcal{U}_B \xrightarrow{S} \mathbf{sSet}/S.B \xrightarrow{P_n F} \mathcal{T}$$

or more explicitly,

$$P_n F(X) = \operatorname{holim}_{(\mathbf{C}_{B,n} \downarrow S.X)^{\text{op}}} F(\underline{i} \times \Delta^p)$$

The natural transformation  $F \xrightarrow{P_n} P_n F$  is then induced by a collection of maps  $F(X) \longrightarrow F(\underline{i} \times \Delta^p)$  for each map  $\underline{i} \times \Delta^p \longrightarrow X$ .

When  $X = \underline{i}$ , the object  $\underline{i} \times \Delta[0] \xrightarrow{\cong} S.\underline{i}$  is initial in  $(\mathbf{C}_{B,n} \downarrow S.\underline{i})^{\text{op}}$ , so the homotopy limit is obtained by evaluating at this initial object (Prop. 2.6.2). This proves property (4), that  $F \longrightarrow P_n F$  is an equivalence on  $\mathcal{U}_{B,n}^{\text{op}}$ .

Next we'll tackle property (1), that  $P_n F$  is a homotopy functor. Let  $\mathcal{U}_n = \mathcal{U}_{*,n}$  be the category of finite unbased sets  $\underline{0}, \dots, \underline{n}$  and all maps between them. Notice that we can define a functor  $\Delta : \mathcal{U}_n^{\text{op}} \longrightarrow \mathbf{Cat}$  taking  $\underline{i}$  to  $\Delta_{X^i}$ . Each map  $\underline{i} \longleftarrow \underline{j}$  goes to the functor  $\Delta_{X^i} \longrightarrow \Delta_{X^j}$  arising from the map  $X^i \longrightarrow X^j$ , whose definition is obvious once we observe that  $X^i \cong \operatorname{Map}(\underline{i}, X)$ . Now take the forwards Grothendieck construction  $\mathcal{U}_n^{\text{op}} \int \Delta$ . This is a category whose objects are elements  $X_p^i$  and whose morphisms  $X_p^i \longrightarrow X_q^j$  are compositions of maps  $X^i \longrightarrow X^j$  from above and maps  $X_p^j \longrightarrow X_q^j$  which are compositions of face and

degeneracy maps. Equivalently, the objects can be described as maps

$$\Delta[p] \times \underline{i} \longrightarrow X.$$

and the morphisms are factorizations

$$\begin{array}{ccc} \Delta[p] \times \underline{i} & \longrightarrow & X. \\ \uparrow & & \parallel \\ \Delta[q] \times \underline{j} & \longrightarrow & X. \end{array}$$

in which the vertical map is a product of  $\underline{j} \longrightarrow \underline{i}$  and some simplicial map  $\Delta[q] \longrightarrow \Delta[p]$ . This is clearly the same category as  $(\mathbf{C}_{B,n}^{\text{op}} \downarrow X)^{\text{op}}$ , so we have a new way to write our definition of  $P_n F(X.)$ :

$$P_n F(X.) = \text{holim}_{\mathcal{U}_n^{\text{op}} \int \Delta} F(\underline{i} \times \Delta^p)$$

Now Prop. 2.6.7 gives the following:

$$\text{holim}_{\mathcal{U}_n \int \Delta_{X^i}} F(\Delta^p \times \underline{i}) \simeq \text{holim}_{(\underline{i} \leftarrow \underline{j}) \in \text{al}\mathcal{U}_n^{\text{op}}} \left( \text{holim}_{\Delta_{X^i}} F(\underline{j} \times \Delta^p) \right)$$

The term inside the parentheses can be rewritten

$$\text{holim}_{\Delta_{X^i}} F(\underline{j} \times \Delta^p) \simeq \text{holim}_{\Delta_{X^i}^{\text{nd}}} F(\underline{j} \times \Delta^p)$$

and this defines a homotopy functor in  $X$ . by Prop. 2.6.9. The homotopy limit of these is also a homotopy functor, and using the naturality statement in Prop. 2.6.7 we conclude that  $P_n F(X.)$  is a homotopy functor. In fact, we have proven something stronger than (1), that  $P_n F$  actually takes weak equivalences of simplicial sets to weak equivalences.

Now we can prove (2). From [Goo91], each strongly co-Cartesian cube of spaces over  $B$  is weakly equivalent to a pushout cube formed by a cofibrant space  $A$  and an  $(n + 1)$ -tuple of spaces  $X_0, \dots, X_n$  over  $B$ , each with a cofibration  $A \longrightarrow X_i$ . Applying singular simplices  $S.$ , we get a cube of simplicial sets

$$T \rightsquigarrow S. \left( \bigcup_{s \in T} X_s \right)$$

where the  $\bigcup$  is shorthand for pushout of spaces along  $A$ . By easy induction, this cube is equivalent to the pushout cube of simplicial sets

$$T \rightsquigarrow \bigcup_{s \in T} S.X_s$$

where the  $\bigcup$  is shorthand for pushout of simplicial sets along  $S.A$ . Since  $P_n F$  is a homotopy functor on simplicial sets, applying  $P_n F$  to both cubes gives equivalent results. Therefore it suffices to show that  $P_n F$  takes a pushout cube of simplicial sets to a Cartesian cube of spaces.

So let  $S$  be any set with cardinality strictly larger than  $n$ , let  $A \in \mathbf{sSet}$  be a simplicial set, and for each element  $s \in S$ , let  $X_s \in \mathbf{sSet}$  be a simplicial set containing  $A$ . Then there is a pushout cube which assigns each subset  $T \subset S$  to the simplicial set  $\bigcup_{t \in T} X_t$ , which is shorthand for the pushout of the  $X_t$  along  $A$ . We want to show that  $P_n F$  takes this to a Cartesian cube. In other words, the natural map

$$\operatorname{holim}_{\Delta[p] \times \underline{i} \rightarrow \bigcup_S X_s} F(\underline{i} \times \Delta^p) \longrightarrow \operatorname{holim}_{(T \subsetneq S)^{\text{op}}} \left( \operatorname{holim}_{\Delta[p] \times \underline{i} \rightarrow \bigcup_T X_s} F(\underline{i} \times \Delta^p) \right)$$

is an equivalence. Using dual Thomason, we rewrite the right-hand side as

$$\operatorname{holim}_{(T, \Delta[p] \times \underline{i} \rightarrow \bigcup_T X_s)} F(\underline{i} \times \Delta^p)$$

where each object of the indexing category is a proper subset  $T \subsetneq S$ , integers  $p \geq 0$  and  $0 \leq i \leq n$ , and a map  $\Delta[p] \times \underline{i} \rightarrow \bigcup_T X_s$ . A map between two objects looks like

$$\begin{array}{ccc} T, & \underline{i} \times \Delta[p] \longrightarrow & \bigcup_T X_s \\ \text{subset} \uparrow & \uparrow & \uparrow \\ U, & \underline{j} \times \Delta[q] \longrightarrow & \bigcup_U X_s \end{array}$$

This category maps forward into  $\mathcal{U}_n^{\text{op}} \int \Delta_{(\bigcup_S X_s)^i}$ , in which a map between two objects is given by the data

$$\begin{array}{ccc} \underline{i} \times \Delta[p] \longrightarrow & \bigcup_S X_s & \\ \uparrow & \uparrow & \\ \underline{j} \times \Delta[q] \longrightarrow & \bigcup_S X_s & \end{array}$$

This functor  $\alpha$  forgets the data of  $T$  and includes  $\bigcup_T X_s^i$  into  $\bigcup_S X_s^i$ . The natural map of homotopy limits

$$\operatorname{holim}_{\Delta[p] \times \underline{i} \rightarrow \bigcup_S X_s} F(\underline{i} \times \Delta^p) \longrightarrow \operatorname{holim}_{(T, \Delta[p] \times \underline{i} \rightarrow \bigcup_T X_s)} F(\underline{i} \times \Delta^p)$$

is induced by a pullback of diagrams along  $\alpha$ , so we just have to show that  $\alpha$  is homotopy initial. Given an object  $\underline{j} \times \Delta[q] \xrightarrow{\varphi} \bigcup_S X_s$  in the target category, the overcategory  $(\alpha \downarrow \varphi)$  has as its objects the factorizations of  $\varphi$

$$\underline{j} \times \Delta[q] \longrightarrow \underline{i} \times \Delta[p] \longrightarrow \bigcup_T X_s \longrightarrow \bigcup_S X_s$$

where  $T \subsetneq S$  must be a *proper* subset of  $S$ .

Let us give a terminal object for this overcategory. Since we are working with simplicial sets instead of spaces, each  $q$ -simplex lands *inside* one of the sets  $X_s$  in the pushout. Therefore there is a smallest subset  $T \subset S$  such that  $\underline{j} \times \Delta[q] \xrightarrow{\varphi} \bigcup_S X_s$  lands inside  $\bigcup_T X_s$ , and since  $j \leq n < |S|$ , this subset is proper. This gives a terminal object for the overcategory  $(\alpha \downarrow \varphi)$ , so it's contractible, which finishes (2).

Finally we check (3). Let  $X$  be a CW complex. We want to show that the natural map

$$\operatorname{holim}_{\Delta[p] \times \underline{i} \rightarrow S.X} F(\underline{i} \times \Delta^p) \longrightarrow \operatorname{holim}_{(\text{finite } X' \subset X)^{\text{op}}} \left( \operatorname{holim}_{\Delta[p] \times \underline{i} \rightarrow S.X'} F(\underline{i} \times \Delta^p) \right)$$

is an equivalence. Using dual Thomason, we rewrite the right-hand side as

$$\operatorname{holim}_{(\text{finite } X' \subset X, \Delta[p] \times \underline{i} \rightarrow S.X')} F(\underline{i} \times \Delta^p)$$

where each object of the indexing category is a finite subcomplex  $X' \subset X$ , integers  $p \geq 0$  and  $0 \leq i \leq n$ , and a map  $\Delta[p] \times \underline{i} \rightarrow S.X'$ . A map between two objects looks like

$$\begin{array}{ccc} X', & \Delta[p] \times \underline{i} \longrightarrow & S.X' \\ \text{inclusion} \uparrow & \uparrow & \uparrow \\ X'', & \Delta[q] \times \underline{j} \longrightarrow & S.X'' \end{array}$$

This category maps forward into  $\mathcal{U}_n^{\text{op}} \int \Delta_{(S.X)}^i$ , in which a map between two objects looks

like

$$\begin{array}{ccc} \underline{i} \times \Delta[p] & \longrightarrow & S.X \\ \uparrow & & \uparrow \\ \underline{j} \times \Delta[q] & \longrightarrow & S.X \end{array}$$

This functor  $\alpha$  forgets the data of  $X'$  and includes  $X'$  into  $X$ . The natural map of homotopy limits defined above is again induced by a pullback of diagrams along  $\alpha$ , so we just have to show that  $\alpha$  is homotopy initial. Given an object  $\underline{j} \times \Delta[q] \xrightarrow{\varphi} S.X$  in the target category, the overcategory  $(\alpha \downarrow \varphi)$  has as its objects the factorizations of  $\varphi$

$$\underline{j} \times \Delta[q] \longrightarrow \underline{i} \times \Delta[p] \longrightarrow S.X' \longrightarrow S.X$$

where  $X' \subset X$  must be a finite subcomplex. But of course each  $q$ -simplex lands inside a unique smallest subcomplex; taking the union over all  $\underline{j}$  gives a smallest finite subcomplex containing the image of  $\Delta^q \times \underline{j}$ . This gives a terminal object for the overcategory  $(\alpha \downarrow \varphi)$ , so it's contractible and we are done proving (3).

### 2.7.2 $P_n F$ for Based Spaces

The argument mimics the one above, so we will only point out what is different. The category  $\mathbf{C}_n$  becomes a subcategory of based simplicial sets consisting of objects of the form

$$(\underline{i} \times \Delta[p])_+, \quad p \geq 0 \text{ and } 0 \leq i \leq n.$$

with one such object for each choice of  $p$  and  $i$ . Each map

$$(\underline{j} \times \Delta[q])_+ \longrightarrow (\underline{i} \times \Delta[p])_+$$

is a choice of simplicial map  $\Delta[q] \longrightarrow \Delta[p]$  and map of finite based sets  $\underline{j}_+ \longrightarrow \underline{i}_+$ . Intuitively,  $\mathbf{C}_n$  is a simplicial fattening of  $\mathcal{T}_n \cong \mathcal{T}_n$ . If  $X$  is a based simplicial set, define

$$P_n F(X) = \operatorname{holim}_{(\mathbf{C}_n \downarrow X)^{\text{op}}} F((\underline{i} \times \Delta^p)_+)$$

Abusing notation, define  $P_n F$  on spaces as the composite

$$\mathcal{T} \xrightarrow{S} \mathbf{sSet}_* \xrightarrow{P_n F} \mathcal{S}p$$

The category  $\mathcal{U}_n$  of finite sets is replaced by the category  $\mathcal{T}_n$  of finite based sets. As before, there is a functor  $\Delta : \mathbf{U}_{*,n}^{\text{op}} \cong \mathcal{T}_n^{\text{op}} \rightarrow \mathbf{Cat}$  taking  $\underline{i}_+$  to  $\Delta_{X^i}$ , and we can rewrite  $P_n F(X.)$  as

$$P_n F(X.) = \text{holim}_{\mathcal{T}_n^{\text{op}} \int \Delta} F((\underline{i} \times \Delta^p)_+)$$

To show that  $P_n F$  is homotopy invariant we rewrite it as

$$\text{holim}_{\mathcal{T}_n^{\text{op}} \int \Delta} F(\Delta^p \times \underline{i}) \simeq \text{holim}_{(\underline{i}_+ \leftarrow \underline{j}_+) \in \mathcal{a}\mathcal{T}_n^{\text{op}}} \left( \text{holim}_{\Delta_{X^i}} F((\underline{j} \times \Delta^p)_+) \right)$$

which proves (1). The proof of (2) and (3) is the same as in the unbased case.

### 2.7.3 Difficulties with Retractive Spaces over $B$

The above proof does not work when generalized to retractive spaces over  $B$ . We may define  $\mathbf{U}_{B,n}$  as the subcategory of spaces *under*  $B$  consisting of spaces of the form

$$\underline{i} \amalg B, \quad 0 \leq i \leq n$$

So a map  $\underline{i} \amalg B \rightarrow \underline{j} \amalg B$  must act as the identity on  $B$ , but the points in  $\underline{i}$  may map into  $\underline{j}$  or anywhere into  $B$ . Then we may define  $\mathbf{U}_{B,n}^{\text{op}} \int \Delta$ , and then define  $P_n F$  as a homotopy limit over this category. The proof of (1), (2) and (3) is then straightforward. However, our argument for (4) does not work because there aren't enough maps in  $\mathbf{U}_{B,n}^{\text{op}} \int \Delta$  to make our desired object initial.

Examining this shortcoming, it seems one must enrich  $\mathbf{U}_{B,n}^{\text{op}}$  and use an enriched version of the above theorems on homotopy limits. This is not entirely straightforward, since in order to define  $P_n F$  here, one must deal with the concept of a “diagram” that is indexed not by a simplicially enriched category but by a simplicial object in  $\mathbf{Cat}$ .

We will avoid doing this, and instead we will handle the case of  $F : \mathcal{R}_{B,\text{fin}}^{\text{op}} \rightarrow \mathcal{S}p$  in section 2.8.2 using splitting theorems that only hold for functors into spectra.

## 2.8 Spectra and Cross Effects

From here onwards we will only consider functors from retractive spaces over  $B$  to spectra. In this section the word *spectra* will refer to prespectra, though the arguments will also work

for coordinate-free orthogonal spectra that have nondegenerately based levels [MMSS01]. Let  $\text{fib}$  denote homotopy fiber and  $\text{cofib}$  denote (reduced) homotopy cofiber. For spaces, these have the usual definition

$$\begin{aligned}\text{fib}(X \longrightarrow Y) &= X \times_Y \text{Map}_*(I, Y) \\ \text{cofib}(X \longrightarrow Y) &= (X \wedge I) \cup_X Y\end{aligned}$$

and for spectra these definitions are applied to each level separately.

We begin this section with some standard facts about spectra and splitting. Recall that the natural map  $X \vee Y \longrightarrow X \times Y$  is an equivalence when  $X$  and  $Y$  are spectra. Comparison of cofiber and fiber sequences then gives the following:

**Proposition 2.8.1.** *Suppose that  $X$ ,  $X'$ , and  $Y$  are spectra with maps*

$$X \xrightarrow{i} Y \xrightarrow{p} X'$$

*such that  $p \circ i$  is an equivalence. Then there are natural equivalences of spectra*

$$X \vee \text{fib}(p) \xrightarrow{\sim} Y \xrightarrow{\sim} X \times \text{cofib}(i)$$

*which also yield an equivalence  $\text{fib}(p) \xrightarrow{\sim} \text{cofib}(i)$ .*

**Corollary 2.8.2.** *If  $X$  is a retract of  $Y$  then  $Y \simeq X \vee Z$  where*

$$Z \simeq \text{fib}(Y \longrightarrow X) \simeq \text{cofib}(X \longrightarrow Y)$$

**Corollary 2.8.3.** *If  $X$  is a well-based space then there is a natural equivalence*

$$\Sigma^\infty(X_+) \simeq \Sigma^\infty(X \vee S^0)$$

**Corollary 2.8.4.** *If  $\mathcal{R}_B^{(\text{op})} \xrightarrow{F} \mathcal{S}p$  is any covariant or contravariant functor then there is a splitting of functors*

$$F(X) \simeq F(B) \times \overline{F}(X)$$

*where  $\overline{F}(X)$  can be defined as the fiber of  $F(X) \longrightarrow F(B)$  or the cofiber of  $F(B) \longrightarrow F(X)$ . This also holds if  $F$  is only defined on finite CW complexes.*

We want a slight generalization of these results to  $n$ -dimensional cubes of retracts. First

recall the higher-order versions of homotopy fiber and homotopy cofiber from [Goo91]. If  $F$  is a  $n$ -cube of spectra then we can think of it as a map between two  $(n - 1)$ -cubes. The *total homotopy fiber*  $\text{tfib}(F)$  is inductively defined as the homotopy fiber of the map between the total homotopy fibers of these two  $(n - 1)$ -cubes. For a 0-cube consisting of the space  $X$ , we define the total fiber to be  $X$ . Therefore the total fiber of a 1-cube  $X \rightarrow Y$  is  $\text{fib}(X \rightarrow Y)$ .

The *total homotopy cofiber*  $\text{tcofib}(F)$  has a similar inductive definition. Recall from [Goo91] that a cube is Cartesian iff its total fiber is weakly contractible, and co-Cartesian iff its total cofiber is weakly contractible. From this it quickly follows that a cube of spectra is Cartesian iff it is co-Cartesian.

If  $F$  is a functor  $\mathcal{R}_B^{\text{op}} \rightarrow \mathcal{S}p$ , the  $n$ th *cross effect*  $\text{cross}_n F(X_1, \dots, X_n)$  is defined as in [Goo03] to be the total fiber of the cube

$$S \subset \underline{n} \rightsquigarrow F\left(\bigcup_{i \in S} X_i\right)$$

whose maps come from inclusions of subsets of  $\underline{n}$ . Here the big union denotes pushout along  $B$ ; one can think of it as a fiberwise wedge sum. Since  $F$  is contravariant, the initial vertex of this cube corresponds to the full subset  $S = \underline{n}$ . Note that there is a natural map

$$\text{cross}_n F(X_1, \dots, X_n) \xrightarrow{i_n} F\left(\bigcup_{i \in \underline{n}} X_i\right)$$

Similarly, the  $n$ th *co-cross effect*  $\text{cocross}_n F(X_1, \dots, X_n)$  is defined as in [McC01] and [Chi10] to be the total cofiber of the cube with the same vertices

$$S \subset \underline{n} \rightsquigarrow F\left(\bigcup_{i \in S} X_i\right)$$

where the maps come from the opposites of inclusions of subsets of  $\underline{n}$ . Each inclusion  $S \subseteq T$  results in a collapsing map

$$\bigcup_{i \in S} X_i \longleftarrow \bigcup_{i \in T} X_i$$

which becomes

$$F\left(\bigcup_{i \in S} X_i\right) \longrightarrow F\left(\bigcup_{i \in T} X_i\right)$$

Note that the final vertex of this cube corresponds to  $S = \underline{n}$ , so there is a natural map

$$F\left(\bigcup_{i \in \underline{n}} X_i\right) \xrightarrow{p_n} \text{cocross}_n F(X_1, \dots, X_n)$$

It is known that the cross effect and co-cross effect are equivalent, when  $F$  is a functor from spectra to spectra ([Chi10], Lemma 2.2). A similar argument gives the following.

**Proposition 2.8.5.** *If  $\mathcal{R}_B^{\text{op}} \xrightarrow{F} Sp$  is any contravariant functor, then the composite*

$$\text{cross}_n F(X_1, \dots, X_n) \xrightarrow{i_n} F\left(\bigcup_{i \in \underline{n}} X_i\right) \xrightarrow{p_n} \text{cocross}_n F(X_1, \dots, X_n)$$

*is an equivalence. Furthermore,  $F(\bigcup X_i)$  splits into a sum of cross-effects:*

$$\begin{aligned} F\left(\bigcup_{i \in \underline{n}} X_i\right) &\simeq \prod_{S \subseteq \underline{n}} \text{cocross}_{|S|} F(X_s : s \in S) \\ &\simeq \prod_{S \subseteq \underline{n}} \text{cross}_{|S|} F(X_s : s \in S) \\ &\simeq \bigvee_{S \subseteq \underline{n}} \text{cross}_{|S|} F(X_s : s \in S) \end{aligned}$$

*The analogous result also holds for covariant functors, and for functors defined only on finite CW complexes.*

**Remark.** This does *not* assume that  $F$  is a homotopy functor.

*Proof.* The argument is by induction on  $n$ . We form the maps

$$\bigvee_{S \subseteq \underline{n}} \text{cross}_{|S|} F(X_s : s \in S) \longrightarrow F\left(\bigcup_{i \in \underline{n}} X_i\right) \longrightarrow \prod_{S \subseteq \underline{n}} \text{cocross}_{|S|} F(X_s : s \in S)$$

and observe that the composite is an equivalence. Therefore the middle contains either of the outside terms as a summand. We use the alternate definitions of  $\text{tfib}$  and  $\text{tcfib}$  found

in [Goo91] to identify the leftover summand with  $\text{cross}_{|S|}F$  and  $\text{cocross}_{|S|}F$ , which proves that they are equivalent and that  $F$  splits into a sum of cross effects.  $\square$

This generalizes the following well known result: (cf. [Bro69], [Coh80])

**Corollary 2.8.6** (Binomial Theorem for Suspension Spectra). *If  $X$  and  $Y$  are well-based spaces then the obvious projection maps yield a splitting*

$$\Sigma^\infty(X \times Y) \xrightarrow{\sim} \Sigma^\infty(X \wedge Y) \times \Sigma^\infty X \times \Sigma^\infty Y$$

If  $X_1, \dots, X_n$  are well-based spaces then we get a more general splitting

$$\Sigma^\infty \prod_{i=1}^n X_i \xrightarrow{\sim} \prod_{\emptyset \neq S \subseteq \underline{n}} \Sigma^\infty \bigwedge_{i \in S} X_i$$

and in particular if  $X$  is well-based then

$$\Sigma^\infty X^n \simeq \bigvee_{i=1}^n \binom{n}{i} \Sigma^\infty X^{\wedge i}$$

**Remark.** The corollary also follows easily if we start with

$$\Sigma^\infty(X_+) \simeq \Sigma^\infty(X \vee S^0)$$

From there the proof is suggested by the facts

$$\begin{aligned} (x+1)(y+1) - 1 &= xy + x + y \\ (x+1)^n - 1 &= \sum_{i=1}^n \binom{n}{i} x^i \end{aligned}$$

We are now in a position to prove the existence of  $P_n F$  for functors from retractive spaces into spectra. First we'll give a result that motivates the construction.

### 2.8.1 An Equivalence Between $[\mathbf{G}_n^{\text{op}}, \mathcal{S}p]$ and $[\mathbf{M}_n^{\text{op}}, \mathcal{S}p]$

Let  $\mathbf{G}_n = \mathcal{T}_n$  be the category of based sets  $\underline{0}_+, \dots, \underline{n}_+$  and based maps between them.  $\mathbf{G}_n$  is the opposite category of Segal's category  $\Gamma$ . As before, let  $\mathbf{M}_n$  be the category of

unbased sets  $\underline{0} = \emptyset, \underline{1}, \dots, \underline{n}$  and *surjective* maps between them. If  $\mathbf{I}$  is a category then let  $[\mathbf{I}, \mathcal{S}p]$  denote the homotopy category of diagrams of spectra indexed by  $\mathbf{I}$ .

The maps in  $\mathbf{G}_n$  are generated by inclusions, collapses, rearrangements, and maps that fold two points into one. From the last section, a diagram of spectra indexed by  $\mathbf{G}_n$  will split into a sum of cross effects. The first two classes of maps (inclusions and collapses) will simply include or collapse these summands. Therefore our diagram has redundancies. If we throw out the redundancies, only the last two classes of maps (rearrangements and folds) still carry interesting information. But these are exactly the maps that generate the smaller category  $\mathbf{M}_n$ . We have just given a heuristic argument for the following known result:

**Proposition 2.8.7.** *There is an equivalence of homotopy categories*

$$[\mathbf{G}_n, \mathcal{S}p] \xrightarrow{C} [\mathbf{M}_n, \mathcal{S}p]$$

*obtained by taking cross-effects*

$$CF(\underline{i}) = \text{cross}_i F(\underline{1}_+, \dots, \underline{1}_+)$$

*Its inverse is obtained by taking sums*

$$[\mathbf{G}_n, \mathcal{S}p] \xleftarrow{P} [\mathbf{M}_n, \mathcal{S}p]$$

$$PG(\underline{i}_+) = \bigvee_{j=0}^i \binom{i}{j} G(j)$$

*There is also an equivalence of homotopy categories*

$$[\mathbf{G}_n^{\text{op}}, \mathcal{S}p] \simeq [\mathbf{M}_n^{\text{op}}, \mathcal{S}p]$$

*obtained from co-cross effects and products*

$$CF(\underline{i}) = \text{cocross}_i F(\underline{1}_+, \dots, \underline{1}_+)$$

$$PG(\underline{i}_+) = \prod_{j=0}^i \binom{i}{j} G(j)$$

**Remark.** The author learned a version of this result from Greg Arone. A similar result

for diagrams of abelian groups was done by Pirashvili [Pir00]. Helmstutler [Hel08] gives a more sophisticated treatment that handles both abelian groups and spectra in the same uniform way. He gives a Quillen equivalence between the two categories of diagrams with the projective model structure. This is of course stronger than just an equivalence of homotopy categories, but we may think of the above result as a very explicit description of the derived functors. This perspective was essential in making the correct guess for  $P_n F$  in section 2.3 above, and it motivates our proof of Thm. 2.8.8 below.

*Proof.* We define diagrams that extend the above constructions on objects. The essential ingredient is to define maps between the various cubes that show up in the definition of total homotopy fiber and cofiber found in [Goo91]. These maps of cubes  $I^{\underline{i}} \rightarrow I^{\underline{j}}$  are all generalized diagonal maps coming from maps of sets  $\underline{i} \leftarrow \underline{j}$ . Then it is easy to define a natural equivalence of diagrams  $CPG \rightarrow G$ . On the other hand, Prop. 2.8.5 gives an equivalence  $PCF(\underline{i}_+) \rightarrow F(\underline{i}_+)$  for each object  $\underline{i}_+ \in \mathbf{G}_n$ , but these equivalences do not commute with the maps of  $\mathbf{G}_n$ . Instead, we define an isomorphism  $PCF \rightarrow F$  in the homotopy category of diagrams. To do this, we choose for each arrow  $\underline{i}_+ \rightarrow \underline{j}_+$  of  $\mathbf{G}_n$  a contractible space of maps

$$PCF(\underline{i}_+) \rightarrow F(\underline{j}_+)$$

that agrees in a natural way with compositions, and such that on the identity arrows  $\underline{i}_+ = \underline{i}_+$  we choose only equivalences

$$PCF(\underline{i}_+) \rightarrow F(\underline{i}_+)$$

Our chosen spaces of maps  $PCF(\underline{i}_+) \rightarrow F(\underline{j}_+)$  end up being products of cubes, the same cubes that appear in the definition of total homotopy fiber above. This gives the desired equivalence of homotopy categories.

The contravariant case is similar, but we will give one more detail here since it is needed in the next section. Let  $F : \mathbf{G}_n^{\text{op}} \rightarrow \mathcal{S}p$  be a diagram. For each map  $\underline{i}_+ \leftarrow \underline{j}_+$  in  $\mathbf{G}_n$ , we use the diagonal map  $I^{\underline{i}} \rightarrow I^{\underline{j}}$  to define

$$\bigvee_{S \subset \underline{i}} I_+^{i-S} \wedge F(S_+) \rightarrow \bigvee_{T \subset \underline{j}} I_+^{j-T} \wedge F(T_+)$$

taking the summand for  $S \subset \underline{i}$  to the summand for  $f^{-1}(S) \subset \underline{j}$ . This passes to a well-defined map on the co-cross effects of  $F$ , which gives the arrows of the diagram  $CF$ .  $\square$

### 2.8.2 $P_n F$ for Retractive Spaces over $B$ into Spectra

Let us consider homotopy functors

$$\mathcal{R}_{B,\text{fin}}^{\text{op}} \xrightarrow{F} \mathcal{S}p$$

from finite retractive spaces into spectra. Our previous construction of  $P_n F$  was roughly the same as a mapping space of diagrams indexed by  $\mathbf{U}_{B,n}^{\text{op}}$ , the spaces under  $B$  with at most  $n$  points. When  $B \neq *$ , this approach calls for more technology because  $\mathbf{U}_{B,n}$  needs to be enriched. However, the equivalence  $[\mathbf{G}_n^{\text{op}}, \mathcal{S}p] \simeq [\mathbf{M}_n^{\text{op}}, \mathcal{S}p]$  suggests that we could just eliminate the inclusion and collapse maps in  $\mathbf{U}_{B,n}$ . This leads to the category  $\mathbf{M}_n$  again, which does not need to be enriched.

So we replace our diagrams

$$\begin{aligned} \mathbf{U}_{B,n} &\longrightarrow \mathcal{S}p \\ \underline{i} \amalg B &\rightsquigarrow X^i \\ \underline{i} \amalg B &\rightsquigarrow F(\underline{i} \amalg B) \end{aligned}$$

with the diagrams of co-cross effects

$$\begin{aligned} \mathbf{M}_n &\longrightarrow \mathcal{S}p \\ \underline{i} &\rightsquigarrow X^{\bar{\wedge} i} \\ \underline{i} &\rightsquigarrow \text{cocross}_i F(\underline{1} \amalg B, \dots, \underline{1} \amalg B) \end{aligned}$$

where  $\bar{\wedge}$  is the external smash product from Def. 2.3.3. We are being sloppy about the existence of maps into  $B^i$ , but this gives enough intuition to suggest that we try the following construction on retractive simplicial sets  $X$  over  $S.B$ :

$$\begin{aligned} E_n F(X) &= \text{holim}_{(\underline{i} \leftarrow \underline{j}) \in \mathbf{aM}_n^{\text{op}}} \left( \text{holim}_{\Delta_{X, \bar{\wedge} i}} \text{cocross}_j F(\Delta^p \amalg B, \dots, \Delta^p \amalg B) \right) \\ &\simeq \text{holim}_{\mathbf{M}_n^{\text{op}} \int \Delta_{X, \bar{\wedge} i}} \text{cocross}_i F(\Delta^p \amalg B, \dots, \Delta^p \amalg B) \end{aligned}$$

As before, the equivalence comes from Prop. 2.6.7. Here  $X^{\bar{\wedge} i}$  is a simplicial set containing  $(S.B)^i$  as a retract, whose fiber over a simplex in  $(S.B)^i$  is the smash product of the fibers

in  $X$ . The homotopy type of  $X^{\bar{i}}$  is homotopy invariant in  $X$  by the same argument as Prop. 2.3.8 above. As before, we extend  $E_n F$  to spaces by  $E_n F(X) := E_n F(S.X)$ .

Each surjective map  $\underline{i} \leftarrow \underline{j}$  induces a cofibration  $X^{\bar{i}} \rightarrow X^{\bar{j}}$ . This determines a functor  $\Delta : \mathbf{M}_n^{\text{op}} \rightarrow \mathbf{Cat}$  by that sends  $\underline{i}$  to the category  $\Delta_{X^{\bar{i}}}$ . The diagram

$$\mathbf{M}_n^{\text{op}} \int \Delta_{X^{\bar{i}}} \xrightarrow{\text{cocross}_i F} \mathcal{S}p$$

is then defined by

$$\begin{array}{ccc} \underline{i}, & \Delta[p] \longrightarrow X^{\bar{i}} & \rightsquigarrow \text{cocross}_i F(\Delta^p \amalg B, \dots, \Delta^p \amalg B) \\ \uparrow & \uparrow & \downarrow \\ \underline{j}, & \Delta[q] \longrightarrow X^{\bar{j}} & \rightsquigarrow \text{cocross}_j F(\Delta^q \amalg B, \dots, \Delta^q \amalg B) \end{array}$$

The map of co-cross effects is defined in the proof of Prop. 2.8.7 above. We can show that  $E_n F$  is  $n$ -excisive by proving properties (1), (2), and (3) from section 2.7. Property (1) follows from the above equivalences, and property (3) is straightforward. We will do (2) in detail.

As before, we can start with a pushout cube of simplicial sets, with initial vertex  $A \in \mathbf{sSet}_{S,B}$ . It's indexed by a set  $S$ , so for each element  $s \in S$ , let  $X_s \in \mathbf{sSet}_{S,B}$  be a simplicial set containing  $A$  (and also containing  $S.B$  as a retract). Then there is a pushout cube which assigns each subset  $T \subset S$  to the simplicial set  $\bigcup_{t \in T} X_t$ , which is shorthand for the pushout of the  $X_t$  along  $A$ . We want to show that  $E_n F$  takes this to a Cartesian cube; in other words, the natural map

$$\begin{aligned} & \text{holim}_{\Delta[p] \rightarrow (\bigcup_S X_s)^{\bar{i}}} \text{cocross}_i F(\Delta^p \amalg B, \dots, \Delta^p \amalg B) \\ & \longrightarrow \text{holim}_{(T \subsetneq S)^{\text{op}}} \left( \text{holim}_{\Delta[p] \rightarrow (\bigcup_T X_s)^{\bar{i}}} \text{cocross}_i F(\Delta^p \amalg B, \dots, \Delta^p \amalg B) \right) \end{aligned}$$

is an equivalence. Using dual Thomason, we rewrite the right-hand side as

$$\text{holim}_{(T, \underline{i}, \Delta[p] \rightarrow (\bigcup_T X_s)^{\bar{i}})} \text{cocross}_i F(\Delta^p \amalg B, \dots, \Delta^p \amalg B)$$

where each object of the indexing category is a proper subset  $T \subsetneq S$ , integers  $p \geq 0$  and

$0 \leq i \leq n$ , and a map  $\Delta[p] \rightarrow (\bigcup_T X_s)^{\bar{\wedge} i}$ . A map between two objects looks like

$$\begin{array}{ccccc}
 & T, & \dot{i}, & \Delta[p] \longrightarrow & (\bigcup_T X_s)^{\bar{\wedge} i} \\
 & \uparrow & \uparrow & & \downarrow \\
 \text{subset} & & & & (\bigcup_T X_s)^{\bar{\wedge} j} \\
 & U, & \dot{j}, & \Delta[q] \longrightarrow & (\bigcup_U X_s)^{\bar{\wedge} j}
 \end{array}$$

As before, this category maps forward into  $\mathbf{M}_n^{\text{op}} \int \Delta_{(\bigcup_S X_s)^{\bar{\wedge} i}}$ , in which a map between two objects looks like

$$\begin{array}{ccc}
 \dot{i}, & \Delta[p] \longrightarrow & (\bigcup_S X_s)^{\bar{\wedge} i} \\
 \uparrow & & \downarrow \\
 \dot{j}, & \Delta[q] \longrightarrow & (\bigcup_S X_s)^{\bar{\wedge} j}
 \end{array}$$

This functor  $\alpha$  forgets the data of  $T$  and includes  $(\bigcup_T X_s)^{\bar{\wedge} i}$  into  $(\bigcup_S X_s)^{\bar{\wedge} i}$ . The natural map of homotopy limits defined above is again induced by a pullback of diagrams along  $\alpha$ , so we just have to show that  $\alpha$  is homotopy initial. Given an object  $\Delta[q] \xrightarrow{\varphi} (\bigcup_S X_s)^{\bar{\wedge} j}$  in the target category, the overcategory  $(\alpha \downarrow \varphi)$  has as its objects the factorizations of  $\varphi$

$$\Delta[q] \longrightarrow \Delta[p] \longrightarrow \left( \bigcup_T X_s \right)^{\bar{\wedge} i} \longrightarrow \left( \bigcup_S X_s \right)^{\bar{\wedge} j}$$

where  $T \subsetneq S$  must be a *proper* subset of  $S$ .

Let us give a terminal object for this overcategory. Either the map out of  $\Delta^q$  hits the basepoint section, in which case we take  $T = \emptyset$ , or it misses the basepoint section, in which case it gives a  $j$ -tuple of simplices in  $\bigcup_S X_s$ , each of which lands *inside* one of the sets  $X_s$  in the pushout. Therefore there is a smallest subset  $T \subset S$  such that  $\Delta[q] \xrightarrow{\varphi} (\bigcup_S X_s)^{\bar{\wedge} j}$  lands inside  $(\bigcup_T X_s)^{\bar{\wedge} j}$ , and since  $j \leq n < |S|$ , this subset is proper. This gives a terminal object for the overcategory  $(\alpha \downarrow \varphi)$ , so it's contractible, which finishes (2).

We might now expect that  $F \rightarrow E_n F$  is an equivalence on  $\mathcal{R}_{B,n}^{\text{op}}$ . This turns out to be false, but Corollary 2.8.4 suggests the following fix. Define a new functor

$$P_n F(X) = \overline{E_n F}(X) \times F(0_B)$$

Note that  $P_n F(X)$  is  $n$ -excisive because it is a homotopy limit of  $n$ -excisive functors.

Now let  $X = \underline{j} \amalg B$ . Then  $X^{\wedge i} \cong (\underline{j})^i \amalg B$ . We can partition  $\Delta_{X^{\wedge i}}$  into two categories, one in which the simplex lands in the basepoint section and another in which the simplex misses the basepoint section. This leads to a partition of  $\mathbf{M}_n^{\text{op}} \int \Delta$  into three categories, one in which there are no simplices, one in which the simplices land in  $B$ , and one in which the simplices miss  $B$ . The homotopy limit of the first two is  $E_n F(0_B)$ , which contains the homotopy limit of the first  $F(0_B)$ . The homotopy limit of the last category is therefore  $\overline{E_n F}(\underline{j} \amalg B)$ . This last category contains a homotopy initial subcategory of objects  $\Delta[0] \times \underline{i} \hookrightarrow \underline{j}$ , with  $i \neq 0$  and  $\underline{i} \hookrightarrow \underline{j}$  an order-preserving inclusion. Therefore

$$\overline{E_n F}(\underline{j} \amalg B) \simeq \operatorname{holim}_{0 \neq \underline{i} \hookrightarrow \underline{j}} \operatorname{cocross}_i F(\underline{1} \amalg B, \dots, \underline{1} \amalg B)$$

But the only surjective maps between subsets of  $\underline{j}$  that respect the inclusion into  $\underline{j}$  are identity maps. So this homotopy limit is an ordinary product:

$$\overline{E_n F}(\underline{j} \amalg B) \simeq \prod_{0 \neq \underline{i} \subset \underline{j}} \operatorname{cocross}_i F(\underline{1} \amalg B, \dots, \underline{1} \amalg B)$$

$$P_n F(\underline{j} \amalg B) \simeq \prod_{\underline{i} \subset \underline{j}} \operatorname{cocross}_i F(\underline{1} \amalg B, \dots, \underline{1} \amalg B)$$

Using our splitting result (Prop. 2.8.5), this shows that  $F(\underline{j} \amalg B) \rightarrow P_n F(\underline{j} \amalg B)$  is an equivalence. This finishes the proof that  $P_n F$  exists for  $F$  from retractive spaces over  $B$  into spectra:

**Theorem 2.8.8.** *If  $F : \mathcal{R}_{B, \text{fin}}^{\text{op}} \rightarrow \mathcal{S}p$  is a homotopy functor, then there is a universal  $n$ -excisive approximation  $P_n F$ , and  $F \rightarrow P_n F$  is an equivalence on spaces with at most  $n$  points.*

## Chapter 3

# Coassembly and duality in $THH$

In this chapter we study the topological Hochschild homology ( $THH$ ) of ring spectra and spectral categories that are associated to contravariant forms of algebraic  $K$ -theory of spaces. The simplest example is  $THH(DX_+)$ , where  $DX_+$  is the functional dual of the unbased CW complex  $X$ . In this case we show that the known equivalence of spectra (cf. [Cam14], [AF])

$$D(THH(DX_+)) \simeq \Sigma_+^\infty LX \simeq THH(\Sigma_+^\infty \Omega X)$$

can be extended to an equivalence of cyclotomic spectra. In the process of setting this result up we review cyclic spaces and spectra, orthogonal  $G$ -spectra and fixed points, and the cyclic bar construction. We also prove along the way a rigidity result for the geometric fixed points  $\Phi^G X$  which appears to be new.

Next we demonstrate a splitting on  $THH(DX_+)$  when  $X$  is a reduced suspension, and use this splitting to recover the stable splitting of the free loop space

$$\Sigma^\infty L\Sigma X \simeq \bigvee_{n=1}^{\infty} \Sigma^\infty S_+^1 \wedge_{C_n} X^{\wedge n}$$

found in [Coh87]. We analyze the case of  $DS_+^1$ , calculating  $TC$  and demonstrating that the coassembly map

$$K(DS^1) \longrightarrow \text{Map}(S^1, K(\mathbb{S}))$$

is not split surjective; however  $K(DS^1) \not\cong \forall(S^1)$  and so this does not completely count out

the possibility of a “dual”  $A$ -theory Novikov conjecture.

Finally we move to the case  $X = BG$  when  $G$  is a finite group, and study the  $THH$  functor associated to  $\mathbb{V}(BG)$ . We demonstrate that the composite of a certain assembly map with coassembly results in a transfer map. This has the surprising corollary that when  $G$  is a finite  $p$ -group the coassembly map

$$\mathbb{V}(BG) \longrightarrow F(BG_+, \mathbb{V}(*))$$

is split surjective after  $p$ -completion, as a map of coarse  $S^1$ -spectra.

### 3.1 Cyclic spaces and cocyclic spaces

A *cyclic set* is a simplicial set with extra structure, which allows the geometric realization to carry a natural  $S^1$ -action [Con83]. One may use the analogous concepts of *cyclic spaces* and *cyclic spectra* to efficiently build some rather sophisticated spaces and spectra with  $S^1$ -actions. In this section we will review the theory of cyclic sets, and extend the existing theory to cocyclic spaces and (co)cyclic orthogonal spectra. This is all standard material from [DHK85], [Jon87], [BHM93], and [Mad95] or a straightforward generalization thereof, but we make an effort to be definite and explicit in areas where our later proofs require it.

#### 3.1.1 The category $\mathbf{\Delta}$ and the natural circle action.

To begin, recall the category  $\mathbf{\Delta}$  and the notion of geometric realization:

**Definition 3.1.1.**  $\mathbf{\Delta}$  is a category with one object  $[n] = \{0, 1, \dots, n\}$  for each  $n \geq 0$ . The morphisms  $\mathbf{\Delta}([m], [n])$  are the functions  $f : [m] \rightarrow [n]$  which preserve the total ordering.

The category  $\mathbf{\Delta}$  is generated by the *coface maps*

$$d^i : [n - 1] \longrightarrow [n], \quad 0 \leq i \leq n$$

$$d^i(j) = \begin{cases} j & \text{if } j < i \\ j + 1 & \text{if } j \geq i \end{cases}$$

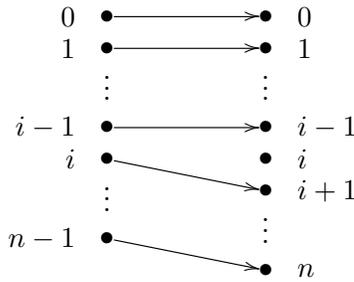


Figure 3.1: The coface map  $d^i$ .

and *codegeneracy maps*

$$s^i : [n + 1] \longrightarrow [n], \quad 0 \leq i \leq n$$

$$s^i(j) = \begin{cases} j & \text{if } j \leq i \\ j - 1 & \text{if } j > i \end{cases}$$

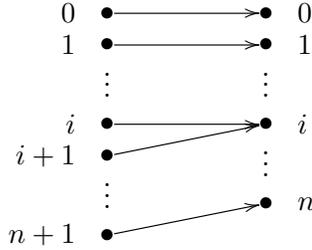


Figure 3.2: The codegeneracy map  $s^i$ .

**Definition 3.1.2.** A *simplicial object* of  $\mathbf{C}$  is a contravariant functor  $X_\bullet : \Delta^{\text{op}} \rightarrow \mathbf{C}$ .

We are particularly interested in the case where  $\mathbf{C}$  is either based spaces or orthogonal spectra. We will use the fact that any simplicial object  $X_\bullet$  is a coequalizer of representable ones:

$$\bigvee_{m,n} \Delta(\bullet, [m])_+ \wedge \Delta([m], [n])_+ \wedge X_n \rightrightarrows \bigvee_n \Delta(\bullet, [n])_+ \wedge X_n \rightarrow X_\bullet$$

Because of this, a left adjoint like geometric realization is determined by what it does to the simplicial based set  $\Delta[n]_+ = \Delta(\bullet, [n])_+$ . As usual, we let  $\Delta^n$  be the convex hull of the standard basis vectors in  $\mathbb{R}^{n+1}$ , which has coordinates

$$\Delta^n = \left\{ t \in \mathbb{R}^{n+1} : \sum t_i = 1, \quad t_i \geq 0 \quad \forall i \right\}$$

Then  $\Delta([m], [n])$  takes each function  $f : [m] \rightarrow [n]$  to the unique linear map  $\Delta^m \rightarrow \Delta^n$  given by  $f$  on the vertices. Then geometric realization is the unique left adjoint which takes  $\Delta[n]_+$  to  $\Delta_+^n$ :

**Definition 3.1.3.** The *geometric realization* of a simplicial based space  $X_\bullet : \Delta^{\text{op}} \rightarrow \mathbf{Top}_*$  is the coequalizer of

$$\prod_{m,n} \Delta^m \times \Delta([m], [n]) \times X_n \rightrightarrows \prod_n \Delta^n \times X_n$$

or, equivalently, the coequalizer of

$$\bigvee_{m,n} \Delta_+^m \wedge \mathbf{\Delta}([m], [n])_+ \wedge X_n \rightrightarrows \bigvee_n \Delta_+^n \wedge X_n$$

The geometric realization of simplicial orthogonal spectra is given by the same construction applied to each spectrum level.

Now we review Connes' cyclic category  $\mathbf{\Lambda}$ , which contains  $\mathbf{\Delta}$  as a wide subcategory [Con83]. This means that  $\mathbf{\Lambda}$  has the same objects, one object  $[n]$  for each natural number  $n \geq 0$ . The morphisms can be described in several equivalent ways:

- A map  $[m] \rightarrow [n]$  in  $\mathbf{\Lambda}$  is an equivalence class of increasing functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  satisfying

$$f(a + m + 1) = f(a) + n + 1$$

subject to the relation  $f \sim f + n + 1$ . Each map in  $\mathbf{\Delta}$  determines such a function  $f$  on the subset  $\{0, \dots, m\}$  which may then be extended in a periodic way.

- A map  $[m] \rightarrow [n]$  in  $\mathbf{\Lambda}$  is an equivalence class of increasing degree 1 maps  $S^1 \rightarrow S^1$  sending the subgroup  $\mathbb{Z}/(m + 1)$  into the subgroup  $\mathbb{Z}/(n + 1)$ , up to any homotopy preserving this condition. In particular, the images of the points in  $\mathbb{Z}/(m + 1)$  cannot move during the homotopy. This is the original definition by Connes [Con83].
- Let  $[n]$  denote the free category on the arrows

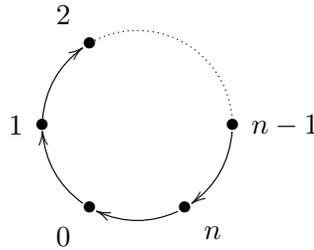


Figure 3.3: The circle category  $[n]$ .

The geometric realization of  $[n]$  is homotopy equivalent to a circle. Then  $\mathbf{\Lambda}([m], [n])$  consists of those functors  $[m] \rightarrow [n]$  which give a degree 1 map on the geometric realizations.

- $\mathbf{\Lambda}$  is the free category on  $\mathbf{\Delta}$  and a *cycle map*  $\tau_n : [n] \rightarrow [n]$  for each  $n \geq 0$ , subject to the additional relations found in [BHM93]:

$$\begin{aligned} \tau_n d^i &= d^{i-1} \tau_{n-1} & 1 \leq i \leq n \\ \tau_n d^0 &= d^n \\ \tau_n s^i &= s^{i-1} \tau_{n+1} & 1 \leq i \leq n \\ \tau_n s^0 &= s^n \tau_{n+1}^2 & 1 \leq i \leq n \\ \tau_n^{n+1} &= \text{id} \end{aligned}$$

In the above definitions  $\tau_n$  corresponds to  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  sending  $x$  to  $x - 1$ , or the circle map  $S^1 \rightarrow S^1$  which multiplies by  $e^{-2\pi i/(n+1)}$ , or the functor

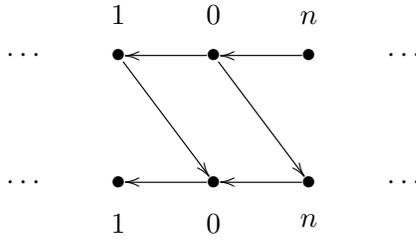


Figure 3.4: The cycle map  $\tau_n$ .

- $\mathbf{\Lambda}$  is the free category on  $\mathbf{\Delta}$  and an *extra degeneracy map*  $s^{n+1} : [n+1] \rightarrow [n]$  for each  $n \geq 0$ , subject to the additional relations found in [DHK85]:

$$\begin{aligned} s^i s^{n+1} &= s^n s^i & 0 \leq i \leq n \\ s^{n+1} d^i &= d^i s^n & 1 \leq i \leq n \\ (s^{n+1} d^0)^{n+1} &= \text{id} \end{aligned}$$

In the above definitions  $s^{n+1}$  corresponds to the function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  which is the identity on  $\{0, \dots, n+1\}$  and is extended periodically. It also corresponds to functor  $[n+1] \rightarrow [n]$  which preserves the arrow  $i \rightarrow i+1$  for all  $0 \leq i \leq n$  but sends the arrow  $n+1 \rightarrow 0$  to  $\text{id}_0$ :

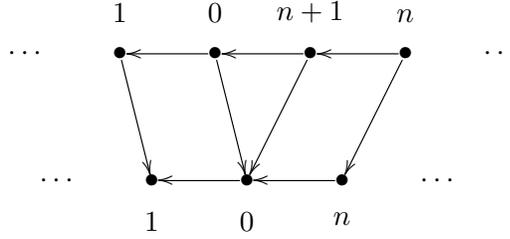


Figure 3.5: The extra degeneracy map  $s^{n+1}$ .

The operations  $s^{n+1}$  and  $\tau_n$  determine each other:

$$\begin{aligned} \tau_n^{-1} &= s^{n+1}d^0 \\ s^{n+1} &= s^0\tau_n^{-1} \end{aligned}$$

**Definition 3.1.4.** The *cyclic category*  $\mathbf{\Lambda}$  is the category defined in any of the above ways.

It is clear that each morphism  $f \in \mathbf{\Lambda}([m], [n])$  gives a well-defined map of sets  $\mathbb{Z}/(m + 1) \rightarrow \mathbb{Z}/(n + 1)$ . This rule respects composition (i.e. it defines a functor  $\mathbf{\Lambda} \rightarrow \mathbf{Set}$ ). Conversely, each map of sets  $\mathbb{Z}/(m + 1) \rightarrow \mathbb{Z}/(n + 1)$  comes from at most one such  $f$ , unless this map of sets is constant, in which case there are  $n + 1$  choices for  $f$ .

**Definition 3.1.5.** A *cyclic based space* is a functor  $X_\bullet : \mathbf{\Lambda}^{\text{op}} \rightarrow \mathbf{Top}_*$ . The *geometric realization*  $|X_\bullet|$  is defined by restricting  $X_\bullet$  to  $\mathbf{\Delta}^{\text{op}}$  and taking the geometric realization of the resulting simplicial space.

The geometric realization of a cyclic space has extra structure which we now examine following [DHK85].

Let  $\Lambda[n] = \mathbf{\Lambda}(-, [n])$  denote the contravariant functor on  $\mathbf{\Lambda}$  represented by the object  $[n]$ . Then  $\Lambda[n]$  is a cyclic set, called the *standard cyclic  $n$ -simplex*. It has one  $k$ -simplex for every point in the set  $\mathbf{\Lambda}([k], [n])$ . Partition  $\mathbf{\Lambda}([k], [n])$  into two subsets

$$\mathbf{\Lambda}([k], [n]) = \mathbf{\Delta}([k], [n]) \cup \mathbf{\Delta}([k], [n])^c$$

The first subset  $\mathbf{\Delta}([k], [n])$  contains all functors in the first figure below which restrict to a functor pictured in the second figure below.

These are  $k$ -simplices in  $\Delta^n$  in the usual sense, and they are classified as  $(k + 1)$ -tuples of increasing integers

$$(i_0, \dots, i_k) \quad 0 \leq i_j \leq n \quad i_j \leq i_{j+1}$$

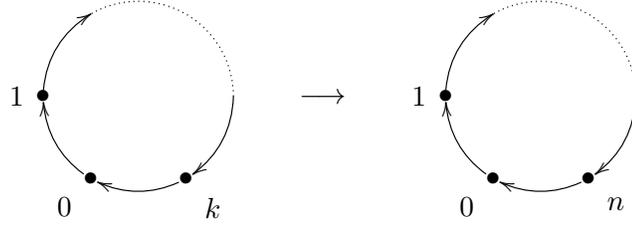


Figure 3.6: A map  $[k]_{\Lambda} \rightarrow [n]_{\Lambda}$ .

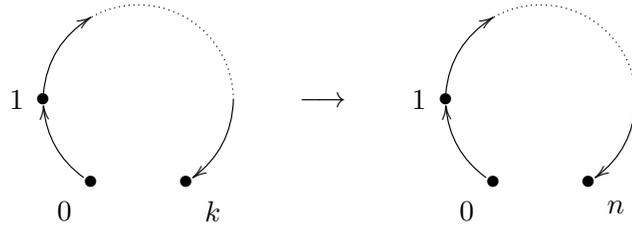


Figure 3.7: A map  $[k]_{\Delta} \rightarrow [n]_{\Delta}$ .

Each face map deletes one integer from the set, and each degeneracy map duplicates one integer in the set. (This is the simplicial structure so we are not allowing the extra degeneracy here.)

Each element of the complement  $\Delta([k], [n])^c$  is some functor  $f : [k] \rightarrow [n]$  which sends some unique arrow  $j \rightarrow j + 1$  with  $0 \leq j$  and  $j + 1 \leq k$  to a composition of arrows including  $n \rightarrow 0$ . These functors are classified by two ascending sets of integers in  $\{0, \dots, n\}$ , one for the image of  $0 \rightarrow \dots \rightarrow j$  and one of the image of  $j + 1 \rightarrow \dots \rightarrow k$ , and the last integer from the second set cannot be greater than the first integer of the first set. We may rewrite this as a  $(k + 1)$ -tuple of ordered pairs

$$((0, i_0), (0, i_1), \dots, (0, i_j), (1, i_{j+1}), \dots, (1, i_k)) \quad i_a \in \{0, \dots, n\}$$

subject to the ordering condition

$$i_{j+1} \leq i_{j+2} \leq \dots \leq i_k \leq i_0 \leq i_1 \leq \dots \leq i_j$$

As before, each face map deletes an ordered pair from this set and each degeneracy maps duplicates one pair. It is possible for a face map to bring us back over to the first set  $\Delta([k], [n])$ , by deleting the last pair of the form  $(0, -)$  or the last pair of the form  $(1, -)$ .

At this point it is natural start thinking about the simplicial complex  $P_n$  on the totally

ordered set

$$\{(0, 0), (0, 1), \dots, (0, n), (1, 0), (1, 1), \dots, (1, n)\}$$

which has a  $k$ -simplex for any  $(k + 1)$ -tuple chosen from this set and satisfying the same ordering condition as before. Restricting to tuples of the form  $(0, -)$  gives a subcomplex  $P_n^0$  isomorphic to  $\Delta[n] = \mathbf{\Delta}(-, [n])$ . Restricting to tuples of the form  $(1, -)$  gives a second subcomplex  $P_n^1$  isomorphic to  $\Delta[n]$ . The remaining simplices correspond with those in  $\mathbf{\Delta}(-, [n])^c \subset \mathbf{\Lambda}(-, [n])$ , so if we identify  $P_n^0 \sim P_n^1$  we recover the simplicial set  $\mathbf{\Lambda}[n]$ .

We provide below some pictures of  $P_n$  to build geometric intuition.

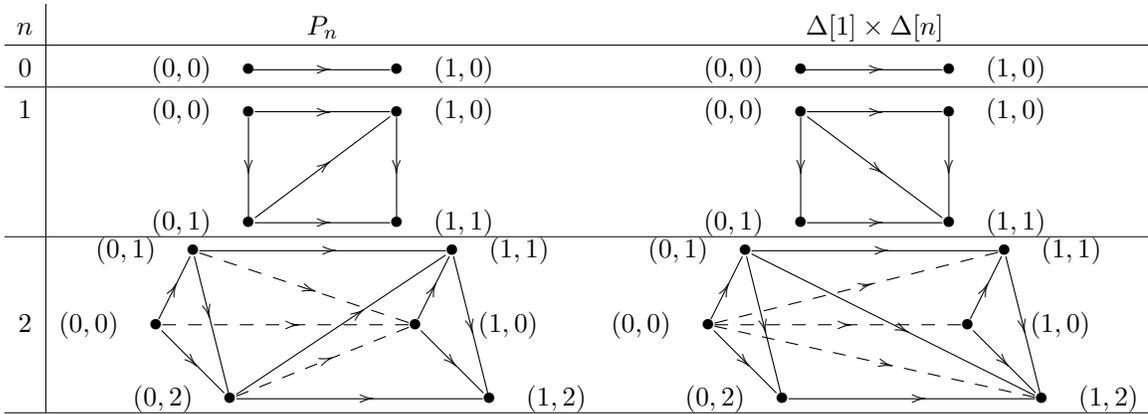


Figure 3.8: Sketch of  $P_n$  for  $n = 0, 1,$  and  $2$ .

There is an evident bijection between the vertices of  $P_n$  and the vertices of the simplicial prism  $[0, 1] \times \Delta^n$ . This map of vertices extends by linear interpolation to a continuous map of topological spaces

$$g : |P_n| \longrightarrow [0, 1] \times \Delta^n$$

The above pictures suggest that  $g$  is a homeomorphism. This is easily proven by flipping the interval  $[0, 1]$  and comparing the resulting CW complex to the product  $|\Delta[1] \times \Delta[n]|$ . Once this is established, we identify subcomplexes together to get a homeomorphism

$$\begin{aligned} |\mathbf{\Lambda}[n]| &\cong |P_n| / (|P_n^0| \sim |P_n^1|) \\ &\cong [0, 1] \times \Delta^n / ((0 \times \Delta^n) \sim (1 \times \Delta^n)) \\ &\cong S^1 \times \Delta^n \end{aligned}$$

We define an  $S^1$ -action on  $|\Lambda[n]|$  by acting on the first coordinate of  $S^1 \times \Delta^n$ :

$$\theta \cdot (\phi, u_0, \dots, u_n) = (\theta + \phi, u_0, \dots, u_n)$$

The maps in  $\mathbf{\Lambda}$  give natural maps between these standard cyclic simplices which are all  $S^1$ -equivariant. The trickiest one to check is  $\tau_n$ , whose action on  $|\Lambda[n]| \cong S^1 \times \Delta^n$  is given by the formula

$$\tau_n(\theta, u_0, u_1, \dots, u_{n-1}, u_n) = (\theta - u_0, u_1, u_2, \dots, u_n, u_0)$$

which is clearly  $S^1$ -equivariant. Since geometric realization commutes with colimits, every cyclic based space  $X_\bullet$  has geometric realization given by the coequalizer

$$\bigvee_{m,n} |\Lambda[n]|_+ \wedge \mathbf{\Lambda}([n], [m])_+ \wedge X_m \rightrightarrows \bigvee_n |\Lambda[n]|_+ \wedge X_n \rightarrow |X_\bullet|$$

Since the  $S^1$ -action on  $|\Lambda[n]|$  commutes with the action of  $\mathbf{\Lambda}$ , it gives a well-defined natural  $S^1$ -action on  $|X_\bullet|$ :

**Theorem 3.1.6.** *The geometric realization  $|X_\bullet|$  of a cyclic based space  $X$  carries a natural based  $S^1$ -action.*

For the purposes of our calculation, we need a different homeomorphism

$$|\Lambda[n]| \cong S^1 \times \Delta^n$$

By abuse of notation let  $g$  denote our original homeomorphism. Then compose  $g$  with the  $S^1$ -equivariant map

$$\begin{aligned} f : S^1 \times \Delta^n &\longrightarrow S^1 \times \Delta^n \\ f(\theta, u_0, \dots, u_n) &= \left( \theta + \frac{0u_0 + 1u_1 + 2u_2 + \dots + nu_n}{n+1}, u_0, \dots, u_n \right) \\ &= \left( \theta - \frac{(n+1)u_0 + nu_1 + \dots + u_n}{n+1}, u_0, \dots, u_n \right) \end{aligned}$$

The following diagram commutes

$$\begin{array}{ccccc}
 |\Lambda[n]| & \xrightarrow[\cong]{g} & S^1 \times \Delta^n & \xrightarrow[\cong]{f} & S^1 \times \Delta^n \\
 \downarrow \tau_n & & \downarrow (\theta - u_0, u_1, u_2, \dots, u_n, u_0) & & \downarrow (\theta - \frac{1}{n+1}, u_1, u_2, \dots, u_n, u_0) \\
 |\Lambda[n]| & \xrightarrow[\cong]{g} & S^1 \times \Delta^n & \xrightarrow[\cong]{f} & S^1 \times \Delta^n
 \end{array}$$

and so under the homeomorphism given by  $f \circ g$ , the cycle map  $\tau_n$  acts by cycling the vertices of  $\Delta^n$  and decreasing the circle coordinate  $\theta$  by  $\frac{1}{n+1}$ . We provide a brief sketch to give intuition for  $f \circ g$ .

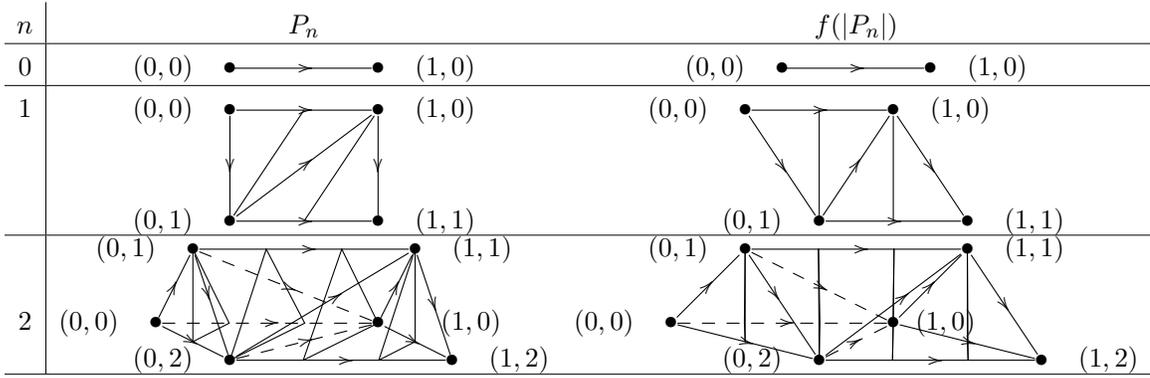


Figure 3.9: Sketch of  $|\Lambda[n]|$  under different coordinate systems.

### 3.1.2 Skeleta and latching objects.

Recall that when  $X_\bullet$  is a simplicial space, the  $n$ th *skeleton*  $\text{Sk}_n X_\bullet$  is obtained by restricting  $X_\bullet$  to the subcategory of  $\Delta^{\text{op}}$  on the objects  $0, \dots, n$  and then taking a left Kan extension back. This may be re-expressed as the coequalizer

$$\bigvee_{k, \ell \leq n} \Delta(\bullet, [k])_+ \wedge \Delta([k], [\ell])_+ \wedge X_\ell \rightrightarrows \bigvee_{k \leq n} \Delta(\bullet, [k])_+ \wedge X_k \rightarrow \text{Sk}_n X_\bullet$$

and so the realization of the skeleton is the coequalizer

$$\bigvee_{k, \ell \leq n} \Delta_+^k \wedge \Delta([k], [\ell])_+ \wedge X_\ell \rightrightarrows \bigvee_{k \leq n} \Delta_+^k \wedge X_k \rightarrow |\text{Sk}_n X_\bullet|$$

Clearly  $|\mathrm{Sk}_n X_\bullet|$  is covered by the two spaces  $\bigvee_{k \leq n-1} \Delta_+^k \wedge X_k$  and  $\Delta_+^n \wedge X_n$ . This leads to a standard pushout square

$$\begin{array}{ccc} L_n X \times \Delta^n \cup_{L_n X \times \partial \Delta^n} X_n \times \partial \Delta^n & \longrightarrow & X_n \times \Delta^n \\ \downarrow & & \downarrow \\ |\mathrm{Sk}_{n-1} X_\bullet| & \longrightarrow & |\mathrm{Sk}_n X_\bullet| \end{array} \quad (3.1)$$

where  $L_n X$  is the  $n$ th *latching object*  $L_n X \subset X_n$ , defined as the subspace consisting of all points lying in the image of some degeneracy map

$$s_i : X_{n-1} \longrightarrow X_n, \quad 0 \leq i \leq n-1$$

The simplicial space  $X_\bullet$  is said to be *Reedy cofibrant* if each  $L_n X \longrightarrow X_n$  is a cofibration in an appropriate sense. To be precise:

**Definition 3.1.7.**  $X_\bullet$  is *Reedy  $q$ -cofibrant* if each  $L_n X \longrightarrow X_n$  is a cofibration in the Quillen model structure on based spaces.  $X_\bullet$  is *Reedy  $h$ -cofibrant* if each  $L_n X \longrightarrow X_n$  is a classical cofibration, i.e. a map satisfying the HEP in the unbased sense.

Of course every Reedy  $q$ -cofibrant space is Reedy  $h$ -cofibrant. The standard result is then

**Theorem 3.1.8.** *For either notion of “cofibration,” if  $X_\bullet$  is Reedy cofibrant then each  $|\mathrm{Sk}_{n-1} X_\bullet| \longrightarrow |\mathrm{Sk}_n X_\bullet|$  is a cofibration and therefore  $|X_\bullet|$  is cofibrant. Moreover if  $X_\bullet, Y_\bullet$  are Reedy cofibrant then any levelwise weak equivalence  $X_\bullet \xrightarrow{\sim} Y_\bullet$  induces an equivalence on all skeleta  $|\mathrm{Sk}_n X_\bullet| \xrightarrow{\sim} |\mathrm{Sk}_n Y_\bullet|$  and therefore an equivalence on realizations  $|X_\bullet| \xrightarrow{\sim} |Y_\bullet|$ .*

The proof of this theorem relies on an induction using the pushout square for each  $1 \leq k \leq n-1$

$$\begin{array}{ccc} s_k \left( \bigcup_{i=0}^{k-1} s_i(X_{n-2}) \right) & \longrightarrow & \bigcup_{i=0}^{k-1} s_i(X_{n-1}) \\ \downarrow & & \downarrow \\ s_k(X_{n-1}) & \longrightarrow & \bigcup_{i=0}^k s_i(X_{n-1}) \end{array}$$

and the usual pushout and pushout-product properties that cofibrations typically satisfy. If one wants to use based  $h$ -cofibrations then one must also assume that all the spaces are well-based.

It is not hard to give a version of this theorem for orthogonal spectra. There is a standard compactly-generated model structure which provides the  $q$ -cofibrations, and the  $h$ -cofibrations are defined as maps having the obvious HEP with respect to the notion of homotopy given by  $X \wedge I_+$ . Then one can show

**Theorem 3.1.9.** *For either notion of “cofibration,” if  $X_\bullet$  is a Reedy cofibrant simplicial orthogonal spectrum then each  $|\mathrm{Sk}_{n-1}X_\bullet| \rightarrow |\mathrm{Sk}_nX_\bullet|$  is a cofibration and therefore  $|X_\bullet|$  is cofibrant. Moreover if  $X_\bullet, Y_\bullet$  are Reedy cofibrant then any levelwise stable equivalence  $X_\bullet \xrightarrow{\sim} Y_\bullet$  induces an equivalence on all skeleta  $|\mathrm{Sk}_nX_\bullet| \xrightarrow{\sim} |\mathrm{Sk}_nY_\bullet|$  and therefore an equivalence on realizations  $|X_\bullet| \xrightarrow{\sim} |Y_\bullet|$ .*

*Proof.* Easy adaptation of the space-level proof for  $q$ -cofibrations. For  $h$ -cofibrations this is a little surprising since we do not assume any of the spectra involved are well-based. The hardest piece of the proof is the statement that if  $f : K \rightarrow L$  is an  $h$ -cofibration of unbased spaces and  $g : A \rightarrow X$  is an  $h$ -cofibration of orthogonal spectra then the pushout-product  $f \square g$  is an  $h$ -cofibration. This follows from the formal pairing result of Schwai z and Vogt quoted in ([MS06], Thm. 4.3.2(i)), together with the fact that  $h$ -cofibrations of unbased spaces are “strong,” but that comes from Str m’s result quoted in ([MS06], Thm 4.4.4(ii)).  $\square$

By analogy with this, and following [BM11a], when  $X_\bullet$  is a cyclic space we define the  $n$ th cyclic skeleton  $\mathrm{Sk}_n^{\mathrm{cyc}}X$  by restricting  $X_\bullet$  to the subcategory of  $\Lambda^{\mathrm{op}}$  on the objects  $0, \dots, n$  and then taking a left Kan extension back. This may be re-expressed as the coequalizer

$$\bigvee_{k, \ell \leq n} \Lambda(\bullet, [k])_+ \wedge \Lambda([k], [\ell])_+ \wedge X_\ell \rightrightarrows \bigvee_{k \leq n} \Lambda(\bullet, [k])_+ \wedge X_k \rightarrow \mathrm{Sk}_n^{\mathrm{cyc}}X_\bullet$$

and so the realization of the skeleton is the coequalizer

$$\bigvee_{k, \ell \leq n} \Lambda_+^k \wedge \Lambda([k], [\ell])_+ \wedge X_\ell \rightrightarrows \bigvee_{k \leq n} \Lambda_+^k \wedge X_k \rightarrow |\mathrm{Sk}_n^{\mathrm{cyc}}X_\bullet|$$

As a matter of convention, we define the  $(-1)$ st cyclic skeleton as the equalizer of the degeneracy and extra degeneracy maps:

$$\mathrm{Sk}_{-1}^{\mathrm{cyc}}X \rightarrow X_0 \rightrightarrows X_1$$

In the simplicial case above the  $(-1)$ st skeleton was simply the initial object, which is  $*$  in the based case.

Next, define the  $n$ th *cyclic latching object*  $L_n^{\text{cyc}}X \subset X_n$  to be the subspace consisting of all points lying in the image of some degeneracy map

$$s_i : X_{n-1} \longrightarrow X_n, \quad 0 \leq i \leq n$$

The 0th latching object is also taken to be  $\text{Sk}_{-1}^{\text{cyc}}X \subset X_0$  rather than being empty. The only difference between  $L_nX$  and  $L_n^{\text{cyc}}X$  is that the *extra* degeneracy is included in  $L_n^{\text{cyc}}X$ ; equivalently  $L_n^{\text{cyc}}X$  is the closure of  $L_nX$  under the action of the cycle map  $t_n$  of order  $n+1$ . Now we give the analogue of the standard pushout square (3.1):

**Proposition 3.1.10.** *For each  $k \geq 0$  there is a natural pushout square*

$$\begin{array}{ccc} L_k^{\text{cyc}}X \times_{C_{k+1}} \Lambda^k \cup_{L_kX \times \partial\Lambda^k} X_k \times_{C_{k+1}} \partial\Lambda^k & \longrightarrow & X_k \times_{C_{k+1}} \Lambda^k \\ \downarrow & & \downarrow \\ |\text{Sk}_{k-1}^{\text{cyc}}X_\bullet| & \longrightarrow & |\text{Sk}_k^{\text{cyc}}X_\bullet| \end{array} \quad (3.2)$$

for unbased cyclic spaces  $X_\bullet$ , and the obvious variant with smash products for based cyclic spaces  $X_\bullet$ .

*Proof.* The square is clearly defined and natural, and the top horizontal map is the inclusion of a subspace. We treat the case  $k = 0$  separately, where the square becomes

$$\begin{array}{ccc} (L_0^{\text{cyc}}X \times S^1) \amalg \emptyset & \longrightarrow & X_0 \times S^1 \\ \downarrow & & \downarrow \\ L_0^{\text{cyc}}X & \longrightarrow & |\text{Sk}_0^{\text{cyc}}X_\bullet| \end{array}$$

which is easily checked to be a pushout. For  $k \geq 1$ , it suffices to check that it is pushout when  $X_\bullet = \Lambda(\bullet, [n])$  is the standard cyclic  $n$ -simplex, because then we can take a colimit of such things to get general cyclic spaces  $X_\bullet$ . Now when  $X_\bullet = \Lambda(\bullet, [n])$  and  $k \geq 1$  the square

may be rewritten

$$\begin{array}{ccc}
 L_k^{\text{cyc}} \Lambda[n] \times_{C_{k+1}} \Lambda^k \amalg (\Lambda_k[n] - L_k^{\text{cyc}} \Lambda[n]) \times_{C_{k+1}} \partial \Lambda^k & \longrightarrow & \Lambda[n] \times_{C_{k+1}} \Lambda^k \\
 \downarrow & & \downarrow \\
 |\text{Sk}_{k-1}^{\text{cyc}} \Lambda[n]| & \longrightarrow & |\text{Sk}_k^{\text{cyc}} \Lambda[n]|
 \end{array}$$

The top map is a disjoint union of some isomorphisms and some nontrivial inclusions, so we can strike out the isomorphisms without changing whether the square is pushout:

$$\begin{array}{ccc}
 (\Lambda_k[n] - L_k^{\text{cyc}} \Lambda[n]) \times_{C_{k+1}} \partial \Lambda^k & \longrightarrow & (\Lambda_k[n] - L_k^{\text{cyc}} \Lambda[n]) \times_{C_{k+1}} \Lambda^k \\
 \downarrow & & \downarrow \\
 |\text{Sk}_{k-1}^{\text{cyc}} \Lambda[n]| & \longrightarrow & |\text{Sk}_k^{\text{cyc}} \Lambda[n]|
 \end{array}$$

The complement of the latching object  $L_k^{\text{cyc}} \Lambda[n]$  consists of maps in  $\Lambda([k], [n])$  for which the  $k+1$  points  $0, \dots, k$  go to distinct points in  $0, \dots, n$ . The  $C_{k+1}$ -action on these maps is free and each orbit has a unique representative that comes from  $\Delta([k], [n])$ , so we can again simplify the square to

$$\begin{array}{ccc}
 (\Delta_k[n] - L_k \Delta[n]) \times \partial \Lambda^k & \longrightarrow & (\Delta_k[n] - L_k \Delta[n]) \times \Lambda^k \\
 \downarrow & & \downarrow \\
 |\text{Sk}_{k-1}^{\text{cyc}} \Lambda[n]| & \longrightarrow & |\text{Sk}_k^{\text{cyc}} \Lambda[n]|
 \end{array}$$

Now one may identify this square as the standard simplicial pushout square for  $\Delta[n]$ , multiplied by the identity map on  $S^1$ . Alternatively, one can enumerate the cells of  $|\text{Sk}_k^{\text{cyc}} \Lambda[n]|$  missing from  $|\text{Sk}_{k-1}^{\text{cyc}} \Lambda[n]|$  and check that the above map precisely attaches those cells. So the square is a pushout and the proof is complete.  $\square$

From here we could discuss different notions of being Reedy cofibrant in a cyclic setting, but we choose to avoid setting up a general theory that we will not use. In practice, it suffices to use subdivision as in the next section to control the homotopy type of  $|X_\bullet|$ . However we will be interested in dualizing  $|X_\bullet|$  and so we wish to give conditions which guarantee that  $|X_\bullet|$  is a cofibrant  $S^1$ -space.

To be definite about the meaning of ‘‘cofibrant’’ we use the following model structure on  $G$ -spaces.

**Theorem 3.1.11.** *If  $G$  is a compact Lie group, there is a fine or genuine model structure on the category of based  $G$ -spaces and based equivariant maps in which*

- *the cofibrations are the retracts of the relative  $G$ -cell complexes*
- *the weak equivalences are the maps  $X \rightarrow Y$  for which each  $X^H \rightarrow Y^H$  is a weak equivalence*
- *the fibrations are the maps  $X \rightarrow Y$  for which each  $X^H \rightarrow Y^H$  is a Serre fibration*

*This model structure is topological, proper, and monoidal. It is compactly generated [MMSS01] by the sets of cofibrations and acyclic cofibrations*

$$\begin{aligned} \mathbf{I} &= \{(G/H \times S^{n-1})_+ \hookrightarrow (G/H \times D^n)_+ : n \geq 0, H \leq G\} \\ \mathbf{J} &= \{(G/H \times D^n)_+ \hookrightarrow (G/H \times D^n \times I)_+ : n \geq 0, H \leq G\} \end{aligned}$$

We prove the following result as a warm-up to a spectrum-level result needed for our dualization theory:

**Proposition 3.1.12.** *If  $X_\bullet$  is a cyclic space and each cyclic latching map  $L_n^{\text{cyc}} X \rightarrow X_n$  is a cofibration of  $C_{n+1}$ -spaces then  $|X_\bullet|$  is a cofibrant  $S^1$ -space.*

*Proof.* It suffices to show that each map of cyclic skeleta

$$|\text{Sk}_{n-1}^{\text{cyc}} X| \longrightarrow |\text{Sk}_n^{\text{cyc}} X|$$

is an  $S^1$ -cofibration. The  $(-1)$ -skeleton

$$L_0^{\text{cyc}} X = \text{Sk}_{-1}^{\text{cyc}} X$$

is already assumed to be cofibrant, and it has trivial  $S^1$ -action, so it is also  $S^1$ -cofibrant.

For the induction we use the square from Prop. 3.1.10 above

$$\begin{array}{ccc} L_n^{\text{cyc}} X \times_{C_{n+1}} \Lambda^n \cup_{L_n X \times \partial \Lambda^n} X_n \times_{C_{n+1}} \partial \Lambda^n & \longrightarrow & X_n \times_{C_{n+1}} \Lambda^n \\ \downarrow & & \downarrow \\ |\text{Sk}_{n-1}^{\text{cyc}} X_\bullet| & \longrightarrow & |\text{Sk}_n^{\text{cyc}} X_\bullet| \end{array}$$

It suffices to prove that the  $C_{n+1}$ -orbits of the pushout-product of the latching map  $L_n^{\text{cyc}} X \rightarrow X$  and the inclusion  $\partial\Lambda^n \rightarrow \Lambda^n$  is an  $S^1$ -cofibration. We know that the latching map is a  $C_{n+1}$ -cofibration and the inclusion of cyclic simplices is a free  $S^1$ -cofibration. Since the pushout-product and orbits both commute with all colimits, it suffices to show that the simpler pushout-product

$$[(C_{n+1}/C_r \times S^{k-1} \rightarrow C_{n+1}/C_r \times D^k)_+ \square (S^1 \times S^{\ell-1} \rightarrow S^1 \times D^\ell)_+]_{C_{n+1}}$$

is an  $S^1$ -cofibration. By associativity of the pushout-product one may rewrite this as

$$[(C_{n+1}/C_r \times S^1)_+ \wedge (S^{k+\ell-1} \rightarrow D^{k+\ell})_+]_{C_{n+1}}$$

which simplifies to

$$(S^1/C_r)_+ \wedge (S^{k+\ell-1} \rightarrow D^{k+\ell})_+$$

and this is one of the generating  $S^1$ -cofibrations.  $\square$

### 3.1.3 Fixed points and subdivision.

Now we analyze fixed points  $|X_\bullet|^{C_r}$  under the cyclic subgroup of order  $r$ ,  $C_r \leq S^1$ . Of course, these fixed points will always have an  $S^1/C_r$ -action, which we often regard as an  $S^1$ -action by pulling back along the obvious isomorphism of groups

$$\rho_r : S^1 \xrightarrow{\cong} S^1/C_r$$

The previous section suggests that when  $X_\bullet$  is appropriately cofibrant the  $C_r$ -fixed points are built only by the cells coming from simplicial level  $(rk - 1)$  for  $k \geq 1$ . In fact, this is true without any cofibrancy assumptions. As a motivating special case, the reader is invited to prove

**Proposition 3.1.13.** *If  $r \nmid n$  then  $|\Lambda[n-1]|_+ \wedge_{C_n} X_{n-1}$  has trivial  $C_r$ -fixed points. Otherwise  $n = rk$  and there is a homeomorphism*

$$|\Lambda[k-1]|_+ \wedge_{C_k} (X_{n-1})^{C_r} \xrightarrow{\cong} (|\Lambda[n-1]|_+ \wedge_{C_n} X_{n-1})^{C_r}$$

which in  $g$ -coordinates is given by the formula

$$(\theta, u_0, \dots, u_{k-1}; x) \mapsto \left(\theta, \frac{1}{r}u_0, \dots, \frac{1}{r}u_{k-1}, \frac{1}{r}u_0, \dots, \frac{1}{r}u_{k-1}, \dots; x\right)$$

The above suggests that the fixed points of  $|X_\bullet|$  should be the realization of some other cyclic space, and we spend the rest of the section making that precise. Following [BHM93] we define an “edgewise” subdivision functor that interacts well with the cyclic structure:

**Definition 3.1.14.** The  $r$ -fold edgewise subdivision functor a map of categories  $\mathbf{\Delta} \xrightarrow{\text{sd}_r} \mathbf{\Delta}$  which takes  $[k-1]$  to  $[rk-1]$ . Each order-preserving map  $[m-1] \rightarrow [n-1]$  is repeated  $r$  times to give a map  $[rm-1] \rightarrow [rn-1]$ . Given a simplicial space  $X$ , we let the  $r$ -fold edgewise subdivision  $\text{sd}_r X$  denote the simplicial space obtained by composing with  $\text{sd}_r$ .

**Proposition 3.1.15** ([BHM93], Lem 1.1). *There is a natural diagonal map  $D_r$  of geometric realizations*

$$|\text{sd}_r X_\bullet| \xrightarrow{D_r} |X_\bullet|$$

which sends each  $(k-1)$ -simplex in  $X_{rk-1}$  to the corresponding  $(rk-1)$ -simplex in  $X_{rk-1}$  by the diagonal

$$(u_0, \dots, u_{k-1}) \mapsto \left(\frac{1}{r}u_0, \dots, \frac{1}{r}u_{k-1}, \frac{1}{r}u_0, \dots, \frac{1}{r}u_{k-1}, \dots\right)$$

Moreover,  $D_r$  is always a homeomorphism.

Now the subdivision of a cyclic space is no longer cyclic, but it has an action by some new category  $\mathbf{\Lambda}_r^{\text{op}}$ :

**Definition 3.1.16.** The  $r$ -cyclic category  $\mathbf{\Lambda}_r$  is the subcategory of  $\mathbf{\Lambda}$  on the objects of the form  $[rk-1]$ ,  $k \geq 1$ , generated by all maps in the image of  $\text{sd}_r : \mathbf{\Delta} \rightarrow \mathbf{\Delta}$  in addition to the cycle maps (cf. [BHM93] Def 1.5). When working in  $\mathbf{\Lambda}_r$  we relabel the object  $[rk-1]$  as  $[k-1]$ .

The  $r$ -cyclic category  $\mathbf{\Lambda}_r$  contains  $\mathbf{\Delta}$  as a subcategory in an obvious way, so any  $r$ -cyclic object is a simplicial object with extra structure. We have now defined a commuting square of injective functors

$$\begin{array}{ccc} \mathbf{\Delta} & \xrightarrow{\text{sd}_r} & \mathbf{\Delta} \\ \downarrow & & \downarrow \\ \mathbf{\Lambda}_r & \xrightarrow{\text{sd}_r} & \mathbf{\Lambda} \end{array}$$

**Corollary 3.1.17.** *If  $X_\bullet$  is a cyclic object then its  $r$ -fold subdivision is naturally an  $r$ -cyclic object.*

The  $r$ -cyclic category  $\mathbf{A}_r$  has a simpler equivalent description:  $\mathbf{A}_r([k-1], [n-1])$  consists of all increasing functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that  $f(x+k) = f(x) + n$ , up to the equivalence relation  $f \sim f + rn$ . Repeating the above analysis, the standard  $n$ -simplex  $\mathbf{A}_r(-, [n])$  has realization given by  $r$  copies of our  $P_n$  glued end-to-end in a circular fashion so as to make a space homeomorphic to  $S^1 \times \Delta^n$  (cf. [Jon87]). This allows us to define a  $S^1 = \mathbb{R}/\mathbb{Z}$  action on each standard  $n$ -simplex. This circle action respects the structure maps of  $\mathbf{A}_r$  just as before; the coface and codegeneracy maps are easy and the cycle map is now given by formula

$$\tau_{rn-1}(\theta, u_0, u_1, \dots, u_{n-1}) = \left(\theta - \frac{1}{r}u_0, u_1, u_2, \dots, u_{n-1}, u_0\right)$$

We easily get a natural  $S^1$ -action on the realization of any  $r$ -cyclic space  $Y_\bullet$ .

By the above formula, the action of  $C_r \leq S^1$  on  $\Lambda_r[n-1]$  is generated by  $\tau_{rn-1}^n$ . When we pass to the geometric realization, applying  $\tau_{rn-1}^n \wedge \text{id}$  to  $\Lambda_r[n-1]_+ \wedge Y_{n-1}$  has the same effect as applying  $\text{id} \wedge t_{rn-1}^n$  to  $\Lambda_r[n-1]_+ \wedge Y_{n-1}$ , but this map makes sense even if we only remember the simplicial structure on  $Y_\bullet$ . Therefore the  $C_r$  action on  $|Y_\bullet|$  comes from the simplicial map

$$t_{rn-1}^n : Y_{n-1} \longrightarrow Y_{n-1}$$

In summary:

**Theorem 3.1.18.** *The realization of any  $r$ -cyclic space  $Y_\bullet$  carries a natural  $S^1$ -action for which the action of  $C_r \leq S^1$  is the realization of a simplicial map.*

It is also straightforward but tedious to check that

**Proposition 3.1.19.** *For any cyclic space  $X_\bullet$ , we regard  $\text{sd}_r X_\bullet$  as  $r$ -cyclic and define a circle action on its realization as above. Then the diagonal homeomorphism*

$$|\text{sd}_r X_\bullet| \xrightarrow{D_r^{C_r}} |X_\bullet|$$

is  $S^1$ -equivariant.

Now that we know how to subdivide any cyclic space without changing the realization or  $S^1$  action at all, we are free to analyze the fixed points. It's not hard to check that  $\tau_{rn-1}^n$

commutes with all maps in  $\mathbf{\Lambda}_r$ , so the levelwise  $C_r$ -fixed points of a  $\mathbf{\Lambda}_r^{\text{op}}$ -space  $Y_\bullet$  is another  $\mathbf{\Lambda}_r^{\text{op}}$ -space  $Y_\bullet^{C_r}$ . The cycle maps  $t_{rn-1}$  at level  $n-1$  of  $Y_\bullet^{C_r}$  now have order  $n$ , so the action of each map of  $\mathbf{\Lambda}_r$  factors through the quotient functor

$$P_r : \mathbf{\Lambda}_r([m-1], [n-1]) \longrightarrow \mathbf{\Lambda}([m-1], [n-1])$$

which takes a function  $f : \mathbb{Z} \longrightarrow \mathbb{Z}$  up to  $f \sim f + rn$  and mods out by the stronger equivalence relation  $f \sim f + n$ .

**Corollary 3.1.20.** *If  $Y_\bullet$  is an  $r$ -cyclic space then  $Y_\bullet^{C_r}$  is a cyclic space in a canonical way.*

The quotient functor  $P_r$  does nothing to the subcategory  $\mathbf{\Delta}$ :

$$\begin{array}{ccc} \mathbf{\Delta} & \xlongequal{\quad} & \mathbf{\Delta} \\ \downarrow & & \downarrow \\ \mathbf{\Lambda}_r & \xrightarrow{P_r} & \mathbf{\Lambda} \end{array}$$

So if  $X_\bullet$  is a cyclic space, there is a canonical isomorphism between the underlying simplicial spaces of  $X_\bullet$  and  $P_r X_\bullet := X_\bullet \circ P_r$ . However upon realizations, this isomorphism is not equivariant. Instead, we have

**Lemma 3.1.21.** *If  $X_\bullet$  is a cyclic space, then on  $|P_r X_\bullet|$  the subgroup  $C_r \leq S^1$  acts trivially. The canonical isomorphism*

$$|X_\bullet| \cong |P_r X_\bullet|$$

*becomes equivariant if we pull back one of the  $S^1$ -actions along the isomorphism of groups  $\rho_r : S^1 \xrightarrow{\cong} S^1/C_r$ :*

$$|X_\bullet| \cong \rho_r^* |P_r X_\bullet|$$

**Corollary 3.1.22.** *If  $Y_\bullet$  is an  $r$ -cyclic space then when  $Y_\bullet^{C_r}$  is regarded as an  $r$ -cyclic space there is a canonical  $S^1$ -equivariant homeomorphism*

$$|Y_\bullet^{C_r}| \cong |Y_\bullet|^{C_r}$$

*and when  $Y_\bullet^{C_r}$  is regarded as a cyclic space there is a canonical  $S^1$ -equivariant homeomorphism*

$$|Y_\bullet^{C_r}| \cong \rho_r^* |Y_\bullet|^{C_r}$$

Assembling a few of these results gives a useful result for describing  $C_r$ -fixed points of  $|X_\bullet|$  as the realization of another cyclic space:

**Corollary 3.1.23.** *If  $X_\bullet$  is a cyclic space then  $\text{sd}_r X_\bullet^{C_r}$  may be regarded as a cyclic space, and the homeomorphisms*

$$|\text{sd}_r X_\bullet^{C_r}| \cong \rho_r^* |P_r(\text{sd}_r X_\bullet^{C_r})| \cong \rho_r^* (|\text{sd}_r X_\bullet|^{C_r}) \xrightarrow{D_r^{C_r}} \rho_r^* (|X_\bullet|^{C_r})$$

are all  $S^1$ -equivariant.

Finally we remark that subdivision can be iterated, giving a commuting triangle of  $S^1$ -equivariant homeomorphisms

$$\begin{array}{ccc} |\text{sd}_{rs} X_\bullet| & & \\ \downarrow D_r & \searrow D_{rs} & \\ |\text{sd}_s X_\bullet| & \xrightarrow{D_s} & |X_\bullet| \end{array}$$

The homeomorphisms of the above proposition do nothing to the simplicial structure, other than the final application of the diagonal map, so it follows quickly that

**Proposition 3.1.24.** *The homeomorphisms of the above proposition make this triangle commute:*

$$\begin{array}{ccc} |(\text{sd}_{rs} X_\bullet)^{C_{rs}}| & & \\ \downarrow \cong & \searrow \cong & \\ \rho_r^* |(\text{sd}_s X_\bullet)^{C_s}|^{C_r} & \xrightarrow{\cong} & \rho_{rs}^* |X_\bullet|^{C_{rs}} \end{array}$$

### 3.1.4 Cocyclic spaces.

The previous section dualizes easily to cocyclic spaces. We recall the basic definitions first.

**Definition 3.1.25.** A *cosimplicial object* of  $\mathbf{C}$  is a covariant functor  $X^\bullet : \Delta \rightarrow \mathbf{C}$ .

Any cosimplicial space  $X_\bullet$  is canonically expressed as an equalizer

$$X^\bullet \rightarrow \prod_n \text{Map}(\Delta_\bullet^n, X^n) \rightrightarrows \prod_{m,n} \text{Map}(\Delta_\bullet^m \times \Delta(m,n), X^n)$$

So a right adjoint out of cocyclic spaces is determined by what it does to the cosimplicial space  $\text{Map}(\Delta_{\bullet}^n, X^n)$ . The totalization is the unique right adjoint which takes  $\text{Map}(\Delta_{\bullet}^n, A)$  to  $\text{Map}(\Delta^n, A)$ :

**Definition 3.1.26.** The *totalization*  $\text{Tot}(X^{\bullet})$  of a cosimplicial based space is the equalizer of

$$\prod_n \text{Map}(\Delta^n, X^n) \rightrightarrows \prod_{m,n} \text{Map}(\Delta^m \times \Delta(m, n), X^n)$$

The totalization of a cosimplicial orthogonal spectrum is given by the same construction applied to each spectrum level.

If  $X^{\bullet}$  is not just cosimplicial, but cocyclic, then it may be expressed as an equalizer in cocyclic spaces

$$X^{\bullet} \rightarrow \prod_n \text{Map}(\Lambda[n]_{\bullet}, X^n) \rightrightarrows \prod_{m,n} \text{Map}(\Lambda[m]_{\bullet} \times \mathbf{\Lambda}(m, n), X^n)$$

and therefore the totalization is an equalizer

$$\text{Tot}(X^{\bullet}) \rightarrow \prod_n \text{Map}(|\Lambda[n]|, X^n) \rightrightarrows \prod_{m,n} \text{Map}(|\Lambda[m]| \times \mathbf{\Lambda}(m, n), X^n)$$

The mapping spaces  $\text{Map}(|\Lambda[n]|, X^n)$  all have a natural  $S^1$ -action in which  $\theta \in S^1$  sends  $f$  to  $f \circ \theta^{-1}$ . The maps of the above equalizer are duals of maps that we already know to be  $S^1$ -equivariant, so they are equivariant as well, proving

**Theorem 3.1.27.** *If  $X^{\bullet}$  is a cocyclic space then  $\text{Tot}(X^{\bullet})$  has a natural  $S^1$ -action. If  $X^{\bullet} = \text{Map}(E_{\bullet}, X)$  for a cyclic space  $E_{\bullet}$  and space  $X$  then the canonical homeomorphism*

$$\text{Tot}(X^{\bullet}) \cong \text{Map}(|E_{\bullet}|, X)$$

*is  $S^1$ -equivariant.*

A useful example to keep in mind is  $E_{\bullet} = S_{\bullet}^1$ . Then  $X^{\bullet} = \text{Map}(S_{\bullet}^1, X)$  is usual cosimplicial model for the free loop space  $LX$ , but in fact this is a cocyclic space and its totalization gives the usual circle action on  $LX$ .

Now we move on to subdivision. A cosimplicial or cocyclic space  $X^{\bullet}$  may be composed with  $\text{sd}_r$  to give a cosimplicial or  $r$ -cocyclic space  $\text{sd}_r X^{\bullet}$ . The dual diagonal is a natural

map defined the totalization of each cosimplicial space as a map of equalizer systems

$$\begin{array}{ccc}
 \text{Tot}(X^\bullet) & \xrightarrow{D_r} & \text{Tot}(\text{sd}_r X^\bullet) \\
 \downarrow & & \downarrow \\
 \prod_{k \geq 1} \text{Map}(\Delta^{k-1}, X^{k-1}) & \xrightarrow{\quad} & \prod_{k \geq 1} \text{Map}(\Delta^{k-1}, X^{rk-1}) \\
 \Downarrow & & \Downarrow \\
 \prod_{k, \ell \geq 1} \text{Map}(\Delta^{k-1} \times \Delta(k-1, \ell-1), X^{\ell-1}) & \xrightarrow{\quad} & \prod_{k \geq 1} \text{Map}(\Delta^{k-1} \times \Delta(k-1, \ell-1), X^{r\ell-1})
 \end{array}$$

The horizontal map takes the function  $\Delta^{rk-1} \rightarrow X^{rk-1}$  at level  $rk-1$  and composes with the diagonal  $\Delta^{k-1} \rightarrow \Delta^{rk-1}$  that we defined in the section on cyclic spaces. When  $X^\bullet = \text{Map}(\Delta[n]_\bullet, X)$  this cosimplicial diagonal map is easily seen to be  $\text{Map}(-, X)$  applied to the simplicial diagonal map on  $\Delta[n]_\bullet$ , which we already know is a homeomorphism, but  $X^\bullet$  is an equalizer of such things and so

**Proposition 3.1.28.** *When  $X^\bullet$  is a comsimplicial space there is a natural dual diagonal map  $D_r$  of totalizations*

$$\text{Tot}(X^\bullet) \xrightarrow{D_r} \text{Tot}(\text{sd}_r X^\bullet)$$

which is a homeomorphism.

As before, if  $X^\bullet$  is a cocyclic object then its  $r$ -fold subdivision  $\text{sd}_r X^\bullet$  is naturally an  $r$ -cocyclic object. If  $Y^\bullet$  is any  $r$ -cocyclic object then it is an equalizer of duals of the standard  $r$ -cyclic simplices

$$Y^\bullet \rightarrow \prod_n \text{Map}(\Lambda_r[n]_\bullet, Y^n) \rightrightarrows \prod_{m, n} \text{Map}(\Lambda_r[m]_\bullet \times \Lambda_r(m, n), Y^n)$$

and therefore its totalization is an equalizer

$$\text{Tot}(Y^\bullet) \rightarrow \prod_n \text{Map}(|\Lambda_r[n]|, Y^n) \rightrightarrows \prod_{m, n} \text{Map}(|\Lambda_r[m]| \times \Lambda_r(m, n), Y^n)$$

So the natural  $S^1$ -action we defined on  $|\Lambda_r[n]|$  passes to a natural  $S^1$ -action on  $\text{Tot}(Y^\bullet)$ . As before,  $t_{rn-1}^n$  at level  $n-1$  defines a cosimplicial endomorphism of  $Y^\bullet$  which on each standard piece  $\text{Map}(\Lambda_r[k]_\bullet, Y^k)$  is the action of the generator of  $C_r \leq S^1$ . In summary:

**Theorem 3.1.29.** *The realization of any  $r$ -cocyclic space  $Y^\bullet$  carries a natural  $S^1$ -action*

for which the action of  $C_r \leq S^1$  is the realization of a cosimplicial map.

Now the dual of the diagonal map is the dual of an equivariant map on each standard space  $\text{Map}(\Lambda_r[k]_\bullet, Y^k)$ , so

**Proposition 3.1.30.** *For any cocyclic space  $X^\bullet$ , the diagonal homeomorphism*

$$\text{Tot}(X^\bullet) \xrightarrow{D_r} \text{Tot}(\text{sd}_r X^\bullet)$$

is  $S^1$ -equivariant.

The remaining results go through with almost no modification to the proof:

**Corollary 3.1.31.** *If  $Y^\bullet$  is an  $r$ -cocyclic space then  $(Y^\bullet)^{C_r}$  is a cocyclic space in a canonical way.*

**Lemma 3.1.32.** *If  $X^\bullet$  is a cocyclic space, then on  $\text{Tot}(P_r X^\bullet)$  the subgroup  $C_r \leq S^1$  acts trivially. The canonical isomorphism*

$$\text{Tot}(X^\bullet) \cong \text{Tot}(P_r X^\bullet)$$

becomes equivariant if we pull back one of the  $S^1$ -actions along the isomorphism of groups  $\rho_r : S^1 \xrightarrow{\cong} S^1/C_r$ :

$$\text{Tot}(X^\bullet) \cong \rho_r^* \text{Tot}(P_r X^\bullet)$$

**Corollary 3.1.33.** *If  $Y^\bullet$  is an  $r$ -cocyclic space then when  $(Y^\bullet)^{C_r}$  is regarded as an  $r$ -cocyclic space there is a canonical  $S^1$ -equivariant homeomorphism*

$$\text{Tot}((Y^\bullet)^{C_r}) \cong \text{Tot}(Y^\bullet)^{C_r}$$

and when  $(Y^\bullet)^{C_r}$  is regarded as a cocyclic space there is a canonical  $S^1$ -equivariant homeomorphism

$$\text{Tot}((Y^\bullet)^{C_r}) \cong \rho_r^* \text{Tot}(Y^\bullet)^{C_r}$$

**Corollary 3.1.34.** *If  $X^\bullet$  is a cocyclic space then  $(\text{sd}_r X^\bullet)^{C_r}$  may be regarded as a cocyclic space, and the homeomorphisms*

$$\text{Tot}((\text{sd}_r X^\bullet)^{C_r}) \cong \rho_r^* \text{Tot}(\text{sd}_r X^\bullet)^{C_r} \xleftarrow{D_r^{C_r}} \rho_r^* \text{Tot}(X^\bullet)^{C_r}$$

are all  $S^1$ -equivariant.

So, as before, the  $C_7$ -fixed points of the totalization of  $X^\bullet$  are themselves the totalization of some other cocyclic space  $(\mathrm{sd}_7 X^\bullet)^{C_7}$ .

## 3.2 Orthogonal $G$ -spectra and equivariant smash powers

We will need equivariant spectra for two purposes in this work: first, to understand the cyclotomic structure on  $THH$  itself, and second, to understand the equivariant structures related to  $\mathbb{V}(BG)$ . In this section we review enough of the theory to handle cyclic and cocyclic spectra. We also review equivariant smash powers, and give a new rigidity result for the functors that relate smash powers and geometric fixed points. In the next section we will use this result to give a satisfactory account of how cyclic and cocyclic orthogonal spectra behave, and later on it will be essential for checking some of the technical lemmas for our construction of  $D(THH(R))$ .

### 3.2.1 Basic definitions, model structures, and fixed points.

Recall from [MM02] and [HHR09] the most fundamental definitions:

**Definition 3.2.1.** If  $G$  is a fixed compact Lie group, an *orthogonal  $G$ -spectrum* is a sequence of based spaces  $\{X_n\}_{n=0}^\infty$  equipped with

- A continuous action of  $G \times O(n)$  on  $X_n$  for each  $n$
- A  $G$ -equivariant structure map  $\Sigma X_n \rightarrow X_{n+1}$  for each  $n$

such that the composite

$$S^p \wedge X_n \rightarrow \dots \rightarrow S^1 \wedge X_{(p-1)+n} \rightarrow X_{p+n}$$

is  $O(p) \times O(n)$ -equivariant.

**Definition 3.2.2.** Let  $U$  be a complete  $G$ -universe. The category  $\mathbf{J}_G$  has objects the finite-dimensional  $G$ -representations  $V \subset U$ , and the mapping spaces  $\mathbf{J}_G(V, W)$  are the Thom spaces  $O(V, W)^{W-V}$  consisting of linear isometric inclusions  $V \rightarrow W$  with choices of point in the orthogonal complement  $W - V$ . The group  $G$  acts on  $O(V, W)^{W-V}$  by conjugating the map and acting on the point in  $W - V$ .

**Definition 3.2.3.** A  $\mathbf{J}_G$ -space is an equivariant functor  $\mathbf{J}_G$  into based  $G$ -spaces and nonequivariant maps. That is, each  $V$  goes to a based space  $X(V)$  and for each pair  $V, W$  the map

$$\mathbf{J}_G(V, W) \rightarrow \text{Map}_*(X(V), X(W))$$

is equivariant.

**Proposition 3.2.4.** *Every  $\mathbf{J}_G$ -space gives an orthogonal  $G$ -spectrum by restricting to  $V = \mathbb{R}^n$ ; denote this functor  $\mathcal{I}_U^{\mathbb{R}^\infty}$ . Conversely, given an orthogonal  $G$ -spectrum  $X$  one may define a  $\mathbf{J}_G$ -space by the rule*

$$X(V) = X_n \wedge_{O(n)} O(\mathbb{R}^n, V)_+, \quad n = \dim V$$

Denote this functor  $\mathcal{I}_{\mathbb{R}^\infty}^U$ . Then  $\mathcal{I}_{\mathbb{R}^\infty}^U$  and  $\mathcal{I}_U^{\mathbb{R}^\infty}$  are inverse equivalences of categories.

**Definition 3.2.5.** Given a  $G$ -representation  $V$  and based  $G$ -space  $A$ , the free spectrum  $F_V A$  is the  $\mathbf{J}_G$ -space

$$(F_V A)(W) := \mathbf{J}_G(V, W) \wedge A$$

For fixed  $V$ , the functor  $A \mapsto F_V A$  is the left adjoint to the functor that evaluates a  $\mathbf{J}_G$ -space at  $V$ .

**Proposition 3.2.6.** *There is a standard stable model structure on the category of orthogonal  $G$ -spectra in which*

- The cofibrations are the retracts of the cell complex spectra built out of the cells

$$\{F_V((G/H \times S^{k-1})_+) \hookrightarrow F_V((G/H \times D^k)_+) : k \geq 0, H \leq G, V \subset U\}$$

- The weak equivalences are the maps inducing isomorphisms on the stable homotopy groups

$$\pi_k^H(X) = \begin{cases} \operatorname{colim}_{V \subset U} \pi_k(\operatorname{Map}_*^H(S^V, X(V))), & k \geq 0 \\ \operatorname{colim}_{V \subset U} \pi_0(\operatorname{Map}_*^H(S^{V-\mathbb{R}^{|k|}}, X(V))), & k < 0, \mathbb{R}^k \subset V \end{cases}$$

- The fibrations are the maps for which each level fixed point map  $X(V)^H \rightarrow Y(V)^H$  is a Serre fibration and each square

$$\begin{array}{ccc} X(V)^H & \longrightarrow & (\Omega^W X(V+W))^H \\ \downarrow & & \downarrow \\ Y(V)^H & \longrightarrow & (\Omega^W Y(V+W))^H \end{array}$$

is a homotopy pullback square.

This model structure is topological, proper, and monoidal. It is compactly generated [MMSS01] by the maps

$$\begin{aligned} \mathbf{I} &= \{F_V((G/H \times S^{k-1})_+) \hookrightarrow F_V((G/H \times D^k)_+) : k \geq 0, H \leq G, V \subset U\} \\ \mathbf{J} &= \{F_V((G/H \times D^k)_+) \hookrightarrow F_V((G/H \times D^k \times I)_+) : k \geq 0, H \leq G, V \subset U\} \\ &\cup \{(F_V((G/H \times S^{k-1})_+) \hookrightarrow F_V((G/H \times D^k)_+)) \square (F_W(S^W) \hookrightarrow \text{Cyl}(F_W S^W \longrightarrow F_0 S^0))\} \end{aligned}$$

where  $\square$  denotes the pushout-product.

Now we move on to different notions of fixed points. Recall that if  $X$  is a  $G$ -space and  $H \leq G$  is a subgroup, then the fixed point subspace  $X^H$  has a natural action by only the normalizer  $NH \leq G$ . Of course  $H$  acts trivially and so we are left with a natural action by the *Weyl group*

$$WH = NH/H \cong \text{Aut}_G(G/H)$$

When  $X$  is a  $G$ -spectrum there are two natural notions of  $H$ -fixed points, each of which gives a  $WH$ -spectrum:

**Definition 3.2.7.** For a  $\mathbf{J}_G$ -space  $X$  and a subgroup  $H \leq G$ , the *categorical fixed points*  $X^H$  are the  $\mathbf{J}_{WH}$ -space which on each  $H$ -fixed  $G$ -representation  $V \subset U^H \subset U$  is just the fixed points  $X(V)^H$ . More simply, if  $X$  is an orthogonal  $G$ -spectrum then  $X^H$  is obtained by taking  $H$ -fixed points levelwise.

**Proposition 3.2.8.** *The categorical fixed points are a Quillen right adjoint from  $G$ -spectra to  $WH$ -spectra. Their right-derived functor is called the genuine fixed points.*

**Definition 3.2.9.** If  $X$  is a  $\mathbf{J}_G$ -space and  $H \leq G$  then the *geometric fixed points*  $\Phi^H X$  are defined as the coequalizer

$$\bigvee_{V,W} F_{WH} S^0 \wedge \mathbf{J}_G^H(V,W) \wedge X(V)^H \rightrightarrows \bigvee_V F_{VH} S^0 \wedge X(V)^H \longrightarrow \Phi^H X$$

These are naturally  $\mathbf{J}_{WH}$ -spaces on the complete  $WH$ -universe  $U^H$ .

**Theorem 3.2.10.** *The geometric fixed points  $\Phi^H$  satisfy the following technical properties:*

1. There is a natural isomorphism of  $WH$ -spectra

$$\Phi^H F_V A \cong F_{V^H} A^H$$

2.  $\Phi^H$  commutes with all coproducts, pushouts along a levelwise closed inclusion, and filtered colimits along levelwise closed inclusions.

3.  $\Phi^H$  preserves all cofibrations, acyclic cofibrations, and weak equivalences between cofibrant objects.

4. If  $H \leq K \leq G$  then  $\Phi^H$  commutes with the change-of-groups from  $G$  down to  $K$ .

5. There is a canonical commutation map

$$\Phi^G(X \wedge Y) \xrightarrow{\alpha} \Phi^G X \wedge \Phi^G Y$$

which is an isomorphism when  $X$  or  $Y$  is cofibrant (cf. [BM13]).

Though it does not seem to appear in the literature, the iterated fixed points map of [BM13] easily generalizes:

**Proposition 3.2.11.** *If  $H \leq K \leq NH \leq G$  then there is a natural iterated fixed points map*

$$\Phi^K X \xrightarrow{\text{it}} \Phi^{K/H} \Phi^H X$$

which is an isomorphism when  $X = F_V A$ , and therefore an isomorphism on all cofibrant spectra. When  $H$  and  $K$  are normal this is a map of  $G/K$ -spectra.

### 3.2.2 The Hill-Hopkins-Ravenel norm isomorphism.

When  $X$  is an orthogonal spectrum, the smash product  $X^{\wedge n}$  has an action of  $C_n \cong \mathbb{Z}/n$  which rotates the factors. This makes  $X^{\wedge n}$  into an orthogonal  $C_n$ -spectrum. It is natural to guess that the geometric fixed points of this  $C_n$ -action should be  $X$  itself, and in fact there is natural diagonal map

$$X \xrightarrow{\Delta} \Phi^{C_n} X^{\wedge n}$$

When  $X$  is cofibrant, this map is an *isomorphism*.

More generally, if  $G$  is a finite group,  $H \leq G$ , and  $X$  is an orthogonal  $H$ -spectrum, we can define a smash product of copies of  $X$  indexed by  $G$

$$N_H^G X := \bigwedge_{g_i H \in G/H} (g_i H)_+ \wedge_H X \cong \bigwedge^{|G/H|} X$$

This construction is the *multiplicative norm* defined by Hill, Hopkins, and Ravenel. This can be given a reasonably obvious  $G$ -action, but on closer inspection the action is dependent on some fixed choice of representatives  $g_i H$  for each left coset of  $H$  ([Boh14], [HHR09]). Unfortunately, changing our choice of representatives changes this action, but up to natural isomorphism it turns out to be the same. We therefore implicitly assume that such representatives have been chosen. The general form of the above observation about  $X^{\wedge n}$  is then

**Theorem 3.2.12** (Hill, Hopkins, Ravenel). *There is a natural diagonal of  $WH$ -spectra*

$$\Phi^H X \xrightarrow{\Delta} \Phi^G N_H^G X$$

When  $X$  is cofibrant,  $\Delta$  is an isomorphism.

This appears in [ABG<sup>+</sup>14], Thm 2.33 and earlier in the proof of [HHR09], Prop B.96. We will reproduce the proof here since it is surprisingly short.

*Proof.* It is conceptually useful to start by checking that on the space level, the indexed smash product of  $A$  over  $G/H$  has fixed points  $A^H$ :

$$A^H \xrightarrow{\cong} (N_H^G A)^G \cong \left( \bigwedge^{|G/H|} A \right)^G$$

The map from left to right is the diagonal:

$$a \in A^H \mapsto (a, \dots, a)$$

Now for the spectrum-level argument. We start by taking the coequalizer presentation

of the orthogonal  $H$ -spectrum  $X$

$$\bigvee_{V,W} F_W S^0 \wedge \mathbf{J}_H(V,W) \wedge X(V) \rightrightarrows \bigvee_V F_V S^0 \wedge X(V) \longrightarrow X$$

and taking  $\Phi^G N_H^G$  of everything in sight. Since  $\Phi^G N_H^G$  commutes with wedges and smashes up to isomorphism, this gives

$$\begin{aligned} \bigvee_{V,W} \Phi^G N_H^G F_W S^0 \wedge (N_H^G \mathbf{J}_H(V,W))^G \wedge (N_H^G X(V))^G &\rightrightarrows \bigvee_V \Phi^G N_H^G F_V S^0 \wedge (N_H^G X(V))^G \\ &\longrightarrow \Phi^G N_H^G X \end{aligned}$$

which simplifies to

$$\bigvee_{V,W} \Phi^G N_H^G F_W S^0 \wedge \mathbf{J}_H^H(V,W) \wedge X(V)^H \rightrightarrows \bigvee_V \Phi^G N_H^G F_V S^0 \wedge X(V)^H \longrightarrow \Phi^G N_H^G X$$

As a diagram, this is no longer guaranteed to be a coequalizer system, but it still commutes. We can simplify using the string of isomorphisms

$$\begin{aligned} \Phi^G N_H^G F_V A &\cong \Phi^G F_{\text{Ind}_H^G V}(N_H^G A) \\ &\cong F_{(\text{Ind}_H^G V)^G}(N_H^G A)^G \\ &\cong F_{V^H} A^H \end{aligned}$$

for any based  $H$ -space  $A$  and  $H$ -representation  $V$ . This gives

$$\bigvee_{V,W} F_{W^H} S^0 \wedge \mathbf{J}_H^H(V,W) \wedge X(V)^H \rightrightarrows \bigvee_V F_{V^H} S^0 \wedge X(V)^H \longrightarrow \Phi^G N_H^G X$$

and the coequalizer of the first two terms is exactly  $\Phi^H X$ . The universal property of the coequalizer then gives us a map

$$\Phi^H X \longrightarrow \Phi^G N_H^G X$$

and we take this as the definition of the diagonal map.

Now consider the special case when  $X = F_V A$ . The inclusion of the term

$$F_{VH} S^0 \wedge A^H$$

into the above coequalizer system maps forward isomorphically to  $\Phi^H X$ , and so we can evaluate the diagonal map by just examining this term. But back at the top of our proof, the inclusion of the term

$$\Phi^G N_H^G F_V S^0 \wedge (N_H^G A)^G$$

also maps forward isomorphically to  $\Phi^G N_H^G X$ . Therefore up to isomorphism, the diagonal map becomes the string of maps we used to connect  $F_{VH} S^0 \wedge A^H$  to  $\Phi^G N_H^G F_V S^0 \wedge (N_H^G A)^G$ , but these maps were all isomorphisms. Therefore the diagonal is an isomorphism when  $X = F_V A$ . Since both sides preserve coproducts, pushouts along closed inclusions, and sequential colimits along closed inclusions, we get by induction that the diagonal is an isomorphism for all cofibrant  $X$ .  $\square$

**Remark.** In [ABG<sup>+</sup>14], Def 2.17. the diagonal map is extended to a more general setting, which includes that if  $H \leq K \leq G$  are normal subgroups a natural map

$$N^{G/K} \Phi^H X \xrightarrow{\Delta} \Phi^K N_H^G X$$

We will use this more general diagonal map when we check compatibility between our cyclotomic maps below.

### 3.2.3 A rigidity theorem for geometric fixed points

In this section, we give a result which helps simplify our work on the cyclic bar construction and its dual. It should be of independent interest because it assures us that all of the natural transformations that we know relating geometric fixed points and smash powers of orthogonal  $G$ -spectra are canonical in a very strong sense. To be clear, though, this is a point-set statement about orthogonal spectra and not a statement about any derived space of natural transformations.

To state it, let  $G\mathbf{Sp}^O$  denote the category of orthogonal  $G$ -spectra, and let **Free** be the full subcategory on the free spectra  $F_V A$ , for all  $G$ -representations  $V$  and based  $G$ -spaces

A. Let

$$\Phi^G \circ \wedge : \prod^k \mathbf{Free} \longrightarrow \mathbf{Sp}^O$$

denote the composite of  $k$ -fold smash product followed by geometric fixed points, with  $k \geq 1$ .

**Proposition 3.2.13.** *The only natural endomorphisms of  $\Phi^G \circ \wedge$  are zero and the identity.*

*Proof.* Consider a natural transformation  $T : \Phi^G \circ \wedge \longrightarrow \Phi^G \circ \wedge$ . On  $(F_0 S^0, F_0 S^0, \dots, F_0 S^0)$ ,  $T$  gives a map of spectra

$$F_0 S^0 \longrightarrow F_0 S^0$$

which is determined by at level 0 a choice of point in  $S^0$ . So there are only two such maps, the identity and zero.

Assume that  $T$  is the identity on this object. Then consider  $T$  on  $(F_{V_1} S^0, F_{V_2} S^0, \dots, F_{V_k} S^0)$ :

$$F_{V_1^G + V_2^G + \dots + V_k^G} S^0 \longrightarrow F_{V_1^G + V_2^G + \dots + V_k^G} S^0$$

Let  $m_i := \dim V_i^G$  and fix an isomorphism between  $\mathbb{R}^{m_i}$  and  $V_i^G$ . Then this map is determined by what it does at level  $m_1 + \dots + m_k$ :

$$O(m_1 + \dots + m_k)_+ \longrightarrow O(m_1 + \dots + m_k)_+$$

which in turn is determined by the image of the identity point, which is some element  $P \in O(m_1 + \dots + m_k)_+$ . Now for any point  $(t_1, \dots, t_k) \in S^{m_1} \wedge \dots \wedge S^{m_k}$  we can choose maps of spectra  $F_{V_i} S^0 \longrightarrow F_0 S^0$  which at level  $V_i$  send the nontrivial point of  $S^0$  to the point  $t_i \in S^{m_i} \cong (S^{V_i})^G$ . Since  $T$  is a natural transformation, this square commutes for all choices of  $(t_1, \dots, t_k)$ :

$$\begin{array}{ccc} O(m_1 + \dots + m_k)_+ & \xrightarrow{\cdot P} & O(m_1 + \dots + m_k)_+ \\ \downarrow \text{ev}_{(t_1, \dots, t_k)} & & \downarrow \text{ev}_{(t_1, \dots, t_k)} \\ S^{m_1 + \dots + m_k} & \xrightarrow{\text{id}} & S^{m_1 + \dots + m_k} \end{array}$$

Therefore  $P = \text{id}$  and  $T$  acts as the identity on  $(F_{V_1} S^0, F_{V_2} S^0, \dots, F_{V_k} S^0)$ .

Finally let  $A_1, \dots, A_k$  be a sequence of  $G$ -spaces and consider  $T$  on  $(F_{V_1} A_1, \dots, F_{V_k} A_k)$ . Each collection of choices of point  $a_i \in A_i^G$  gives a sequence of maps  $F_{V_i} S^0 \longrightarrow F_{V_i} A_i$ , and

applying  $T$  to this sequence of maps gives a commuting square

$$\begin{array}{ccc} F_{V_1^G + \dots + V_k^G} S^0 \wedge \dots \wedge S^0 & \xrightarrow{\text{id}} & F_{V_1^G + \dots + V_k^G} S^0 \wedge \dots \wedge S^0 \\ \downarrow F_{\dots(a_1, \dots, a_k)} & & \downarrow F_{\dots(a_1, \dots, a_k)} \\ F_{V_1^G + \dots + V_k^G} A_1^G \wedge \dots \wedge A_k^G & \xrightarrow{?} & F_{V_1^G + \dots + V_k^G} A_1^G \wedge \dots \wedge A_k^G \end{array}$$

From inspection of level  $m_1 + \dots + m_k$ , the bottom map must be the identity on the point  $\text{id} \wedge (a_1, \dots, a_k)$ . But this is true for all  $(a_1, \dots, a_k)$  and so the bottom map is the identity. Therefore  $T$  is the identity on  $(F_{V_1} A_1, \dots, F_{V_k} A_k)$ , so it is the identity on every object in  $\prod^k \mathbf{Free}$ .

For the second case, we assume  $T$  is zero on  $(F_0 S^0, \dots, F_0 S^0)$  and follow the same steps as before, concluding that  $T$  is zero on  $(F_{V_1} S^0, \dots, F_{V_k} S^0)$  and then it is zero on  $(F_{V_1} A_1, \dots, F_{V_k} A_k)$ .  $\square$

To derive corollaries, we say that a functor  $\phi : \prod^k \mathbf{GSp}^O \rightarrow \mathbf{Sp}^O$  is *rigid* if restricting to the subcategory  $\prod^k \mathbf{Free}$  gives an injective map on natural transformations out of  $\phi$ . In other words, a natural transformation out of  $\phi$  is determined by its behavior on the subcategory  $\mathbf{Free}$ . The above implies

**Corollary 3.2.14.** *If  $\phi_1$  and  $\phi_2$  are functors  $\prod^k \mathbf{GSp}^O \rightarrow \mathbf{Sp}^O$  which when restricted to the subcategory  $\prod^k \mathbf{Free}$  are separately isomorphic to  $\Phi^G \circ \wedge$ , and  $\phi_1$  is rigid, then there is at most one nonzero natural transformation  $\phi_1 \rightarrow \phi_2$ .*

We next check

**Proposition 3.2.15.**  $\wedge \circ (\Phi^G, \dots, \Phi^G)$  is a rigid functor.

*Proof.* We will show that  $(X, Y) \rightsquigarrow \Phi^G X \wedge \Phi^G Y$  is a rigid functor.  $\Phi^G X \wedge \Phi^G Y$  is a smash product of a double coequalizer and so may be written as a coequalizer of two maps into

$$\bigvee_{V, W} F_{V^G} S^0 \wedge F_{W^G} S^0 \wedge X(V)^G \wedge Y(W)^G$$

Therefore a map  $\Phi^G X \wedge \Phi^G Y \rightarrow Z$  is determined by the image of each of these terms for each  $V$  and  $W$ . For a fixed choice of  $V$  and  $W$ , we can replace  $X$  by  $F_V X(V)$  and  $Y$  by  $F_W Y(W)$ . Then on the  $(V, W)$  summand above, this replacement map is

$$F_{V^G} S^0 \wedge F_{W^G} S^0 \wedge (O(V)_+ \wedge X(V))^G \wedge (O(W)_+ \wedge Y(W))^G$$

$$\longrightarrow F_{VG}S^0 \wedge F_{WG}S^0 \wedge X(V)^G \wedge Y(W)^G$$

This is a surjective map of spaces and so the map into  $Z$  is determined by the composites

$$\Phi^G(F_V X(V)) \wedge \Phi^G(F_W Y(W)) \longrightarrow \Phi^G X \wedge \Phi^G Y \longrightarrow Z$$

for all  $V$  and  $W$ . Therefore  $\Phi^G X \wedge \Phi^G Y$  is rigid; the general case of a  $k$ -fold smash product is an easy generalization of this argument. In particular  $X \rightsquigarrow \Phi^G X$  is also a rigid functor.  $\square$

**Remark.** The author does not know at the moment whether  $\Phi^G \circ \wedge$  is a rigid functor in general. It is not trivially true because fixed points do not commute with coequalizers. The issue is whether the fixed points of  $(X \wedge Y)(V)$  are covered by the fixed points of  $F_{V'}X(V')$  and  $F_W Y(W)$  for varying  $V'$  and  $W$ .

The above results give new rigidity statements for the maps relating geometric fixed points and smash powers:

**Theorem 3.2.16.** *Let  $X$  and  $Y$  denote arbitrary  $G$ -spectra. Then the commutation map*

$$\Phi^G X \wedge \Phi^G Y \xrightarrow{\alpha} \Phi^G(X \wedge Y)$$

*is the only nonzero natural transformation from  $\Phi^G X \wedge \Phi^G Y$  to  $\Phi^G(X \wedge Y)$ .*

**Remark.** If  $X$  and  $Y$  are  $G$ -spectra and  $H \leq G$  then there is more than one natural map

$$\Phi^H X \wedge \Phi^H Y \longrightarrow \Phi^H(X \wedge Y)$$

because we could for instance post-compose  $\alpha_H$  with  $\mathcal{I}_{\mathbb{R}^\infty}^U g$ ,  $g \in Z(G)$ . However  $\alpha_H$  is the only natural transformation that respects the forgetful functor to  $H$ -spectra; in other words it is the one that is natural with respect to all the  $H$ -equivariant maps of spectra and not just the  $G$ -equivariant ones. Similar considerations apply to the iterated fixed points map below.

**Theorem 3.2.17.** *Let  $X$  denote an arbitrary  $H$ -spectrum with  $H \leq G$ . Then the Hill-Hopkins-Ravenel diagonal map*

$$\Phi^H X \xrightarrow{\Delta} \Phi^G N_H^G X$$

*is the only such map that is both natural and nonzero.*

**Theorem 3.2.18.** *If  $X$  is a  $G$ -spectrum and  $N \leq G$  is a normal subgroup, then the iterated fixed points map*

$$\Phi^G X \xrightarrow{\text{it}} \Phi^{G/N} \Phi^N X$$

*is characterized by the property that it is natural in  $X$  and nonzero.*

We end with five more corollaries that will be particularly useful for the present work; they served as the motivation for the above results.

**Proposition 3.2.19.** *If  $X$  and  $Y$  are  $G$ -spectra and  $N \leq G$  is a normal subgroup, then the following square commutes:*

$$\begin{array}{ccc} \Phi^G X \wedge \Phi^G Y & \xrightarrow{\alpha_G} & \Phi^G(X \wedge Y) \\ \downarrow \text{it} \wedge \text{it} & & \downarrow \text{it} \\ \Phi^{G/N} \Phi^N X \wedge \Phi^{G/N} \Phi^N Y & \xrightarrow{\alpha_{G/N}} \Phi^{G/N}(\Phi^N X \wedge \Phi^N Y) \xrightarrow{\Phi^{G/N} \alpha_N} & \Phi^{G/N} \Phi^N(X \wedge Y) \end{array}$$

**Proposition 3.2.20.** *If  $X$  is an ordinary spectrum and  $m, n \geq 0$  then the following square commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\Delta_{C_{mn}}} & \Phi^{C_{mn}} X^{\wedge mn} \\ \downarrow \Delta_{C_m} & & \downarrow \text{it} \\ \Phi^{C_m} X^{\wedge m} & \xrightarrow{\Phi^{C_m}(\Delta_*)} & \Phi^{C_{mn}/C_n} \Phi^{C_n} X^{\wedge mn} \end{array}$$

Here  $\Delta_*$  is the generalized HHR diagonal

$$N^{C_{mn}/C_n} X \xrightarrow{\Delta_*} \Phi^{C_n} N^{C_{mn}} X$$

found in [ABG<sup>+</sup>14], Def 2.17.

**Proposition 3.2.21.** *If  $X$  is an ordinary spectrum and  $m, n \geq 0$  then the square of non-equivariant spectra*

$$\begin{array}{ccc} X^{\wedge m} & \xrightarrow{\Delta_*} & \Phi^{C_n} X^{\wedge mn} \\ \parallel & & \downarrow \cong \\ X^{\wedge m} & \xrightarrow{\Delta_n} & \Phi^{C_n} X^{\wedge mn} \end{array}$$

*commutes when  $\cong$  is any natural isomorphism; for example one may pick an isomorphism of  $C_n$ -sets*

$$C_{mn} \cong C_m \times C_n$$

and apply  $\Phi^{C_n}$  to the resulting map

$$N^{C_{mn}} X \xrightarrow{\cong} N^{C_m \times C_n} X$$

**Proposition 3.2.22.** *If  $X$  is an  $G$ -spectrum and  $g \in Z(G)$ , then multiplication by  $g$  on the trivial representation levels passes to a map of  $\mathbf{J}_G$ -spaces*

$$X \xrightarrow{\mathcal{I}_{\mathbb{R}\infty}^U g} X$$

which on fixed points

$$\Phi^G X \xrightarrow{\Phi^G \mathcal{I}_{\mathbb{R}\infty}^U g} \Phi^G X$$

is the identity map.

**Proposition 3.2.23.** *If  $X$  and  $Y$  are orthogonal spectra, then the self-map of orthogonal  $C_r$ -spectra*

$$f : N^{C_r}(X \wedge Y) \cong X^{\wedge r} \wedge Y^{\wedge r} \longrightarrow X^{\wedge r} \wedge Y^{\wedge r}$$

which rotates only the  $Y$  factors but not the  $X$  factors fits into a commuting triangle (cf. [ABG<sup>+</sup>14] Prop. 2.20)

$$\begin{array}{ccc}
 & \Phi^{C_r}(X^{\wedge r} \wedge Y^{\wedge r}) & \\
 & \Delta \nearrow & \downarrow \Phi^{C_r} \mathcal{I}_{\mathbb{R}\infty}^U f \\
 X \wedge Y & & \\
 & \Delta \searrow & \\
 & \Phi^{C_r}(X^{\wedge r} \wedge Y^{\wedge r}) & 
 \end{array}$$

### 3.3 Cyclic spectra, cocyclic spectra, and the cyclic bar construction

Cyclic and cocyclic orthogonal spectra work by applying the space-level constructions to each level separately. This gives orthogonal spectra with  $S^1$ -actions whose naïve fixed points behave exactly as in the space case. However this by itself is not very helpful, because we want to build the cyclic bar construction for spectra, and the naïve fixed points of an equivariant smash power are not well-behaved.

From the previous section we know that the geometric fixed points of a smash product are relatively simple. Therefore we will focus our energy on the relationship between cyclic and cocyclic structures, subdivision, and the geometric fixed points. This gives us the results we need to define the cyclic bar construction for ring spectra and spectrally-enriched categories. Finally we give sufficient conditions for the cyclic bar construction to produce a cofibrant  $S^1$ -spectrum, so that we may dualize it in the next section.

#### 3.3.1 Cyclic spectra.

Let  $X_\bullet$  be a cyclic orthogonal spectrum. Then  $\mathrm{sd}_r X_\bullet$  is an  $r$ -cyclic orthogonal spectrum. At each simplicial level,  $(\mathrm{sd}_r X)_{n-1}$  is an orthogonal spectrum with  $C_r$ -action generated by the  $n$ th power of the cycle map  $t_{rn-1}^n$ . This commutes with all the face, degeneracy, and cycle maps, making  $\mathrm{sd}_r X_\bullet$  an  $r$ -cyclic object in orthogonal  $C_r$ -spectra. Now induce up to a  $\mathbf{J}_{C_r}$ -space and define the geometric fixed points of each simplicial level:

$$\bigvee_{V,W} F_{WC_r} S^0 \wedge \mathbf{J}_{C_r}^{C_r}(V, W) \wedge (\mathrm{sd}_r X(V))_{n-1}^{C_r} \cong \bigvee_V F_{VC_r} S^0 \wedge (\mathrm{sd}_r X(V))_{n-1}^{C_r} \longrightarrow \Phi^{C_r}(\mathrm{sd}_r X)_{n-1}$$

Since geometric fixed points is a functor, we conclude that  $\Phi^{C_r} \mathrm{sd}_r X_\bullet$  is naturally an  $r$ -cyclic orthogonal spectrum. By Prop 3.2.22 above, the  $n$ th power of the cycle map  $t_{rn-1}^n$  acts trivially on these geometric fixed points, establishing that  $\Phi^{C_r} \mathrm{sd}_r X_\bullet$  is a cyclic spectrum. Using  $P_r \Phi^{C_r} \mathrm{sd}_r X_\bullet$  to denote  $\Phi^{C_r} \mathrm{sd}_r X_\bullet$  as an  $r$ -cyclic spectrum, we have the equivariant isomorphisms

$$|\Phi^{C_r} \mathrm{sd}_r X_\bullet| \cong \rho_r^* |P_r \Phi^{C_r} \mathrm{sd}_r X_\bullet| \cong \rho_r^* \Phi^{C_r} |\mathrm{sd}_r X_\bullet| \cong \rho_r^* \Phi^{C_r} |X_\bullet|$$

where the middle map is the canonical commutation of  $\Phi^{C_r}$  with geometric realization. These are all just the maps of Cor 3.1.23 applied to the term  $F_{V^{C_r}} S^0 \wedge X(V)^{C_r}$  in the coequalizer system for  $\Phi^{C_r} X$ .

For that commutation to work it is necessary to think of our object as  $r$ -cyclic, because without the  $\Phi^{C_r}$  the cycle map certainly does not act trivially. In summary:

**Proposition 3.3.1.** *If  $X_\bullet$  is a cyclic spectrum then  $\Phi^{C_r} \text{sd}_r X_\bullet$  is naturally a cyclic spectrum, and there is a natural  $S^1$ -equivariant isomorphism*

$$|\Phi^{C_r} \text{sd}_r X_\bullet| \cong \rho_r^* \Phi^{C_r} |X_\bullet|$$

It's worth pointing out that Prop 3.2.22 is not obviously true here because when  $V$  is a nontrivial representation, the  $n$ th power of the cycle map acts on the fixed points of level  $V$

$$[(\text{sd}_r X)_{n-1}(V)]^{C_r} \cong [(\text{sd}_r X)_{n-1}(\mathbb{R}^m) \wedge_{O(m)} O(\mathbb{R}^m, V)_+]^{C_r}$$

by acting only on the left-hand term  $(\text{sd}_r X)_{n-1}(\mathbb{R}^m)$ . But this is not the  $C_r$ -action, which acts on both terms, and one can see that  $t_{rn-1}^n$  does not in fact act trivially on the  $C_r$ -fixed points at this level. So the fact that it acts trivially on the coequalizer  $\Phi^{C_r}$  is indeed special.

To round out our ability to do homotopy theory, we will apply Thm 3.1.9 to check when a map of cyclic spectra gives a nonequivariant equivalence on the realizations, and use subdivision to check the fixed points. However we need to do a bit more work to make Prop 3.1.12 work on the spectrum level:

**Proposition 3.3.2.** *If  $X_\bullet$  is a cyclic spectrum and each cyclic latching map  $L_n^{\text{cyc}} X \rightarrow X_n$  is a cofibration of  $C_{n+1}$ -spectra then  $|X_\bullet|$  is a cofibrant  $S^1$ -spectrum.*

*Proof.* It suffices to show that each map of cyclic skeleta

$$|\text{Sk}_{n-1}^{\text{cyc}} X| \longrightarrow |\text{Sk}_n^{\text{cyc}} X|$$

is an  $S^1$ -cofibration. The  $(-1)$ -skeleton

$$L_0^{\text{cyc}} X = \text{Sk}_{-1}^{\text{cyc}} X$$

is already assumed to be cofibrant, and it has trivial  $S^1$ -action, so it is also  $S^1$ -cofibrant.

For the induction we use the square from Prop. 3.1.10

$$\begin{array}{ccc}
 L_n^{\text{cyc}} X \times_{C_{n+1}} \Lambda^n \cup_{L_n X \times \partial \Lambda^n} X_n \times_{C_{n+1}} \partial \Lambda^n & \longrightarrow & X_n \times_{C_{n+1}} \Lambda^n \\
 \downarrow & & \downarrow \\
 |\text{Sk}_{n-1}^{\text{cyc}} X_\bullet| & \longrightarrow & |\text{Sk}_n^{\text{cyc}} X_\bullet|
 \end{array}$$

It suffices to prove that the  $C_{n+1}$ -orbits of the pushout-product of the latching map  $L_n^{\text{cyc}} X \rightarrow X$  and the inclusion  $\partial \Lambda^n \rightarrow \Lambda^n$  is an  $S^1$ -cofibration. We know that the latching map is a  $C_{n+1}$ -cofibration of spectra and the inclusion of cyclic simplices is a free  $S^1$ -cofibration of spaces. Since the pushout-product and orbits both commute with all colimits, it suffices to show that the simpler pushout-product

$$[(F_V(C_{n+1}/C_r \times S^{k-1})_+ \rightarrow F_V(C_{n+1}/C_r \times D^k)_+) \square (S^1 \times S^{\ell-1} \rightarrow S^1 \times D^\ell)_+]_{C_{n+1}}$$

is an  $S^1$ -cofibration of spectra when  $V$  is any  $C_r$ -representation. By associativity of the pushout-product one may rewrite this as

$$[F_V(C_{n+1}/C_r \times S^1)_+ \wedge (S^{k+\ell-1} \rightarrow D^{k+\ell})_+]_{C_{n+1}}$$

which simplifies to

$$[F_V(C_{n+1}/C_r)_+ \wedge_{C_{n+1}} S_+^1] \wedge (S^{k+\ell-1} \rightarrow D^{k+\ell})_+$$

It suffices to show the left-hand term is cofibrant as an  $S^1$ -spectrum, but it is obtained by applying the left Quillen functor  $-\wedge_{C_{n+1}} S_+^1$  to the  $C_{n+1}$ -cofibrant object  $F_V(C_{n+1}/C_r)_+$ , so it is cofibrant and the result is proved.  $\square$

### 3.3.2 Cocyclic spectra.

Let  $X^\bullet$  be a cocyclic orthogonal spectrum. Then  $\text{sd}_r X^\bullet$  is an  $r$ -cocyclic orthogonal spectrum, and by the same argument as above,  $\Phi^{C_r} \text{sd}_r X^\bullet$  is naturally a cocyclic orthogonal spectrum. As before, we get the string of equivariant maps

$$\text{Tot}(\Phi^{C_r} \text{sd}_r X^\bullet) \cong \rho_r^* \text{Tot}(P_r \Phi^{C_r} \text{sd}_r X^\bullet) \longleftarrow \rho_r^* \Phi^{C_r} \text{Tot}(\text{sd}_r X^\bullet) \cong \rho_r^* \Phi^{C_r} \text{Tot}(X^\bullet)$$

The middle map is the canonical commutation of  $\Phi^{C_r}$  with totalization, but as one might expect it is not an isomorphism.

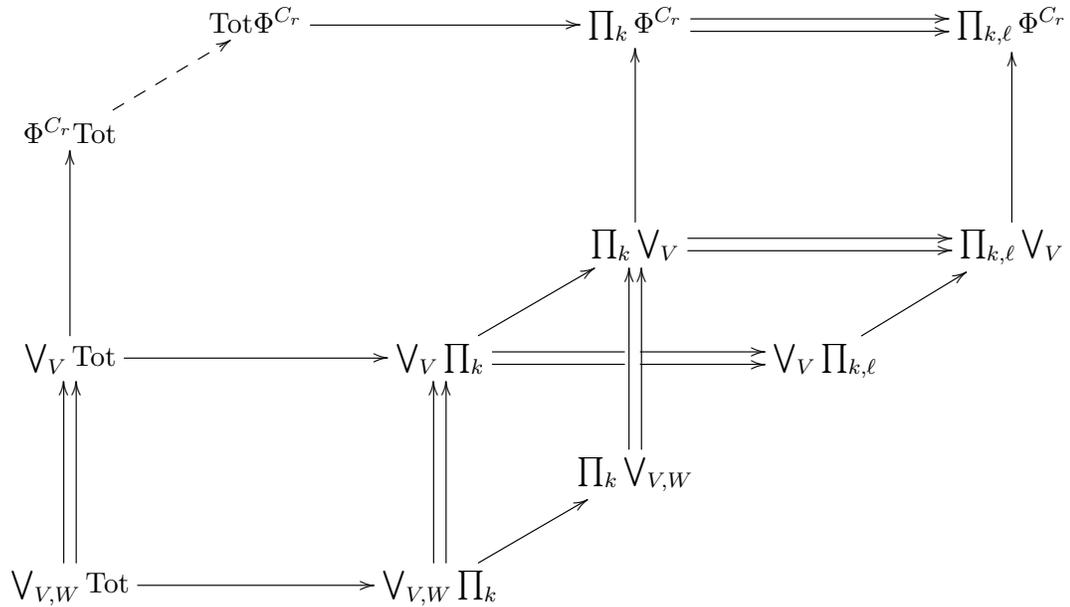
To be specific:

**Proposition 3.3.3.** *There is an interchange map*

$$\Phi^{C_r} \text{Tot}(Z^\bullet) \longrightarrow \text{Tot}(\Phi^{C_r} Z^\bullet)$$

which defines a natural transformation between functors on cosimplicial spectra with  $C_r$ -actions.

*Proof.* The interchange map is given canonically by universal properties, as seen by a long diagram-chase on the shorthand diagram



If  $Z^\bullet \rightarrow \tilde{Z}^\bullet$  is a map of cosimplicial spectra with a  $C_r$ -action, one checks these squares

commute

$$\begin{array}{ccc}
\bigvee_V S^{-V} \wedge \mathrm{Tot}(Z^\bullet)(V)^{C_r} & \longrightarrow & \bigvee_V S^{-V} \wedge \mathrm{Tot}(\tilde{Z}^\bullet)(V)^{C_r} \\
\downarrow & & \downarrow \\
\bigvee_V S^{-V} \wedge \prod_k F(\Delta_+^k, Z^k)(V)^{C_r} & \longrightarrow & \bigvee_V S^{-V} \wedge \prod_k F(\Delta_+^k, \tilde{Z}^k)(V)^{C_r} \\
\downarrow & & \downarrow \\
\prod_k F(\Delta_+^k, \bigvee_V S^{-V} \wedge Z^k(V)^{C_r}) & \longrightarrow & \prod_k F(\Delta_+^k, \bigvee_V S^{-V} \wedge \tilde{Z}^k(V)^{C_r}) \\
\downarrow & & \downarrow \\
\prod_k F(\Delta_+^k, \Phi^{C_r} Z^k) & \longrightarrow & \prod_k F(\Delta_+^k, \Phi^{C_r} \tilde{Z}^k)
\end{array}$$

and then applies the above large diagram to see that the interchange is indeed a natural transformation.  $\square$

In summary we get a weaker form of the result from the previous section:

**Proposition 3.3.4.** *If  $X^\bullet$  is a cocyclic spectrum then  $\Phi^{C_r} \mathrm{sd}_r X^\bullet$  is naturally a cocyclic spectrum, and there is a natural  $S^1$ -equivariant map*

$$\rho_r^* \Phi^{C_r} \mathrm{Tot}(X^\bullet) \longrightarrow \mathrm{Tot}(\Phi^{C_r} \mathrm{sd}_r X^\bullet)$$

The following result is closely related to ([CJ02], Thm 10), and we can use it to produce a large list of examples of cocyclic spectra:

**Proposition 3.3.5.** *If  $X^\bullet$  is a cosimplicial based space and  $E$  is an orthogonal spectrum (or prespectrum, or based space), then the canonical interchange map*

$$E \wedge \prod_k \mathrm{Map}_*(\Delta_+^k, X^k) \longrightarrow \prod_k E \wedge \mathrm{Map}_*(\Delta_+^k, X^k) \longrightarrow \prod_k \mathrm{Map}_*(\Delta_+^k, E \wedge X^k)$$

induces a closed inclusion

$$E \wedge \mathrm{Tot}(X^\bullet) \hookrightarrow \mathrm{Tot}(E \wedge X^\bullet)$$

*If  $X^\bullet$  satisfies the condition that for all  $k$ , the unique map  $\gamma_k : X^k \longrightarrow X^0$  has  $\gamma^{-1}(*) = \{*\}$ , then the above map is a homeomorphism. In particular if  $X^\bullet$  is an unbased cosimplicial space then the analogous map is a homeomorphism.*

*Proof.* It suffices to check this on each spectrum level so assume  $E$  is a based space. Smashing commutes with equalizers ([Str09], 5.3) and so we have a map of equalizer systems

$$\begin{array}{ccc}
 E \wedge \mathrm{Tot}(X^\bullet) & \longrightarrow & \mathrm{Tot}(E \wedge X^\bullet) \\
 \downarrow & & \downarrow \\
 E \wedge \prod_k \mathrm{Map}_*(\Delta_+^k, X^k) & \longrightarrow & \prod_k \mathrm{Map}_*(\Delta_+^k, E \wedge X^k) \\
 \Downarrow & & \Downarrow \\
 E \wedge \prod_{k,\ell,\Delta(k,\ell)} \mathrm{Map}_*(\Delta_+^k, X^\ell) & \longrightarrow & \prod_{k,\ell,\Delta(k,\ell)} \mathrm{Map}_*(\Delta_+^k, E \wedge X^\ell)
 \end{array}$$

We check that the composite from the top-left going down and then right is a closed inclusion. It lands in the image of  $\mathrm{Tot}(E \wedge X^\bullet)$  and so lifts to a closed inclusion

$$E \wedge \mathrm{Tot}(X^\bullet) \hookrightarrow \mathrm{Tot}(E \wedge X^\bullet)$$

Now we check that if  $X^\bullet$  satisfies our condition then this map is surjective. Each point in the totalization gives maps  $\mathrm{Map}(\Delta_+^k, E \wedge X^k)$ , and the only morphism in  $\Delta(k, 0)$  gives the commuting square

$$\begin{array}{ccc}
 \Delta^k & \longrightarrow & E \wedge X^k \\
 \downarrow & & \downarrow \\
 \Delta^0 & \longrightarrow & E \wedge X^0
 \end{array}$$

Now if our map  $\Delta^0 \rightarrow E \wedge X^0$  hits the basepoint, then by our condition all the  $\Delta^k$  must go to the basepoint. Otherwise the map  $\Delta^0 \rightarrow E \wedge X^0$  picks out some unique point  $e \in E$ , and every point in every  $\Delta^k$  must hit something of the form  $(e, -)$ . Therefore each of our maps factors through the closed inclusion  $S^0 = \{e, *\} \hookrightarrow E$ , giving a system of continuous maps that agree with cofaces and codegeneracies; this is our desired preimage in  $E \wedge \mathrm{Tot}(X^\bullet)$ .  $\square$

This implies that the interchange gives an isomorphism of spectra

$$\Sigma_+^\infty LX \cong \mathrm{Tot}(\Sigma_+^\infty X^{\bullet+1})$$

Note that this cosimplicial spectrum is *not* Reedy fibrant, though when  $X$  is finite and simply-connected, a connectivity argument implies that  $\Sigma_+^\infty LX$  will be equivalent to the derived totalization too.

### 3.3.3 The cyclic bar construction.

Let  $R$  be an orthogonal ring spectrum. The *cyclic bar construction on  $R$*  is the cyclic spectrum  $N_{\bullet}^{\text{cyc}} R$  with

$$N_n^{\text{cyc}} R = R^{\wedge(n+1)} = R^{\wedge n} \wedge \underline{R}$$

We underline the last copy of  $R$  since in the simplicial structure it plays a special role. The action of  $\mathbf{A}$  is best visualized by taking the category  $[n]$  and labelling the arrows with copies of  $R$ :

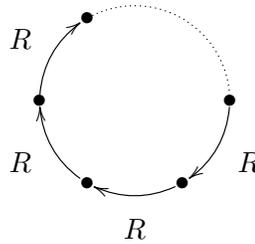


Figure 3.10: The cyclic bar construction.

Each map  $[k] \rightarrow [n]$  induces a map  $R^{\wedge(n+1)} \rightarrow R^{\wedge(k+1)}$  as follows. For each arrow  $i \rightarrow i + 1$ , its image in  $[n]$  is some composition  $j \rightarrow \dots \rightarrow j + \ell$ , which corresponds to  $\ell$  copies of  $R$  in  $R^{\wedge(n+1)}$ , which we smash together and multiply using the product on  $R$ , giving the single copy of  $R$  in slot  $i$  in  $R^{\wedge(k+1)}$ . If there are no arrows then we simply insert a copy of  $\mathbb{S}$  along the identity map of  $R$ .

More generally, if  $\mathbf{C}$  is a *spectral category*, which for us means a category enriched in orthogonal spectra, then the cyclic nerve on  $\mathbf{C}$  is defined as

$$N_n^{\text{cyc}} \mathbf{C} = \bigvee_{c_0, \dots, c_n \in \text{ob } \mathbf{C}} \mathbf{C}(c_0, c_1) \wedge \mathbf{C}(c_1, c_2) \wedge \dots \wedge \mathbf{C}(c_{n-1}, c_n) \wedge \underline{\mathbf{C}(c_n, c_0)}$$

One may think of these objects loosely as “functors” from  $[k]$  into  $\mathbf{C}$ , where ordinary products have been substituted by smash products, and this suggests the correct face, degeneracy, and cycle maps. The face maps multiply adjacent copies of  $\mathbf{C}(c_i, c_{i+1})$ :

$$d_i : N_n^{\text{cyc}} \mathbf{C} \rightarrow N_{n-1}^{\text{cyc}} \mathbf{C}, \quad 0 \leq i \leq n$$

$$\begin{aligned}
d_0 : & \quad \mathbf{C}(c_0, c_1) \wedge \dots \wedge \underline{\mathbf{C}(c_n, c_0)} \longrightarrow \dots \wedge \underline{\mathbf{C}(c_n, c_1)} \\
d_i : & \quad \dots \wedge \mathbf{C}(c_i, c_{i+1}) \wedge \mathbf{C}(c_{i+1}, c_{i+2}) \wedge \dots \longrightarrow \dots \wedge \mathbf{C}(c_i, c_{i+2}) \wedge \dots \\
d_n : & \quad \dots \wedge \mathbf{C}(c_{n-1}, c_n) \wedge \underline{\mathbf{C}(c_n, c_0)} \longrightarrow \dots \wedge \underline{\mathbf{C}(c_{n-1}, c_0)}
\end{aligned}$$

The degeneracy maps and the extra degeneracy map both insert copes of  $\mathbb{S}$  and include into some  $\mathbf{C}(c_i, c_i)$  along the identity map of spectra  $\mathbb{S} \rightarrow \mathbf{C}(c_i, c_i)$ :

$$s_i : N_n^{\text{cyc}} \mathbf{C} \longrightarrow N_{n+1}^{\text{cyc}} \mathbf{C}, \quad 0 \leq i \leq n+1$$

$$\begin{aligned}
s_0 : & \quad \mathbb{S} \wedge \mathbf{C}(c_0, c_1) \wedge \dots \longrightarrow \dots \mathbf{C}(c_0, c_0) \wedge \mathbf{C}(c_0, c_1) \wedge \dots \\
s_i : & \quad \dots \wedge \mathbb{S} \wedge \mathbf{C}(c_i, c_{i+1}) \wedge \dots \longrightarrow \dots \wedge \mathbf{C}(c_i, c_i) \wedge \mathbf{C}(c_i, c_{i+1}) \wedge \dots \\
s_n : & \quad \dots \wedge \mathbb{S} \wedge \underline{\mathbf{C}(c_n, c_0)} \longrightarrow \dots \wedge \mathbf{C}(c_n, c_n) \wedge \underline{\mathbf{C}(c_n, c_0)} \\
s_{n+1} : & \quad \dots \wedge \underline{\mathbf{C}(c_n, c_0)} \wedge \mathbb{S} \longrightarrow \dots \wedge \mathbf{C}(c_n, c_0) \wedge \underline{\mathbf{C}(c_0, c_0)}
\end{aligned}$$

These are enough to determine the action of the cycle map  $t_n = (d_0 s_{n+1})^{-1}$ :

$$t_n : \mathbf{C}(c_0, c_1) \wedge \dots \wedge \underline{\mathbf{C}(c_n, c_0)} \longrightarrow \mathbf{C}(c_n, c_0) \wedge \mathbf{C}(c_0, c_1) \wedge \dots \wedge \underline{\mathbf{C}(c_{n-1}, c_n)}$$

Since the cyclic nerve  $N^{\text{cyc}} \mathbf{C}$  is a cyclic spectrum, its geometric realization is an  $S^1$ -spectrum. In this paper we will call this geometric realization the *topological Hochschild homology* of  $\mathbf{C}$  and denote it  $THH(\mathbf{C})$  for short.

Using relatively recent work on the norm functor, we can say a surprising amount about the geometric fixed points of this construction:

**Proposition 3.3.6.** *If  $\mathbf{C}$  is a spectral category then there are natural maps of  $S^1$ -spectra for  $r \geq 0$*

$$\gamma_r : THH(\mathbf{C}) \longrightarrow \rho_n^* \Phi^{C_r} THH(\mathbf{C})$$

which are compatible in the following sense:

$$\begin{array}{ccc}
T & \xrightarrow{\gamma_{mn}} & \rho_{mn}^* \Phi^{C_{mn}} T \\
\downarrow \gamma_m & & \downarrow \text{it} \\
\rho_m^* \Phi^{C_m} T & \xrightarrow{\rho_m^* \Phi^{C_m} \gamma_n} & \rho_m^* \Phi^{C_m} \rho_n^* \Phi^{C_n} T
\end{array}$$

If every  $\mathbf{C}(c_i, c_j)$  is a cofibrant orthogonal spectrum then every  $\gamma_r$  is an isomorphism.

*Proof.* This is a straightforward extension of the result in [ABG<sup>+</sup>14] for ring spectra, but we will take care to spell out the steps more explicitly. We use from the previous section the isomorphism of  $S^1$ -spectra

$$|\Phi^{C_r} \text{sd}_r N^{\text{cyc}} \mathbf{C}| \xrightarrow{\cong} \rho_r^* \Phi^{C_r} |N^{\text{cyc}} \mathbf{C}|$$

The  $r$ -cyclic spectrum  $\text{sd}_r N^{\text{cyc}} \mathbf{C}$  is at level  $(n-1)$  the wedge of smash products with  $rn$  terms

$$\bigvee_{c_0, \dots, c_{rn-1} \in \text{ob } \mathbf{C}} \mathbf{C}(c_0, c_1) \wedge \dots \wedge \mathbf{C}(c_{rn-1}, c_0)$$

and the  $C_r$ -action is by  $t_{rn-1}^n$ , which rotates this  $rn$ -fold smash product by  $n$  slots.

Let  $V$  be a  $C_r$  representation. Restricting to spectrum level  $V$  of simplicial level  $(n-1)$ , we now have a big wedge indexed by  $rn$ -tuples of objects of  $\mathbf{C}$ . The  $C_r$ -action on the summands is complicated, but on the indices of those summands it is simple: the  $rn$ -tuple of objects gets cycled by  $n$  slots. Therefore any  $C_r$ -fixed point must lie in a wedge summand indexed by some collections of objects of the form

$$c_0, c_1, \dots, c_{n-1}, c_0, c_1, \dots, c_{n-1}, c_0, c_1, \dots, c_{n-1}$$

that is, only  $n$  objects that are repeated  $r$  times. The inclusion of these summands into the rest induces an isomorphism on  $\Phi^{C_r}$ , because it induces a homeomorphism on the fixed points of each spectrum level separately.

Once we have restricted to these summands, the  $C_r$ -action preserves the summands, so we calculate  $\Phi^{C_r}$  of each summand separately. Now we are calculating

$$\Phi^{C_r} (\mathbf{C}(c_0, c_1) \wedge \dots \wedge \mathbf{C}(c_{n-1}, c_0))^{\wedge r}$$

and so we use the Hill-Hopkins-Ravenel norm diagonal

$$\mathbf{C}(c_0, c_1) \wedge \dots \wedge \mathbf{C}(c_{n-1}, c_0) \xrightarrow{\Delta} \Phi^{C_r} (\mathbf{C}(c_0, c_1) \wedge \dots \wedge \mathbf{C}(c_{n-1}, c_0))^{\wedge r}$$

We want to show that these diagonal maps for each  $n \geq 1$  assemble into a map of cyclic spectra

$$N^{\text{cyc}} \mathbf{C} \xrightarrow{\Delta} \Phi^{C_r} \text{sd}_r N^{\text{cyc}} \mathbf{C}$$

(cf. [ABG<sup>+</sup>14], Thm 5.3) We need to check commutativity with the face, degeneracy, and cycle maps. Most of the face and degeneracy maps easily follow because the diagonal is natural. However we run into issues with  $d_0$  and  $t_{rn-1}$ . The  $r$ -fold smash  $(d_0)^{\wedge r}$  of  $d_0$  from the cyclic structure is not the same map as  $d_0$  in the  $r$ -cyclic structure. However, they differ by one or two applications of the map

$$f : (\mathbf{C}(c_0, c_1) \wedge \dots \wedge \mathbf{C}(c_{n-1}, c_0))^{\wedge r} \longrightarrow (\mathbf{C}(c_0, c_1) \wedge \dots \wedge \mathbf{C}(c_{n-1}, c_0))^{\wedge r}$$

which simply takes the factors  $\mathbf{C}(c_{n-1}, c_0)$  and cycles them while leaving all the other terms fixed. (Similarly for  $t_{rn-1}$ .) We have defined  $f$  on the trivial universe and it commutes with the  $C_r$ -action so it passes to a well-defined map  $\mathcal{I}_{\mathbb{R}^\infty}^U f$  on a complete universe. It suffices to show that  $f$  commutes with  $\Delta$ , but we did that in Prop 3.2.23 above.

This proves that the diagonal norm map is a map of cyclic spectra, and we define  $\gamma_r$  to be its geometric realization:

$$|N^{\text{cyc}} \mathbf{C}| \xrightarrow{|\Delta_r|} |\Phi^{C_r} \text{sd}_r N^{\text{cyc}} \mathbf{C}| \xrightarrow{\cong} \rho_r^* \Phi^{C_r} |N^{\text{cyc}} \mathbf{C}|$$

These are all  $S^1$ -equivariant by Prop 3.3.1 above. When all the  $\mathbf{C}(c_i, c_{i+1})$  are cofibrant,  $\gamma_r$  is a realization of isomorphisms at each level, so  $\gamma_r$  is an isomorphism.

Now we check compatibility, and for simplicity we forget the  $S^1$ -equivariance. The compatibility square may be expanded and subdivided

$$\begin{array}{ccccc} |N^{\text{cyc}} \mathbf{C}| & \xrightarrow{\Delta_{mn}} & |\Phi^{C_{mn}} \text{sd}_{mn} N^{\text{cyc}} \mathbf{C}| & \xrightarrow[\cong]{D_{mn}} & \Phi^{C_{mn}} |N^{\text{cyc}} \mathbf{C}| \\ \downarrow \Delta_m & & \downarrow \text{it} & & \downarrow \text{it} \\ |\Phi^{C_m} \text{sd}_m N^{\text{cyc}} \mathbf{C}| & \xrightarrow{\Phi^{C_m} \Delta_*} & |\Phi^{C_{mn}/C_n} \Phi^{C_n} \text{sd}_{mn} N^{\text{cyc}} \mathbf{C}| & \xrightarrow[\cong]{D_{mn}} & \Phi^{C_{mn}/C_n} \Phi^{C_n} |N^{\text{cyc}} \mathbf{C}| \\ \cong \downarrow D_m & & \cong \downarrow D_m & & \parallel \\ \Phi^{C_m} |N^{\text{cyc}} \mathbf{C}| & \xrightarrow{\Phi^{C_m} \Delta_n} & \Phi^{C_m} |\Phi^{C_n} \text{sd}_n N^{\text{cyc}} \mathbf{C}| & \xrightarrow[\cong]{D_n} & \Phi^{C_{mn}/C_n} \Phi^{C_n} |N^{\text{cyc}} \mathbf{C}| \end{array}$$

The right-hand squares commute easily. Prop 3.2.20 tells us that the top-left square commutes. For the bottom-left we pull off the  $\Phi^{C_m}$  and forget the  $C_m$ -actions, leaving us with the commuting square from Prop 3.2.21.  $\square$

Blumberg and Mandell remark in ([BM08], Sec 3) that this kind of cyclic bar construction is often insufficient to give an  $S^1$ -spectrum with the kind of *cyclotomic* structure

described in the previous proposition. We advertise that the existence of this structure is an exciting and newly-discovered feature of the cyclic bar construction in the category of orthogonal spectra, and our treatment here is little more than an elaboration of the work in [ABG<sup>+</sup>14].

Finally, in order to do homotopy theory, we need to know what sorts of maps of categories  $\mathbf{C} \rightarrow \mathbf{D}$  will be sent to weak equivalences under this functor, and we need conditions guaranteeing that  $|N^{\text{cyc}}\mathbf{C}|$  will be cofibrant. For both of these purposes we need to describe the latching maps and cyclic latching maps. Let  $\mathbf{S}$  denote the initial spectrally-enriched category on the objects of  $\mathbf{C}$ :

$$\mathbf{S}(c_i, c_j) = \begin{cases} \mathbb{S} & c_i = c_j \\ * & c_i \neq c_j \end{cases}$$

Then the latching maps can be described concisely in terms of the canonical functor  $\mathbf{S} \rightarrow \mathbf{C}$ :

**Proposition 3.3.7.** *For every  $n \geq 0$  the latching map  $L_n N^{\text{cyc}}\mathbf{C} \rightarrow N_n^{\text{cyc}}\mathbf{C}$  is the wedge of pushout-products*

$$\bigvee_{c_0, \dots, c_n \in \text{ob } \mathbf{C}} (\mathbf{S}(c_0, c_1) \rightarrow \mathbf{C}(c_0, c_1)) \square \dots \square (\mathbf{S}(c_{n-1}, c_n) \rightarrow \mathbf{C}(c_{n-1}, c_n)) \square (* \rightarrow \underline{\mathbf{C}(c_n, c_0)})$$

and the cyclic latching map  $L_n^{\text{cyc}} N^{\text{cyc}}\mathbf{C} \rightarrow N_n^{\text{cyc}}\mathbf{C}$  is the wedge of pushout-products

$$\bigvee_{c_0, \dots, c_n \in \text{ob } \mathbf{C}} (\mathbf{S}(c_0, c_1) \rightarrow \mathbf{C}(c_0, c_1)) \square \dots \square (\underline{\mathbf{S}(c_n, c_0)} \rightarrow \underline{\mathbf{C}(c_n, c_0)})$$

*Proof.* We induct using the usual pushout squares. Or, using the fact that the latching map is a closed inclusion, we identify it as the correct subspace on each spectrum level.  $\square$

**Definition 3.3.8.**  $\mathbf{C}$  is *cofibrant* if every map  $\mathbf{S}(c_i, c_j) \rightarrow \mathbf{C}(c_i, c_j)$  is a cofibration of orthogonal spectra.

This is weaker than the notion of “cofibrant” one would need for a model structure; in particular when  $\mathbf{C}$  has one object it is just the condition that the inclusion of the unit is a cofibration. Now we can give our homotopical results:

**Proposition 3.3.9.** *If  $\mathbf{C}$ ,  $\mathbf{D}$  are cofibrant and  $\mathbf{C} \rightarrow \mathbf{D}$  is a pointwise weak equivalence which is the identity on objects, then it induces an  $\mathcal{F}$ -equivalence of  $S^1$ -spectra  $|N^{\text{cyc}}\mathbf{C}| \rightarrow |N^{\text{cyc}}\mathbf{D}|$ .*

*Proof.* A map of  $S^1$ -spectra  $X \rightarrow Y$  is an  $\mathcal{F}$ -equivalence iff it induces equivalences on the genuine fixed points  $(fX)^{C_n} \rightarrow (fY)^{C_n}$  for all  $n \geq 0$ . By the usual isotropy separation argument this is equivalent to  $X \rightarrow Y$  inducing equivalences on the geometric fixed points  $\Phi^{C_n}(cX) \rightarrow \Phi^{C_n}(cY)$  for all  $n \geq 0$ . By the above, the geometric fixed points are naturally equivalent to the original spectrum. The proposition below assures us that these geometric fixed points are derived. Therefore it suffices to show that  $|N^{\text{cyc}}\mathbf{C}| \rightarrow |N^{\text{cyc}}\mathbf{D}|$  is an ordinary stable equivalence of spectra.

On each simplicial level  $N^{\text{cyc}}\mathbf{C}$  is a wedge of smashes of cofibrant spectra and so the smash products are derived, so  $N^{\text{cyc}}\mathbf{C} \rightarrow N^{\text{cyc}}\mathbf{D}$  is a levelwise weak equivalence. Therefore we only need the simplicial objects to be “proper,” meaning the inclusion of the latching object is an  $h$ -cofibration. But by Prop 3.3.7 and the fact that the pushout-product preserves cofibrations, every latching map is a cofibration of orthogonal spectra, so it is certainly an  $h$ -cofibration.  $\square$

For the next section we will need more control over when the resulting  $S^1$ -spectrum  $|N^{\text{cyc}}\mathbf{C}|$  is cofibrant, which we’ll provide here:

**Proposition 3.3.10.** *If  $\mathbf{C}$  is cofibrant then  $|N^{\text{cyc}}\mathbf{C}|$  is a cofibrant  $S^1$ -spectrum. Moreover the inclusion of each cyclic skeleton into the next is a cofibration of  $S^1$ -spectra.*

*Proof.* By Prop 3.3.2 it suffices to show that the cyclic latching map from Prop 3.3.7

$$\bigvee_{c_0, \dots, c_{n-1} \in \text{ob } \mathbf{C}} (\mathbf{S}(c_0, c_1) \rightarrow \mathbf{C}(c_0, c_1)) \square \dots \square (\mathbf{S}(c_{n-1}, c_0) \rightarrow \mathbf{C}(c_{n-1}, c_0))$$

is a  $C_n$ -cofibration of spectra, and this is proven along the same lines as the argument that iterated pushout-products of cells of orthogonal spectra yield cells of orthogonal  $G$ -spectra:

$$(F_n S_+^{k-1} \rightarrow F_n D_+^k)^{\square G} \cong (F_{n\rho_G} S(k\rho_G)_+ \rightarrow F_{n\rho_G} D(k\rho_G)_+)$$

$\square$

### 3.4 Mapping and dualizing cyclotomic spectra

In this section we use the technology developed thus far to prove that cyclotomic structures can be dualized, and show that the equivalence of ordinary spectra

$$D(\mathrm{THH}(D(X_+))) \simeq \mathrm{THH}(\Sigma_+^\infty \Omega X) \simeq \Sigma_+^\infty LX$$

is actually an equivariant equivalence of cyclotomic spectra (when  $X$  is a finite simply-connected CW complex).

#### 3.4.1 A general framework for dualizing cyclotomic structures.

Recall that a *cyclotomic spectrum* is an orthogonal  $S^1$ -spectrum  $T$  with compatible maps of  $S^1$ -spectra for all  $n \geq 1$

$$c_n : \rho_n^* \Phi^{C_n} T \longrightarrow T$$

for which the composite map

$$\rho_n^* L \Phi^{C_n} T \longrightarrow \rho_n^* \Phi^{C_n} T \longrightarrow T$$

is an  $\mathcal{F}$ -equivalence of  $S^1$ -spectra. To be more specific about the compatibility, we require that for all  $m, n \geq 1$  the square

$$\begin{array}{ccc} \rho_{mn}^* \Phi^{C_{mn}} X & \xrightarrow{c_{mn}} & X \\ \downarrow \text{it} & & \uparrow c_m \\ \rho_m^* \Phi^{C_m} \rho_n^* \Phi^{C_n} X & \xrightarrow{\rho_m^* \Phi^{C_m} c_n} & \rho_m^* \Phi^{C_m} X \end{array}$$

commutes. The left vertical is the canonical iterated fixed points map described in [BM13], Prop 2.4, and it is an isomorphism when  $X$  is cofibrant as an  $S^1$ -spectrum.

A *pre-cyclotomic spectrum* has all the same structure except that the derived  $c_n$  need not be an equivalence.

In contrast to this, we give a more restrictive definition:

**Definition 3.4.1.** A *tight cyclotomic spectrum* is a cofibrant  $S^1$ -spectrum with isomorphisms  $\gamma_n : T \xrightarrow{\cong} \rho_n^* \Phi^{C_n} T$  of  $S^1$ -spectra for all  $n \geq 0$  compatible in the following way:

$$\begin{array}{ccc}
 T & \xrightarrow[\cong]{\gamma_{mn}} & \rho_{mn}^* \Phi^{C_{mn}} T \\
 \cong \downarrow \gamma_m & & \cong \downarrow \text{it} \\
 \rho_m^* \Phi^{C_m} T & \xrightarrow[\cong]{\rho_m^* \Phi^{C_m} \gamma_n} & \rho_m^* \Phi^{C_m} \rho_n^* \Phi^{C_n} T
 \end{array}$$

Here “cofibrant” means in the usual stable model structure on orthogonal  $S^1$ -spectra. This includes the cofibrant spectra in the  $\mathcal{F}$ -model structure too. In particular since  $T$  is cofibrant,  $\rho_n^* \Phi^{C_n} T$  is already derived. So a tight cyclotomic spectrum may be regarded as a cyclotomic spectrum, by taking  $c_n = \gamma_n^{-1}$  and then forgetting that it is an isomorphism. The previous section tells us

**Proposition 3.4.2.** *If  $R$  is an orthogonal ring spectrum and  $\mathbb{S} \rightarrow R$  is a cofibration of orthogonal spectra then  $THH(R)$  is naturally a tight cyclotomic spectrum. If  $\mathbf{C}$  is a spectral category, each  $\mathbf{C}(c_i, c_j)$  is a cofibrant orthogonal spectrum, and each unit  $\mathbb{S} \rightarrow \mathbf{C}(c, c)$  is a cofibration of orthogonal spectra, then  $THH(\mathbf{C})$  is a tight cyclotomic spectrum.*

The point of these definitions is to discuss dualization of cyclotomic structures. Our first result is

**Proposition 3.4.3.** *If  $T$  is a tight cyclotomic spectrum and  $T'$  is pre-cyclotomic then the function spectrum  $F(T, T')$  has a natural pre-cyclotomic structure.*

**Corollary 3.4.4.** *If  $T$  is a tight cyclotomic spectrum then the functional dual  $DT = F(T, \mathbb{S})$  is pre-cyclotomic.*

*Proof.* We define the structure map  $c_r$  as the composite

$$\rho_r^* \Phi^{C_r} F(T, T') \xrightarrow{\bar{\alpha}} F(\rho_r^* \Phi^{C_r} T, \rho_r^* \Phi^{C_r} T') \xrightarrow{F(\gamma_r, c_r)} F(T, T')$$

where  $\bar{\alpha}$  is the “restriction” map adjoint to

$$\rho_r^* \Phi^{C_r} F(T, T') \wedge \rho_r^* \Phi^{C_r} T \xrightarrow{\alpha} \rho_r^* \Phi^{C_r} (F(T, T') \wedge T) \longrightarrow \rho_r^* \Phi^{C_r} T'$$

and  $\alpha$  is the usual commutation of  $\Phi^{C_r}$  with smash products. By the usual rules for equivariant adjunctions,  $c_r$  is automatically  $S^1$ -equivariant.

We verify that these maps are compatible. Clearly they are natural in  $T$  and  $T'$ , so in the diagram

$$\begin{array}{ccccc}
 \rho_{mn}^* \Phi^{C_{mn}} F(T, T') & \xrightarrow{\rho} & F(\rho_{mn}^* \Phi^{C_{mn}} T, \rho_{mn}^* \Phi^{C_{mn}} T') & \xrightarrow{F(\text{id}, \text{id})} & F(\rho_{mn}^* \Phi^{C_{mn}} T, \rho_m^* \Phi^{C_m} \rho_n^* \Phi^{C_n} T') \\
 \downarrow \text{it} & & & & \cong \uparrow F(\text{id}, \text{id}) \\
 \rho_m^* \Phi^{C_m} \rho_n^* \Phi^{C_n} F(T, T') & \longrightarrow & \rho_m^* \Phi^{C_m} F(\rho_n^* \Phi^{C_n} T, \rho_n^* \Phi^{C_n} T') & \longrightarrow & F(\rho_m^* \Phi^{C_m} \rho_n^* \Phi^{C_n} T, \rho_m^* \Phi^{C_m} \rho_n^* \Phi^{C_n} T') \\
 & & \downarrow \cong & & \downarrow \cong \\
 & & \rho_m^* \Phi^{C_m} F(T, \rho_n^* \Phi^{C_n} T') & \longrightarrow & F(\rho_m^* \Phi^{C_m} T, \rho_m^* \Phi^{C_m} \rho_n^* \Phi^{C_n} T') \\
 & & \downarrow & & \downarrow \\
 & & \rho_m^* \Phi^{C_m} F(T, T') & \longrightarrow & F(\rho_m^* \Phi^{C_m} T, \rho_m^* \Phi^{C_m} T') \\
 & & & & \downarrow \\
 & & & & F(T, \rho_m^* \Phi^{C_m} T') \\
 & & & & \downarrow \\
 & & & & F(T, T')
 \end{array}$$

the two small squares automatically commute. The left-most and right-most paths compose to give the two maps we are trying to compare (the wrong-way map along the right edge is an isomorphism because the iterated fixed points map is an isomorphism on the cofibrant spectrum  $T$ ). So, we just need to show that the big rectangle at the top commutes. It is adjoint to

$$\begin{array}{ccccc}
 \rho_{mn}^* \Phi^{C_{mn}} F(T, T') \wedge \rho_{mn}^* \Phi^{C_{mn}} T & \xrightarrow{\alpha} & \rho_{mn}^* \Phi^{C_{mn}} (F(T, T') \wedge T) & \longrightarrow & \rho_{mn}^* \Phi^{C_{mn}} T' \\
 \downarrow \text{it} \wedge \text{it} & & \downarrow \text{it} & & \downarrow \text{it} \\
 \rho_m^* \Phi^{C_m} \rho_n^* \Phi^{C_n} F(T, T') \wedge \rho_m^* \Phi^{C_m} \rho_n^* \Phi^{C_n} T & \xrightarrow{\alpha \circ \alpha} & \rho_m^* \Phi^{C_m} \rho_n^* \Phi^{C_n} (F(T, T') \wedge T) & \longrightarrow & \rho_m^* \Phi^{C_m} \rho_n^* \Phi^{C_n} T'
 \end{array}$$

The right square is by naturality of the iterated fixed points map, and the left square is by Prop 3.2.19.  $\square$

**Remark.** The technical lemmas we have checked here are enough to ensure that the smash product of two op-pre-cyclotomic spectra is op-pre-cyclotomic, and that there is a pairing adjunction between op-pre-cyclotomic spectra and pre-cyclotomic spectra, but we will not spell that out here.

Our main example of interest will be when  $T = |N^{\text{cyc}} \mathbf{C}|$  is the cyclic nerve of a ring or

category. In this case  $F(T, T')$  is clearly the totalization of the cocyclic  $S^1$ -spectrum

$$Y^k = F(N_k^{\text{cyc}} \mathbf{C}, T')$$

The  $S^1$  acts only on the  $T'$ , and  $\mathbf{A}$  acts by composing the map into  $T'$  with the action of  $\mathbf{A}^{\text{op}}$  on the cyclic bar construction. Therefore the totalization carries two commuting  $S^1$ -actions, so we restrict to the diagonal action to get the  $S^1$ -action which extends to the pre-cyclotomic structure defined above. The following result allows us to re-express that pre-cyclotomic structure purely in terms of constructions on the totalization of  $Y^\bullet$ .

**Proposition 3.4.5.** *The structure map on  $\text{Tot}(Y^\bullet) \cong F(|N^{\text{cyc}} \mathbf{C}|, T')$  is equal to the composite of  $S^1$ -equivariant maps*

$$\begin{aligned} \rho_r^* \Phi^{C_r} \text{Tot}(F(N_\bullet^{\text{cyc}} \mathbf{C}, T')) &\xrightarrow{D_r} \rho_r^* \Phi^{C_r} \text{Tot}(F(\text{sd}_r N_\bullet^{\text{cyc}} \mathbf{C}, T')) \\ &\longrightarrow \rho_r^* \text{Tot}(\Phi^{C_r} F(\text{sd}_r N_\bullet^{\text{cyc}} \mathbf{C}, T')) \\ &\xrightarrow{\bar{\alpha}} \rho_r^* \text{Tot}(F(P_r \Phi^{C_r} \text{sd}_r N_\bullet^{\text{cyc}} \mathbf{C}, \Phi^{C_r} T')) \\ &\xrightarrow{\cong} \text{Tot}(F(\Phi^{C_r} \text{sd}_r N_\bullet^{\text{cyc}} \mathbf{C}, \rho_r^* \Phi^{C_r} T')) \\ &\xrightarrow{F(\Delta, c_r)} \text{Tot}(F(N_\bullet^{\text{cyc}} \mathbf{C}, T')) \end{aligned}$$

*Proof.* We compare to the structure map we defined above:

$$\begin{array}{ccc} \rho_r^* \Phi^{C_r} F(|N_\bullet^{\text{cyc}} \mathbf{C}|, T') & \xrightarrow{\cong} & \rho_r^* \Phi^{C_r} \text{Tot}(F(N_\bullet^{\text{cyc}} \mathbf{C}, T')) \\ \downarrow \bar{\alpha} & \searrow^{D_r} & \downarrow D_r \\ F(\rho_r^* \Phi^{C_r} |N_\bullet^{\text{cyc}} \mathbf{C}|, \rho_r^* \Phi^{C_r} T') & \xrightarrow{\cong} & \rho_r^* \Phi^{C_r} F(|\text{sd}_r N_\bullet^{\text{cyc}} \mathbf{C}|, T') \xrightarrow{\cong} \rho_r^* \Phi^{C_r} \text{Tot}(F(\text{sd}_r N_\bullet^{\text{cyc}} \mathbf{C}, T')) \\ & \searrow^{D_r} & \downarrow \bar{\alpha} \\ & & F(\rho_r^* \Phi^{C_r} |\text{sd}_r N_\bullet^{\text{cyc}} \mathbf{C}|, \rho_r^* \Phi^{C_r} T') \xrightarrow{\cong} \rho_r^* \text{Tot}(\Phi^{C_r} F(\text{sd}_r N_\bullet^{\text{cyc}} \mathbf{C}, T')) \\ & & \downarrow \bar{\alpha} \\ & & F(\rho_r^* |P_r \Phi^{C_r} \text{sd}_r N_\bullet^{\text{cyc}} \mathbf{C}|, \rho_r^* \Phi^{C_r} T') \xrightarrow{\cong} \rho_r^* \text{Tot}(F(P_r \Phi^{C_r} \text{sd}_r N_\bullet^{\text{cyc}} \mathbf{C}, \Phi^{C_r} T')) \\ & & \downarrow \cong \\ & & F(|\Phi^{C_r} \text{sd}_r N_\bullet^{\text{cyc}} \mathbf{C}|, \rho_r^* \Phi^{C_r} T') \xrightarrow{\cong} \text{Tot}(F(\Phi^{C_r} \text{sd}_r N_\bullet^{\text{cyc}} \mathbf{C}, \rho_r^* \Phi^{C_r} T')) \\ & & \downarrow F(\Delta, c_r) \\ & & F(|N_\bullet^{\text{cyc}} \mathbf{C}|, T') \xrightarrow{\cong} \text{Tot}(F(N_\bullet^{\text{cyc}} \mathbf{C}, \Phi^{C_r} T')) \end{array}$$

All maps here have been carefully checked to be  $S^1$ -equivariant. The  $\rho_r^*$ 's have to remain out front on the right-hand side for a while longer because  $\Phi^{C_r}F(\mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} \mathbf{C}, T')$  and  $F(\Phi^{C_r} \mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} \mathbf{C}, \Phi^{C_r} T')$  are not cyclic but are merely  $r$ -cyclic (though their realizations still do have trivial  $C_r$ -action). Most of these squares commute easily, but the nontrivial one in the middle can be simplified to the following: if  $X_{\bullet}$  is a simplicial  $C_r$ -spectrum and  $T$  is a  $C_r$ -spectrum then

$$\begin{array}{ccc}
 \Phi^{C_r}F(|X_{\bullet}|, T) & \xrightarrow{\cong} & \Phi^{C_r}\mathrm{Tot}(F(X_{\bullet}, T)) \\
 \downarrow \bar{\alpha} & & \downarrow \\
 F(\Phi^{C_r}|X_{\bullet}|, \Phi^{C_r}T) & & \mathrm{Tot}(\Phi^{C_r}F(X_{\bullet}, T)) \\
 \downarrow \cong & & \downarrow \bar{\alpha} \\
 F(|\Phi^{C_r}X_{\bullet}|, \Phi^{C_r}T) & \xrightarrow{\cong} & \mathrm{Tot}(F(\Phi^{C_r}X_{\bullet}, \Phi^{C_r}T))
 \end{array}$$

commutes. □

Now we know that  $F(T, T')$  has a pre-cyclotomic structure, but this will not be too useful in practice unless we can make  $F(T, T')$  derived so that it carries homotopical meaning. Unfortunately, this is quite difficult to do directly without weakening the structure maps of  $F(T, T')$  to mere zig-zags. However the model structure on cyclotomic and pre-cyclotomic spectra defined in [BM13] allows us to avoid that. It has following attractive property that allows our constructions to be derived:

**Lemma 3.4.6.** *If  $T$  is cofibrant or fibrant in the model\* category on (pre)cyclotomic spectra, then it is also cofibrant or fibrant, respectively, as an orthogonal  $S^1$ -spectrum in the  $\mathcal{F}$ -model structure.*

*Proof.* The fibrant part is true by definition. For the cofibrant part it suffices to check that the monad

$$\mathbb{C}X = \bigvee_{n \geq 1} \rho_n^* \Phi^{C_n} X$$

preserves cofibrant objects in the  $\mathcal{F}$ -model structure. This is true because wedge sums, geometric fixed points, and change of groups all preserve cofibrations. □

In light of this fact, we can replace  $T'$  with a fibrant cyclotomic spectrum  $fT'$ , resulting in the pre-cyclotomic spectrum  $F(T, fT')$  whose underlying  $S^1$ -equivariant spectrum

is derived. Specializing to  $T' = \mathbb{S}$  gives a pre-cyclotomic structure on the derived dual  $F(T, f\mathbb{S})$ .

Now if the underlying  $S^1$ -spectrum of  $T$  is a retract of a finite  $S^1$ -cell spectrum, or equivalently is *dualizable*, then by ([LMSM86], III.1.9) the cyclotomic structure maps of  $F(T, fT')$  are all weak equivalences. Therefore  $F(T, fT')$  is actually cyclotomic, and not just pre-cyclotomic, when  $T$  is finite.

In general  $F(T, fT')$  is not cyclotomic: a counterexample is  $T = \Sigma_+^\infty \mathbb{R}P^\infty$  and  $T' = \mathbb{S}$ . However we will see an example in the next section where  $T$  is infinite and  $F(T, fT')$  is still cyclotomic.

### 3.4.2 The cyclotomic dual of $THH(DX_+)$ is $\Sigma_+^\infty LX$ .

Let  $X$  be a finite based CW complex and let  $D(X_+) = F(X_+, \mathbb{S})$  denote its Spanier-Whitehead dual. Though  $\mathbb{S}$  is not fibrant,  $X$  is compact, and so this spectrum has the correct homotopy type. Unfortunately, though  $DX_+$  is finite, it is not compact. Even worse, a big realization of a simplicial spectrum built out of such things is not compact. So when we dualize it again we will need to map into a fibrant sphere, which slightly complicates the proof below.

$DX_+$  is a commutative ring, with multiplication given by the dual of the diagonal map

$$X_+ \longrightarrow (X \times X)_+$$

Now the levelwise fiber sequence of spectra

$$F(X, \mathbb{S}) \longrightarrow F(X_+, \mathbb{S}) \longrightarrow \mathbb{S}$$

preserves the multiplications coming from the diagonal maps on both  $X$  and  $X_+$ , so we can conclude that the most obvious map

$$\mathbb{S} \vee DX \xrightarrow{\sim} DX_+$$

is an equivalence of ring spectra, where on the left the  $\mathbb{S}$  is the unit and the multiplication on  $F(X, \mathbb{S})$  is the dual of the diagonal

$$X \longrightarrow X \wedge X$$

Let  $cDX$  denote cofibrant replacement of  $DX$  as a unitless ring, so that

$$cDX_+ := \mathbb{S} \vee cDX \longrightarrow \mathbb{S} \vee DX$$

is a particularly nice cofibrant replacement of ring spectra. We'll take as our example of a tight cyclotomic spectrum

$$T = THH(cDX_+)$$

This is equivalent to the derived  $THH$  of the derived dual of  $X_+$ . Our starting point is the result

**Theorem 3.4.7** (Cohen,Campbell). *[Cam14] When  $X$  is a finite simply-connected CW complex there is an equivalence of ordinary spectra*

$$D(THH(D(X_+))) \simeq THH(\Sigma_+^\infty \Omega X) \simeq \Sigma_+^\infty LX$$

when everything is derived and  $LX = \text{Map}(S^1, X)$  is the free loop space.

**Remark.** When  $M$  is a manifold  $DM_+ \simeq M^{-TM}$  is a Thom spectrum, but the analysis of [BCS08] does not apply because the multiplication on  $M^{-TM}$  does not arise from the normal bundle  $M \rightarrow BO$  being a loop map.

We will spend the rest of this section proving a more highly-structured version of that result:

**Theorem 3.4.8.** *Let  $f\mathbb{S}$  be a fibrant replacement of  $\mathbb{S}$  as a cyclotomic spectrum. Then for every unbased space  $X$  there is a natural map of pre-cyclotomic spectra*

$$\Sigma_+^\infty LX \longrightarrow F(THH(cDX_+), f\mathbb{S})$$

The left-hand side is always cyclotomic. When  $X$  is a finite simply-connected CW-complex the right-hand side is cyclotomic and the map is an  $\mathcal{F}$ -equivalence.

**Corollary 3.4.9.** *When  $X$  is a finite simply-connected CW-complex, the equivalence between  $THH(\Sigma_+^\infty \Omega X)$  and the functional dual of  $THH(DX_+)$  is an equivalence of cyclotomic spectra.*

*Proof.* As above, let  $Y^\bullet$  denote the cocyclic  $S^1$ -spectrum

$$Y^k = F(N_k^{\text{cyc}} cDX_+, f\mathbb{S}) = F((cDX_+)^{\wedge(k+1)}, f\mathbb{S})$$

Then the totalization of  $Y^\bullet$  is isomorphic to  $F(|N^{\text{cyc}}cDX_+|, f\mathbb{S})$ , and Prop 3.4.5 gives us a recipe for the pre-cyclotomic structure.

Define a second cosimplicial spectrum  $Z^\bullet$  by

$$Z^k = \Sigma_+^\infty \text{Map}(\mathbf{\Lambda}([k], [0]), X) \cong \Sigma_+^\infty X^{k+1}$$

with  $\mathbf{\Lambda}$  action given by applying  $\Sigma_+^\infty$  to the usual  $\mathbf{\Lambda}^{\text{op}}$  action on the  $\mathbf{\Lambda}([k], [0])$  term. By the same reasoning above, the totalization of  $Z^\bullet$  would be homeomorphic to  $\text{Map}(|\mathbf{\Lambda}[0]|, X) \cong LX$  if not for the  $\Sigma_+^\infty$ . But by Prop 3.3.5, the usual interchange still gives an isomorphism of spectra

$$\Sigma_+^\infty LX \cong \text{Tot}(Z^\bullet)$$

Note that  $Z^\bullet$  is *not* Reedy fibrant, though when  $X$  is finite and simply-connected,  $Y^\bullet$  will turn out to be its fibrant replacement and so  $\Sigma_+^\infty LX$  will be equivalent to the derived totalization too.

Now we construct a cosimplicial map  $Z^\bullet \rightarrow Y^\bullet$ . The evaluation map composed with the product in  $\mathbb{S}$  and fibrant replacement

$$\begin{aligned} (\Sigma_+^\infty X)^{\wedge(k+1)} \wedge c(DX_+)^{\wedge(k+1)} &\longrightarrow (\Sigma_+^\infty X)^{\wedge(k+1)} \wedge (DX_+)^{\wedge(k+1)} \\ &\longrightarrow (\mathbb{S})^{\wedge(k+1)} \longrightarrow \mathbb{S} \longrightarrow f\mathbb{S} \end{aligned}$$

is adjoint to a map

$$Z^k = \Sigma_+^\infty X^{k+1} \xrightarrow{\sim} F((cDX_+)^{\wedge(k+1)}, f\mathbb{S}) = Y^k$$

It clearly commutes with the  $S^1$ -action on each level coming from  $f\mathbb{S}$ . We check that it commutes with the cocyclic structure: for each  $\gamma \in \mathbf{\Lambda}([k], [\ell])$  we have the square

$$\begin{array}{ccc} \text{Map}(\mathbf{\Lambda}[0]_k, X) \cong X^{k+1} & \longrightarrow & F((cDX_+)^{\wedge k+1}, f\mathbb{S}) \\ \downarrow \gamma & & \downarrow \gamma \\ \text{Map}(\mathbf{\Lambda}[0]_\ell, X) \cong X^{\ell+1} & \longrightarrow & F((cDX_+)^{\wedge \ell+1}, f\mathbb{S}) \end{array}$$

which commutes if this one commutes:

$$\begin{array}{ccc}
 X^{k+1} \wedge (cDX_+)^{\ell+1} & \xrightarrow{\gamma \wedge \text{id}} & X^{\ell+1} \wedge (cDX_+)^{\ell+1} \\
 \downarrow \text{id} \wedge \gamma & & \downarrow \\
 X^{k+1} \wedge (cDX_+)^{k+1} & \longrightarrow & \mathbb{S}
 \end{array}$$

Both branches have the same description:  $\gamma$  gives a map from a necklace with  $k+1$  beads and every segment labelled by  $X$  to a necklace with  $\ell+1$  beads and every segment labelled by  $DX$ . Each copy of  $X$  is sent by  $\gamma$  to a string of  $a$  copies of  $DX$ ; we apply the diagonal to  $X_+ \xrightarrow{\Delta} (\prod^a X)_+$  and pair with those  $a$  copies of  $DX$ .

Therefore we have a map of cocyclic  $S^1$ -spectra  $Z^\bullet \rightarrow Y^\bullet$ , with  $S^1$  acting trivially on each cosimplicial level of  $Z^\bullet$ . This gives an equivariant map  $\text{Tot}(Z^\bullet) \rightarrow \text{Tot}(Y^\bullet)$ , and our next task is to check that it respects the pre-cyclotomic structures that we already have on  $\Sigma_+^\infty LX$  and  $\text{Tot}(Y^\bullet)$ :

$$\begin{array}{ccccc}
 \Phi^{C_r} \Sigma_+^\infty LX & \xrightarrow{\cong} & \rho_r^* \Phi^{C_r} \text{Tot}(Z^\bullet) & \longrightarrow & \rho_r^* \Phi^{C_r} \text{Tot}(Y^\bullet) & (3.3) \\
 \downarrow \cong & & \cong \downarrow D_r & & \cong \downarrow D_r & \\
 & & \rho_r^* \Phi^{C_r} \text{Tot}(\text{sd}_r Z^\bullet) & \longrightarrow & \rho_r^* \Phi^{C_r} \text{Tot}(\text{sd}_r Y^\bullet) & \\
 & & \downarrow & & \downarrow & \\
 & & \text{Tot}(\Phi^{C_r} \text{sd}_r Z^\bullet) & \longrightarrow & \rho_r^* \text{Tot}(\Phi^{C_r} \text{sd}_r Y^\bullet) & \\
 & & \cong \uparrow \Delta & & \downarrow F(\Delta, c_r) \circ \bar{\alpha} & \\
 \Sigma_+^\infty LX & \xrightarrow{\cong} & \text{Tot}(Z^\bullet) & \longrightarrow & \text{Tot}(Y^\bullet) &
 \end{array}$$

We start with the left-hand rectangle of (3.3), where everything is a suspension spectrum

and so all maps are completely determined by what they do at spectrum level 0:

$$\begin{array}{ccc}
 LX^{C_r} & \xrightarrow{\cong} & \mathrm{Tot}(Z^\bullet)^{C_r} \\
 \downarrow \cong & & \downarrow \cong \\
 & & \mathrm{Tot}(\mathrm{sd}_r Z^\bullet)^{C_r} \\
 & & \downarrow \\
 & & \mathrm{Tot}((\mathrm{sd}_r Z^\bullet)^{C_r}) \\
 & & \uparrow \cong \\
 LX & \xrightarrow{\cong} & \mathrm{Tot}(Z^\bullet)
 \end{array}$$

Now the horizontal homeomorphisms may be computed by observing that  $\Lambda[0]_k = \mathbf{\Lambda}([k], [0])$  has  $(k+1)$  points  $f_0, \dots, f_k$ , where  $f_i : \mathbb{Z} \rightarrow \mathbb{Z}$  sends 0 through  $i-1$  to 0 and  $i$  through  $k$  to 1 (or if  $i = 0$  it sends 0 through  $k$  to 0). Using our choice of homeomorphism  $|\Lambda[0]| \cong \mathbb{R}/\mathbb{Z}$  from the previous section, the  $k$ -simplex given by  $f_i$  maps down to the circle  $\mathbb{R}/\mathbb{Z}$  by the formula

$$(t_0, \dots, t_k) \mapsto (t_i + \dots + t_k) \sim (1 - (t_0 + \dots + t_{i-1}))$$

Negating the circle and reparametrizing  $\Delta^k \subset \mathbb{R}^k$  as points  $(x_1, \dots, x_k)$  for which  $0 \leq x_1 \leq x_2 \leq \dots \leq x_k \leq 1$  according to the rule  $x_i = t_0 + \dots + t_{i-1}$ , we arrive at the simple rule

$$(f_i, x_1, \dots, x_k) \mapsto x_i, \quad x_0 := 0$$

So now the homeomorphism  $LX \cong \mathrm{Tot}(X^{\bullet+1})$  can be expressed by the formula

$$\Delta^{k-1} \times LX \longrightarrow X^k$$

$$(r_1, \dots, r_{k-1}, \gamma) \mapsto (\gamma(0), \gamma(r_1), \dots, \gamma(r_{k-1}))$$

which is the formula in [CJ02].

Under this change of coordinates, we calculate the map along each branch to be

$$\gamma(-) \longrightarrow (r_1, \dots, r_{k-1}) \mapsto (\gamma(0), \gamma(\frac{1}{r}r_1), \gamma(\frac{1}{r}r_2), \dots, \gamma(\frac{1}{r}r_{k-1}), \gamma(0), \gamma(\frac{1}{r}r_1), \dots)$$

and so the square commutes.

Returning to (3.3), the top and middle squares of the right-hand row automatically commute by the naturality of the cosimplicial diagonal and the interchange map with geometric fixed points. The final square is then

$$\begin{array}{ccc} \mathrm{Tot}(\Phi^{C_r} \mathrm{sd}_r Z^\bullet) & \longrightarrow & \rho_r^* \mathrm{Tot}(\Phi^{C_r} \mathrm{sd}_r Y^\bullet) \\ \cong \uparrow \Delta & & \downarrow F(\Delta, c_r) \circ \bar{\alpha} \\ \mathrm{Tot}(Z^\bullet) & \longrightarrow & \mathrm{Tot}(Y^\bullet) \end{array}$$

The map  $\Delta$  is the cocyclic map

$$\Phi^{C_r} \Sigma_+^\infty X^{rk} \xleftarrow{\cong} \Sigma_+^\infty X^k$$

given by the Hill-Hopkins-Ravenel diagonal; this is almost tautologically cosimplicial. The map  $F(\Delta, c_r) \circ \bar{\alpha}$  is also cocyclic, so to check that this square commutes it suffices to check level  $k - 1$ . This boils down to this rectangle:

$$\begin{array}{ccc} \Phi^{C_r} \Sigma_+^\infty X^{rk} \wedge \Phi^{C_r} (cDX_+)^{\wedge rk} & \xrightarrow{\alpha} & \Phi^{C_r} (\Sigma_+^\infty X^{rk} \wedge (cDX_+)^{\wedge rk}) & \longrightarrow & \Phi^{C_r} \mathbb{S} \\ \uparrow \Delta \wedge \Delta & \nearrow \Delta & & & \downarrow \cong \\ \Sigma_+^\infty X^k \wedge (cDX_+)^{\wedge k} & \longrightarrow & & & \mathbb{S} \end{array}$$

The top triangle commutes because the norm diagonal commutes with smash products. The trapezoid commutes because the inverse of the right-hand isomorphism is the norm diagonal on  $\mathbb{S}$  (in fact there is only one isomorphism  $\mathbb{S} \rightarrow \mathbb{S}$ ), and the norm diagonal is natural. We have finished the proof that  $\mathrm{Tot}(Z^\bullet) \rightarrow \mathrm{Tot}(Y^\bullet)$  is a map of pre-cyclotomic spectra.

In total we now have a map of pre-cyclotomic spectra  $\Sigma_+^\infty LX \rightarrow \mathrm{Tot}(Y^\bullet)$ . Next we check that for each  $r$  the structure map of  $\mathrm{Tot}(Y^\bullet) = F(|N_{\bullet}^{\mathrm{cyc}} cDX_+|, f\mathbb{S})$  is nonequivariantly an equivalence when  $\Phi^{C_r}$  is derived. To be precise, we forget every circle action and just remember the cosimplicial  $C_r$ -action on  $\mathrm{sd}_r Y^\bullet$ , making it a cosimplicial  $C_r$ -spectrum.

Then our structure maps respect the restriction to the  $k$ -skeleton for each  $k \geq 0$ :

$$\begin{array}{ccccc}
 \Phi^{C_r} F(|\mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} cDX_+|, f\mathbb{S}) & \longrightarrow & F(\Phi^{C_r} |\mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} cDX_+|, \Phi^{C_r} f\mathbb{S}) & \longrightarrow & F(|N_{\bullet}^{\mathrm{cyc}} cDX_+|, f\mathbb{S}) \\
 \downarrow & & \downarrow & & \downarrow \\
 \Phi^{C_r} F(|\mathrm{Sk}_k \mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} cDX_+|, f\mathbb{S}) & \rightarrow & F(\Phi^{C_r} |\mathrm{Sk}_k \mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} cDX_+|, \Phi^{C_r} f\mathbb{S}) & \rightarrow & F(|\mathrm{Sk}_k N_{\bullet}^{\mathrm{cyc}} cDX_+|, f\mathbb{S})
 \end{array} \tag{3.4}$$

We prove that the bottom horizontal composite is an equivalence when the left-hand  $\Phi^{C_r}$  is derived, by induction on  $k$ . Since  $\Phi^{C_r}$  preserves fiber sequences, to power the induction it suffices to show that on the fiber of the map from the  $k$ -skeleton into the  $(k-1)$ -skeleton this structure map is an equivalence. On the right-hand term this is the dual of

$$|\mathrm{Sk}_k N_{\bullet}^{\mathrm{cyc}} cDX_+| / |\mathrm{Sk}_{k-1} N_{\bullet}^{\mathrm{cyc}} cDX_+| \cong \Delta^k / \partial \Delta^k \wedge cDX^{\wedge k} \wedge cDX_+$$

because the cofiber of the latching map

$$(\mathbb{S} \longrightarrow cDX_+)^{\square k} \square (* \longrightarrow cDX_+)$$

is the smash product of the cofibers

$$cDX^{\wedge k} \wedge cDX_+$$

To compute the left-hand terms we use the  $C_r$ -equivariant cofiber sequence

$$\mathbb{S} \longrightarrow (cDX_+)^{\wedge r} \longrightarrow \bigvee \binom{r}{1} cDX \vee \bigvee \binom{r}{2} cDX^{\wedge 2} \vee \dots \vee cDX^{\wedge r}$$

to show that the cofiber of the latching map

$$(\mathbb{S} \longrightarrow (cDX_+)^{\wedge r})^{\square k} \square (* \longrightarrow (cDX_+)^{\wedge r})$$

is isomorphic to

$$\left( \bigvee \binom{r}{1} cDX \vee \bigvee \binom{r}{2} cDX^{\wedge 2} \vee \dots \vee cDX^{\wedge r} \right)^{\wedge k} \wedge (cDX_+)^{\wedge r}$$

Therefore the left-hand fiber is the dual of

$$\begin{aligned} & |\mathrm{Sk}_k \mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} cDX_+| / |\mathrm{Sk}_{k-1} \mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} cDX_+| \\ & \cong \Delta^k / \partial \Delta^k \wedge \left( \bigvee \binom{r}{1} cDX \vee \bigvee \binom{r}{2} cDX^{\wedge 2} \vee \dots \vee cDX^{\wedge r} \right)^{\wedge k} \wedge (cDX_+)^{\wedge r} \end{aligned}$$

When  $i < r$  the term  $\bigvee \binom{r}{i} (cDX)^{\wedge i}$  has  $C_r$ -action that acts transitively on the indexing set of the wedge, so the geometric fixed points  $\Phi^{C_r}$  are trivial. On the last term  $i = r$  the norm isomorphism tells us that the geometric fixed points are  $cDX$ . Since  $\Phi^{C_r}$  commutes with wedge sums and smash products, we conclude that the diagonal map induces an isomorphism

$$cDX^{\wedge k} \wedge cDX_+ \xrightarrow{\Delta} \Phi^{C_r} |\mathrm{Sk}_k \mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} cDX_+| / |\mathrm{Sk}_{k-1} \mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} cDX_+|$$

Abbreviating  $|\mathrm{Sk}_k \mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} cDX_+|$  as  $T_k$ , the fiber of maps between levels becomes

$$\begin{aligned} \Phi^{C_r} cF(\Delta^k / \partial \Delta^k \wedge T_k / T_{k-1}, f\mathbb{S}) & \longrightarrow F(\Delta^k / \partial \Delta^k \wedge \Phi^{C_r} T_k / T_{k-1}, \Phi^{C_r} f\mathbb{S}) \\ & \xrightarrow{\cong} F(\Delta^k / \partial \Delta^k \wedge cDX^{\wedge k} \wedge cDX_+, \Phi^{C_r} f\mathbb{S}) \\ & \longrightarrow F(\Delta^k / \partial \Delta^k \wedge cDX^{\wedge k} \wedge cDX_+, f\mathbb{S}) \end{aligned}$$

It is not quite obvious that this composite is an equivalence, because  $\Phi^{C_r} f\mathbb{S}$  is not a fibrant orthogonal spectrum. However since we are not using the ring structure on  $cDX_+$  here, we may replace the reduced ring  $cDX$  by a spectrum with finitely many cells, and apply the above maps. Then the replaced composite is an equivalence. But the replacement is equivalent to the above composite on the first and last terms, and so the above composite must be an equivalence too. This powers the desired induction, and we conclude that the bottom row of (3.4) is an equivalence for all  $k \geq 0$ .

We shorten (3.4) to

$$\begin{array}{ccc} \Phi^{C_r} F(|\mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} cDX_+|, f\mathbb{S}) & \longrightarrow & F(|N_{\bullet}^{\mathrm{cyc}} cDX_+|, f\mathbb{S}) \\ \downarrow & & \downarrow \\ \Phi^{C_r} F(|\mathrm{Sk}_k \mathrm{sd}_r N_{\bullet}^{\mathrm{cyc}} cDX_+|, f\mathbb{S}) & \longrightarrow & F(|\mathrm{Sk}_k N_{\bullet}^{\mathrm{cyc}} cDX_+|, f\mathbb{S}) \end{array}$$

The bottom horizontal map is an equivalence for all  $k \geq 0$ , but  $\Phi^{C_r}$  does not commute with homotopy inverse limits, so we cannot immediately conclude that the top horizontal is an equivalence. However the right vertical is an isomorphism on  $\pi_{\leq k}$  because the fibers of maps between higher levels are all at least  $(k+1)$ -connected. (Since  $X$  is 1-connected  $X^{\wedge k}$  is  $(2k-1)$ -connected, so  $\Omega^{k-1}X^{\wedge k}$  is  $k$ -connected.) Similarly,  $X^{\wedge rk}$  can be given a  $C_r$ -equivariant cell structure in which the lowest-dimensional cells are the diagonal ones, of dimension at least  $2k$ . Then  $\Phi^{C_r}X^{\wedge rk}$  still has dimension at least  $2k$  and so is  $(2k-1)$ -connected. Therefore  $\Omega^{k-1}\Phi^{C_r}X^{\wedge rk}$  is  $k$ -connected and by the same reasoning as before the left vertical is an isomorphism on  $\pi_{\leq k}$ . So we conclude that the top horizontal map is an isomorphism on  $\pi_{\leq k}$ . But this is true for all  $k \geq 0$  so the top map is an equivalence, proving that  $\text{Tot}(Y^\bullet) = F(|N_{\bullet}^{\text{cyc}}cDX_+|, f\mathbb{S})$  is cyclotomic.

Finally,  $\text{Tot}(Z^\bullet) \rightarrow \text{Tot}(Y^\bullet)$  known to be a weak equivalence nonequivariantly. One could check it by noting that  $Y^\bullet$  is Reedy fibrant, so it suffices to show that the derived totalization of  $Z^\bullet$  converges to the ordinary totalization. In fact this rearranges to the observation that the ‘‘Taylor tower’’ of the functor  $F(X) = \Sigma_+^\infty LX$  converges to  $F$ , which is proven for instance in ([Goo91], 4.4). At any rate, any map of cyclotomic spectra which is a weak equivalence on the underlying spectra is automatically an  $\mathcal{F}$ -equivalence of  $S^1$  spectra and so we are done.  $\square$

### 3.5 A stable splitting of $THH(D_+\Sigma X)$

#### 3.5.1 Proof of the splitting

We continue to assume that  $X$  is a finite based CW complex. Now the levelwise fiber sequence

$$F(X, \mathbb{S}) \longrightarrow F(X_+, \mathbb{S}) \longrightarrow \mathbb{S}$$

preserves the multiplications coming from the diagonal maps on both  $X$  and  $X_+$ , so we can conclude that there is an equivalence of ring spectra

$$\mathbb{S} \vee F(X, \mathbb{S}) \xrightarrow{\sim} F(X_+, \mathbb{S})$$

where on the left the  $\mathbb{S}$  is the unit and the multiplication on  $F(X, \mathbb{S})$  is the dual of the diagonal

$$X \longrightarrow X \wedge X$$

Let  $cDX$  denote cofibrant replacement of  $F(X, \mathbb{S})$  as a unitless ring, so that  $\mathbb{S} \vee cDX$  is a ring spectrum for which the unit map is a cofibration. Therefore there is an equivalence of ring spectra

$$\mathbb{S} \vee cD(\Sigma X) \xrightarrow{\sim} D(\Sigma X)_+$$

The ring spectrum on the left is the free ring spectrum on the unitless ring  $E = cD(\Sigma X)$ , which has multiplication given by the diagonal map

$$\Sigma X \longrightarrow \Sigma X \wedge \Sigma X$$

We wish to show that this is equivalent by a zig-zag of unitless rings to the same ring  $E$  but with 0 as the multiplication map. First, we construct an  $A_\infty$  operad whose  $n$ th space is

$$(\mathbb{R}_{\geq 0})^{n-1}$$

The point  $(t_1, \dots, t_{n-1})$  maps  $E^{\wedge n} \longrightarrow E$  by dualizing the map

$$\begin{aligned} S^1 \wedge X &\longrightarrow S^n \wedge X^{\wedge n} \\ (s, x) &\mapsto (s + t_1, s + t_2, \dots, s + t_{n-1}, s, x, x, \dots, x) \end{aligned}$$

Here we think of  $S^n$  as  $\mathbb{R}^n$  modulo the complement of the open set  $(0, 1)^n$ . The composition comes from adding the  $t_i$  together. This operad contains within it the associative operad (all  $t_i$  equal to 0), giving the original multiplication on  $E$ . It also contains a large  $A_\infty$  suboperad (all  $t_i \geq 1$ ) giving only the 0 multiplication. Using the monadic bar construction and these changes of operad, we can build a zig-zag equivalence of ring spectra  $\mathbb{S} \vee E \simeq \mathbb{S} \vee E$  between the multiplication we started with and the 0 multiplication.

Now that we've made the multiplication zero, we calculate  $THH(\mathbb{S} \vee cD(\Sigma X))$ . After we mod out the degeneracies, all that is left at simplicial level  $k$  is

$$(cD(\Sigma X))^{\wedge(k+1)} \vee (\mathbb{S} \wedge (cD(\Sigma X))^{\wedge k})$$

The second term is in the image of the extra degeneracy map, so it lies in the  $S^1$ -orbit of the first term from one level down. Therefore we get

$$THH(\mathbb{S} \vee cD(\Sigma X)) \cong \mathbb{S} \vee \bigvee_{n=1}^{\infty} |\Lambda[n-1]|/\partial|\Lambda[n-1]| \wedge_{C_n} (cD(\Sigma X))^{\wedge n}$$

Using the homeomorphism  $|\Lambda[n-1]| \cong S^1 \times \Delta^{n-1}$  for which the  $C_n$ -action rotates the vertices and decreases the  $S^1$ -coordinate by  $1/n$ , we get

$$\begin{aligned} THH(\mathbb{S} \vee cD(\Sigma X)) &\cong \mathbb{S} \vee \bigvee_{n=1}^{\infty} ((\Delta^{n-1}/\partial\Delta^{n-1}) \wedge S^1_+) \wedge_{C_n} (cD(\Sigma X))^{\wedge n} \\ &\cong \mathbb{S} \vee \bigvee_{n=1}^{\infty} (S^{\bar{\rho}}_{C_n} \wedge S^1_+) \wedge_{C_n} (cD(\Sigma X))^{\wedge n} \end{aligned}$$

where  $\bar{\rho}_{C_n}$  denotes the reduced regular representation. We can further simplify using the equivariant equivalence

$$D(\Sigma X)^{\wedge n} \wedge S^{\bar{\rho}}_{C_n} \xrightarrow{\sim} D(\Sigma(X^{\wedge n}))$$

which then gives

$$THH(D_+\Sigma X) \simeq \mathbb{S} \vee \Sigma^{-1} \left( \bigvee_{n=1}^{\infty} D(X^{\wedge n}) \wedge_{C_n} S^1_+ \right)$$

When written in this last form, the  $S^1$  action is the obvious one.

### 3.5.2 Dualization and examples

Now we dualize:

$$\begin{aligned}
D(\mathrm{THH}(D_+\Sigma X)) &\simeq \mathbb{S} \times \Sigma \left( \prod_{n=1}^{\infty} D(D(X^{\wedge n}) \wedge_{C_n} S_+^1) \right) \\
&\simeq \mathbb{S} \vee \Sigma \left( \prod_{n=1}^{\infty} D[(D(X^{\wedge n}) \wedge S_+^1)_{hC_n}] \right) \\
&\simeq \mathbb{S} \vee \Sigma \left( \prod_{n=1}^{\infty} D[D(X^{\wedge n}) \wedge S_+^1]^{hC_n} \right) \\
&\simeq \mathbb{S} \vee \Sigma \left( \prod_{n=1}^{\infty} D[D(X^{\wedge n}) \wedge S_+^1]^{C_n} \right) \\
&\simeq \mathbb{S} \vee \Sigma \left( \prod_{n=1}^{\infty} F^{C_n}(D(X^{\wedge n}) \wedge S_+^1, \mathbb{S}) \right) \\
&\simeq \mathbb{S} \vee \Sigma \left( \prod_{n=1}^{\infty} F^{C_n}(S_+^1, D(D(X^{\wedge n}))) \right)
\end{aligned}$$

Now assume that  $X$  is finite CW and connected (but not necessarily simply-connected):

$$\begin{aligned}
D(\mathrm{THH}(D_+\Sigma X)) &\simeq \mathbb{S} \vee \Sigma \left( \prod_{n=1}^{\infty} F^{C_n}(S_+^1, D(D(X^{\wedge n}))) \right) \\
&\simeq \mathbb{S} \vee \Sigma \left( \prod_{n=1}^{\infty} F^{C_n}(S_+^1, X^{\wedge n}) \right) \\
&\simeq \mathbb{S} \vee \Sigma \left( \prod_{n=1}^{\infty} \Sigma^{-TS^1} S_+^1 \wedge_{C_n} X^{\wedge n} \right) \\
&\simeq \mathbb{S} \vee \bigvee_{n=1}^{\infty} \Sigma^{\infty} S_+^1 \wedge_{C_n} X^{\wedge n}
\end{aligned}$$

Using the previous section we recover the splitting of the free loop space of a suspension

$$\Sigma_+^{\infty}(L\Sigma X) \simeq \bigvee_{n=1}^{\infty} \Sigma^{\infty} S_+^1 \wedge_{C_n} X^{\wedge n}$$

found in [Coh87].

We end with a few specific choices of  $X$  for added concreteness. The simplest is  $X = S^0$ , which gives

$$THH(D(S_+^1)) \simeq \mathbb{S} \vee \Sigma^{-1} \bigvee_{n=1}^{\infty} \Sigma_+^{\infty}(S^1/C_n)$$

The reader may find it illuminating to rewrite the  $n$ th term in this sum as

$$\Sigma_+^{-1} S^1/C_n \cong \Sigma_+^{-T(S^1/C_n)} S^1/C_n$$

because this is the dual in the  $S^1$ -equivariant stable category of  $\Sigma_+^{\infty} S^1/C_n$ .

When  $\Sigma X = S^3$  we get

$$THH(D(S_+^3)) \simeq \mathbb{S} \vee (\Omega S^{-2} \wedge S_+^1) \vee (\Omega S^{-4} \wedge_{C_2} S_+^1) \vee (\Omega S^{-6} \wedge_{C_3} S_+^1) \vee \dots$$

Nonequivariantly, this may be rewritten

$$\mathbb{S} \vee (S^{-3} \wedge S_+^1) \vee (S^{-5} \wedge S_+^1) \vee (S^{-7} \wedge S_+^1) \vee \dots \simeq \mathbb{S} \vee \bigvee_{k=1}^{\infty} (S^{-2k-1} \wedge S_+^1)$$

and when  $\Sigma X = S^n$  with  $n$  odd we get

$$\begin{aligned} THH(D(S_+^n)) &\simeq \mathbb{S} \vee (S^{-n} \wedge S_+^1) \vee (S^{-2n+1} \wedge S_+^1) \vee (S^{-3n+2} \wedge S_+^1) \vee \dots \\ &\simeq \mathbb{S} \vee \bigvee_{k=1}^{\infty} (S^{-kn+k-1} \wedge S_+^1) \end{aligned}$$

The dual of this is therefore an infinite wedge of spheres. Indeed this agrees with

$$\Sigma_+^{\infty} LS^n \simeq \Sigma_+^{\infty} S^n \vee \Sigma_+^{\infty} \Omega S^n$$

using Snaith splitting for  $\Sigma^{\infty} \Omega S^n$ !

For even spheres  $\Sigma X = S^{2n}$  we instead get

$$\begin{aligned} THH(D(S_+^{2n})) &\simeq \mathbb{S} \vee (S^{-2n} \wedge S_+^1) \vee (S^{-4n} \wedge \mathbb{R}P_+^2) \vee (S^{-6n+2} \wedge S_+^1) \vee (S^{-8n+2} \wedge \mathbb{R}P_+^2) \vee \dots \\ &\simeq \mathbb{S} \vee \bigvee_{k=1}^{\infty} (S^{-(4k-2)n+2(k-1)} \wedge S_+^1) \vee \bigvee_{k=1}^{\infty} (S^{-4kn+2(k-1)} \wedge \mathbb{R}P_+^2) \end{aligned}$$

### 3.5.3 Multiplicative structure

The above splitting for  $THH(D(\Sigma X)_+)$  implies

**Theorem 3.5.1.** *The Bökstedt spectral sequence for  $THH(D(\Sigma X)_+)$  collapses, resulting in the  $E^\infty$ -page*

$$E_{p,q}^\infty \cong E_{p,q}^2 \cong HH_p(C_*(D(\Sigma X)_+))_{\deg q} \cong HH_p(C^*(\Sigma X_+))_{\deg(-q)}$$

This is useful for computing the multiplicative structure on  $THH(D(\Sigma X)_+)$ , which is guaranteed to be a commutative ring spectrum. It gives for instance

$$\pi_*(THH(D_+S^1)) \cong \pi_*(\mathbb{S})[\alpha, \beta]/(\beta^2 = 0)$$

with commutativity in both the ordinary and graded senses, because  $|\alpha_i| = 0$  and  $|\beta| = -1$ .

If  $n$  is even,

$$\pi_*(THH(D_+S^{n+1})) \cong \pi_*(\mathbb{S})[\alpha, \beta]/(\beta^2 = 0)$$

where  $|\alpha_i| = -in$  and  $|\beta| = -n - 1$ .

### 3.6 The calculation of $TC(DS_+^1)$ and its linear approximation

#### 3.6.1 $TC(DS_+^1)$

Now we move on to TC, which is a homotopy limit of genuine fixed point spectra

$$THH^{C_{p^m}}$$

over the restriction and Frobenius maps. We will start with the specific example where the ring spectrum is  $D_+S^1$  and try to see how to generalize from there. From the last section,

$$THH(D_+S^1) \simeq \mathbb{S} \vee \Omega \left( \bigvee_{n=1}^{\infty} \Sigma_+^{\infty} S_{C_n}^1 \right)$$

and we want the homotopy limit over  $F$  and  $R$ . The first sphere factor splits off as a cyclotomic spectrum, giving  $TC(\mathbb{S})$ , which has already been calculated. So we ignore it. We may also ignore the  $\Omega$  because it commutes with  $F$ ,  $R$ , and holims. So our next move is to calculate the genuine  $C_{p^m}$ -fixed points of the  $n$ th summand using tom Dieck splitting:

$$(\Sigma_+^{\infty} S^1)^{C_{p^m}} \simeq \Sigma_+^{\infty} \left( \prod_{0 \leq i \leq m} EC_{p^{m-i}} \times_{C_{p^{m-i}}} (S_{C_n}^1)^{C_{p^i}} \right)$$

Keeping in mind that  $S_{C_n}^1$  is homeomorphic to  $S^1$  with the  $S^1$  action that rotates around  $n$  times, some of the terms disappear and we get

$$\Sigma_+^{\infty} \left( \prod_{0 \leq i \leq m, p^i | n} EC_{p^{m-i}} \times_{C_{p^{m-i}}} S_{C_n}^1 \right)$$

Let  $k$  be the largest integer such that  $p^k | n$ . Then the  $i$ th summand changes form depending on  $k$ :

$$\begin{aligned} k < i & \quad \emptyset \\ k = i & \quad S_{C_{p^{m-i_n}}}^1 \\ i \leq k \leq m & \quad BC_{p^{k-i}} \times S_{C_{p^{m-k_n}}}^1 \\ k \geq m & \quad BC_{p^{m-i}} \times S_{C_n}^1 \end{aligned}$$

These calculations were done using the following

**Lemma 3.6.1.** *Let  $G$  be a finite group, and let  $X$  be an unbased  $G$ -CW complex. Let  $H \leq G$  be the largest subgroup that fixes all of  $X$ , necessarily normal. If  $G/H$  acts freely on  $X$  then*

$$X_{hG} \cong BH \times X_{G/H}$$

*Proof.*

$$\begin{aligned} X_{hG} &:= (EG \times X)_G \\ &\cong ((EG)/H \times X)_{G/H} \\ &\simeq (E(G/H) \times BH \times X)_{G/H} \\ &\simeq BH \times X_{G/H} \end{aligned}$$

□

Now we have a complete description of  $(\Sigma_+^\infty S^1)^{C_{p^m}}$ . Our next task is to compute the Frobenius maps

$$(\Sigma_+^\infty S^1)^{C_{p^m}} \longrightarrow (\Sigma_+^\infty S^1)^{C_{p^{m-1}}}$$

and the inverse limit over all of them. Since  $m$  is going to infinity, we would like to simplify our analysis by always assuming that  $i$  and  $k$  are smaller than  $m$ . Unfortunately, this involves pruning stuff off from infinitely many terms in the holim system, which is not allowed. Still, most instances of the Frobenius map fall into that case:

**Lemma 3.6.2.** *On a typical summand the Frobenius map*

$$\Sigma_+^\infty BC_{p^{k-i}} \times S_{C_{p^{m-k_n}}}^1 \longrightarrow \Sigma_+^\infty BC_{p^{k-i}} \times S_{C_{p^{(m-1)-k_n}}}^1$$

*is simply the transfer*

$$\Sigma_+^\infty S_{C_{p^{m-k_n}}}^1 \longrightarrow \Sigma_+^\infty S_{C_{p^{(m-1)-k_n}}}^1$$

*on the right-hand factor smashed with the identity on the left-hand factor.*

*Proof.* If  $G$  is abelian and  $K \leq H \leq G$  then consider the square

$$\begin{array}{ccc} \pi_*^{G/K}(\Sigma_+^\infty E(G/K) \times X^K) & \xrightarrow{tD} & \pi_*^G(\Sigma_+^\infty X) \\ \downarrow \text{res} & & \downarrow \text{res} \\ \pi_*^{H/K}(\Sigma_+^\infty E(G/K) \times X^K) & \xrightarrow{tD} & \pi_*^H(\Sigma_+^\infty X) \end{array}$$

The maps labeled  $tD$  are components in the tom Dieck splitting, defined in Schwede’s notes. This square commutes, essentially because the definition of  $tD$  involves only restriction and maps of spectra, and the restriction maps are clearly natural. The last thing to check is that our way of identifying the left-hand side with homotopy groups of  $E(G/K) \times_{G/K} X^K$  agrees with the transfer:

$$\begin{array}{ccc} \pi_*^{\{e\}}(\Sigma_+^\infty E(G/K) \times_{G/K} X^K) & \xrightarrow[\sim]{tr} & \pi_*^{G/K}(\Sigma_+^\infty E(G/K) \times X^K) \\ \downarrow tr & & \downarrow res \\ \pi_*^{\{e\}}(\Sigma_+^\infty E(G/K) \times_{H/K} X^K) & \xrightarrow[\sim]{tr} & \pi_*^{H/K}(\Sigma_+^\infty E(G/K) \times X^K) \end{array}$$

WLOG the subgroup  $K$  is trivial:

$$\begin{array}{ccc} \pi_*^{\{e\}}(\Sigma_+^\infty EG \times_G X) & \xrightarrow[\sim]{tr} & \pi_*^G(\Sigma_+^\infty EG \times X) \\ \downarrow tr & & \downarrow res \\ \pi_*^{\{e\}}(\Sigma_+^\infty EG \times_H X) & \xrightarrow[\sim]{tr} & \pi_*^H(\Sigma_+^\infty EG \times X) \end{array}$$

This follows from [Mad95], equation (4.1.6). □

There is one more interesting case:

**Lemma 3.6.3.** *On the summands where  $m = i$  the Frobenius map*

$$\Sigma_+^\infty S_{C_p^m}^1 \longrightarrow \Sigma_+^\infty S_{C_p^m}^1$$

*is an equivalence to the summand  $(m - 1, i - 1)$ .*

*Proof.* Anytime the cyclotomic spectrum  $T$  is a suspension spectrum  $\Sigma_+^\infty X$ , the Frobenius map on the geometric fixed points summand of  $THH^{C_p^m}$  is always

$$\Sigma_+^\infty X \simeq \Sigma_+^\infty (X^{C_p}) \hookrightarrow \Sigma_+^\infty X$$

In the classical case  $\Sigma_+^\infty LX$  this resulted in a  $p$ -fold power map, but here the map is simply an equivalence to another summand! This isn’t totally crazy since the  $S^1/C_p$ -action is still shifted into an  $S^1$ -action that winds around  $p$  times too fast. □

On the next page we give a table in which the entries are arranged so that the Frobenius

maps (not drawn) go vertically up the columns. Entries in grey have  $k > m$ . They would be irrelevant to TC, if not for the fact that there are infinitely many and they go infinitely far down. On just about every summand  $m > i$  and the Frobenius map goes up and is a transfer. However, when  $m = i$  we get a different Frobenius map which is an equivalence onto the summand with  $m$  and  $i$  each decreased by one. One may use the given indices to check this table against

$$\begin{array}{ll}
 k < i & \emptyset \\
 k = i & S_{C_{p^{m-i}n}}^1 \\
 i \leq k \leq m & BC_{p^{k-i}} \times S_{C_{p^{m-k}n}}^1 \\
 k \geq m & BC_{p^{m-i}} \times S_{C_n}^1
 \end{array}$$

summand	1			2			...			$p-1$			$p$			...			$p^2$		
	$k=0$			$k=0$			$k=0$			$k=0$			$k=1$			$k=2$					
	$i=0$			$i=0$			$i=0$			$i=0$			$i=1$			$i=0$					
$m=0$	$\Sigma_+^\infty S^1$	$\Sigma_+^\infty S_{C_2}^1$	...	$\Sigma_+^\infty S_{C_{p-1}}^1$	–	$\Sigma_+^\infty S_{C_p}^1$	...	–	–	$\Sigma_+^\infty S_{C_{p^2}}^1$	...	–	–	$\Sigma_+^\infty S_{C_{p^2}}^1$	...	–	–	$\Sigma_+^\infty S_{C_{p^2}}^1$			
$m=1$	$\Sigma_+^\infty S_{C_p}^1$	$\Sigma_+^\infty S_{C_{2p}}^1$	...	$\Sigma_+^\infty S_{C_{(p-1)p}}^1$	$\Sigma_+^\infty S_{C_p}^1$	$\vee$	$\Sigma_+^\infty BC_p \times S_{C_p}^1$	...	–	$\Sigma_+^\infty S_{C_{p^2}}^1$	$\vee$	$\Sigma_+^\infty BC_p \times S_{C_p}^1$	...	$\Sigma_+^\infty S_{C_{p^2}}^1$	$\vee$	$\Sigma_+^\infty BC_p \times S_{C_p}^1$	$\vee$	$\Sigma_+^\infty BC_p \times S_{C_{p^2}}^1$			
$m=2$	$\Sigma_+^\infty S_{C_{p^2}}^1$	$\Sigma_+^\infty S_{C_{2p^2}}^1$	...	$\Sigma_+^\infty S_{C_{(p-1)p^2}}^1$	$\Sigma_+^\infty S_{C_{p^2}}^1$	$\vee$	$\Sigma_+^\infty BC_p \times S_{C_{p^2}}^1$	...	$\Sigma_+^\infty S_{C_{p^2}}^1$	$\vee$	$\Sigma_+^\infty BC_p \times S_{C_{p^2}}^1$	$\vee$	$\Sigma_+^\infty S_{C_{p^2}}^1$	$\vee$	$\Sigma_+^\infty BC_p \times S_{C_{p^2}}^1$	$\vee$	$\Sigma_+^\infty BC_{p^2} \times S_{C_{p^2}}^1$	$\vee$	$\Sigma_+^\infty BC_{p^2} \times S_{C_{p^2}}^1$		
$m=3$	$\Sigma_+^\infty S_{C_{p^3}}^1$	$\Sigma_+^\infty S_{C_{2p^3}}^1$	...	$\Sigma_+^\infty S_{C_{(p-1)p^3}}^1$	$\Sigma_+^\infty S_{C_{p^3}}^1$	$\vee$	$\Sigma_+^\infty BC_p \times S_{C_{p^3}}^1$	...	$\Sigma_+^\infty S_{C_{p^3}}^1$	$\vee$	$\Sigma_+^\infty BC_p \times S_{C_{p^3}}^1$	$\vee$	$\Sigma_+^\infty S_{C_{p^3}}^1$	$\vee$	$\Sigma_+^\infty BC_p \times S_{C_{p^3}}^1$	$\vee$	$\Sigma_+^\infty BC_{p^2} \times S_{C_{p^3}}^1$	$\vee$	$\Sigma_+^\infty BC_{p^2} \times S_{C_{p^3}}^1$		
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$m=\infty$	$\Sigma^\infty \Sigma_+*$	$\Sigma^\infty \Sigma_+ BC_2$	...	$\Sigma^\infty \Sigma_+ BC_{p-1}$	$\Sigma^\infty \Sigma_+*$	$\vee$	$\Sigma^\infty \Sigma_+ BC_p$	...	$\Sigma^\infty \Sigma_+*$	$\vee$	$\Sigma^\infty \Sigma_+ BC_p$	$\vee$	$\Sigma^\infty \Sigma_+*$	$\vee$	$\Sigma^\infty \Sigma_+ BC_p$	$\vee$	$\Sigma^\infty \Sigma_+ BC_{p^2}$	$\vee$	$\Sigma^\infty \Sigma_+ BC_{p^2}$		

Table 3.1: The splitting of  $THH(D_+ S^1)^{C_{p^n}}$ .

It looks like we'll soon encounter inverse limits of  $C_p$ -transfer maps, so we recall [Mad95], Lemma 4.4.9 (an extension of [BHM93], Lemma 5.15):

**Lemma 3.6.4.** *For any  $S^1$ -spectrum  $T$ , the  $S^1$ -transfer induces a  $p$ -adic equivalence*

$$\Sigma T_{hS^1} \longrightarrow \operatorname{holim}_{\leftarrow} T_{hC_{p^n}}$$

Now we will discuss in detail two approaches to this calculation. The first one follows [BHM93]. When we look at the holim system for TC of  $\Sigma_+^\infty LX$ ,

$$\begin{array}{ccccccc}
 LX & & & & & & \\
 \uparrow \Delta_p & \swarrow \text{tr} & & & & & \\
 LX^{C_p} & & LX_{hC_p} & & & & \\
 \uparrow \Delta_p & \swarrow \text{tr} & \swarrow \text{tr} & & & & \\
 LX^{C_{p^2}} & & (LX^{C_p})_{hC_p} & & LX_{hC_{p^2}} & & \\
 \uparrow \Delta_p & \swarrow \text{tr} & \swarrow \text{tr} & \swarrow \text{tr} & & & \\
 LX^{C_{p^3}} & & (LX^{C_{p^2}})_{hC_p} & & (LX^{C_p})_{hC_{p^2}} & & LX_{hC_{p^3}} \\
 \vdots & & \vdots & & \vdots & & \vdots
 \end{array}$$

it becomes natural to separate out the behavior in the first column from the rest. This is easiest to do if we calculate  $TR$  (the inverse limit in the vertical direction). Observe that  $R$  splits, so

$$TR(D_+S^1) \simeq \prod_{i=0}^{\infty} THH(D_+S^1)_{hC_{p^i}}$$

That was easy! Next we need the homotopy fiber of  $F - \text{id}$ . Using the above lemmas, we know these maps as well.  $F$  shifts from level  $i$  in the product to  $i - 1$ , but it sends the 0th factor to itself. To correct this weirdness, we separate out the 0th factor:

$$\begin{array}{ccccc}
 \bigvee_{n=1}^{\infty} \Sigma_+^\infty S_{C_n}^1 & \longrightarrow & \prod_{i=0}^{\infty} \Sigma THH(D_+S^1)_{hC_{p^i}} & \longrightarrow & \prod_{i=1}^{\infty} \Sigma THH(D_+S^1)_{hC_{p^i}} \\
 \downarrow F - \text{id} & & \downarrow F - \text{id} & & \downarrow F - \text{id} \\
 \bigvee_{n=1}^{\infty} \Sigma_+^\infty S_{C_n}^1 & \longrightarrow & \prod_{i=0}^{\infty} \Sigma THH(D_+S^1)_{hC_{p^i}} & \longrightarrow & \prod_{i=1}^{\infty} \Sigma THH(D_+S^1)_{hC_{p^i}}
 \end{array}$$

In the left column,  $F$  takes summand  $n$  to summand  $pn$  by an equivalence. The fiber of the middle column is the suspension of the desired  $TC$ . The fiber of the right-hand column is the homotopy limit of

$$\dots \xrightarrow{\text{transfer}} \Sigma THH(D_+ S^1)_{hC_{p^3}} \xrightarrow{\text{transfer}} \Sigma THH(D_+ S^1)_{hC_{p^2}} \xrightarrow{\text{transfer}} \Sigma THH(D_+ S^1)_{hC_p}$$

which by the above lemma is

$$\Sigma^2 THH(D_+ S^1)_{hS^1}$$

Using

$$\begin{aligned} S^1_{hS^1} &\simeq * \\ (S^1_{C_n})_{hS^1} &\simeq BC_n \end{aligned}$$

and the fact that orbits commute with wedges, we get a pullback square after  $p$ -completion

$$\begin{array}{ccc} TC(D_+ S^1)_p^\wedge & \longrightarrow & \Sigma^\infty \Sigma_+ \mathbb{C}P^\infty \times \bigvee_{n=1}^\infty \Sigma_+^\infty BC_n \\ \downarrow & & \downarrow \\ \mathbb{S} \times \Sigma^{-1} \bigvee_{n=1}^\infty \Sigma_+^\infty S^1_{C_n} & \xrightarrow{\Delta_p - \text{id}} & \mathbb{S} \times \Sigma^{-1} \bigvee_{n=1}^\infty \Sigma_+^\infty S^1_{C_n} \end{array}$$

Now, since  $C_n$  acts freely on the 1-sphere, its cohomology is 2-periodic:

$$H^*(BC_n) \cong \mathbb{Z} \quad 0 \quad \mathbb{Z}/n \quad 0 \quad \mathbb{Z}/n \quad 0 \quad \mathbb{Z}/n \quad \dots$$

When  $p^k$  is the largest power of  $p$  dividing  $n$ , use the covering map

$$BC_n \longrightarrow BC_{p^k}$$

and its stable transfer to form a  $p$ -adic equivalence  $(\Sigma_+^\infty BC_n)_p^\wedge \simeq (\Sigma_+^\infty BC_{p^k})_p^\wedge$ . This leads to the simplification

$$\begin{array}{ccc} TC(D_+ S^1)_p^\wedge & \longrightarrow & \Sigma^\infty \Sigma_+ \mathbb{C}P^\infty \times \bigvee_{n=1}^\infty \Sigma_+^\infty BC_{p^k} \\ \downarrow & & \downarrow \\ \mathbb{S} \times \Sigma^{-1} \bigvee_{n=1}^\infty \Sigma_+^\infty S^1_{C_{p^k}} & \xrightarrow{\Delta_p - \text{id}} & \mathbb{S} \times \Sigma^{-1} \bigvee_{n=1}^\infty \Sigma_+^\infty S^1_{C_{p^k}} \end{array}$$

where our convention is that  $p^k$  is the highest power of  $p$  dividing  $n$ . The square splits into

an infinite wedge of squares. The easiest is

$$\begin{array}{ccc} \mathbb{S} \vee \Sigma \mathbb{C}P_{-1}^{\infty} & \longrightarrow & \Sigma^{\infty} \Sigma_+ \mathbb{C}P^{\infty} \\ \downarrow & & \downarrow \\ \mathbb{S} & \xrightarrow{0} & \mathbb{S} \end{array}$$

The rest may be divided according to the equivalence class of  $n$ , where two positive integers are equivalent if they differ by a factor of  $p^k$ . Each equivalence class gives the same pullback square

$$\begin{array}{ccc} X & \longrightarrow & \bigvee_{k=1}^{\infty} \Sigma_+^{\infty} BC_{p^k} \\ \downarrow & & \downarrow \\ \Sigma^{-1} \bigvee_{k=1}^{\infty} \Sigma_+^{\infty} S_{C_{p^k}}^1 & \xrightarrow{\Delta_p - \text{id}} & \Sigma^{-1} \bigvee_{n=1}^{\infty} \Sigma_+^{\infty} S_{C_{p^k}}^1 \end{array}$$

Remember that the “power map”  $\Delta_p$  actually takes the  $k$ th summand to the  $k + 1$ st summand by an equivalence. This allows us to simplify the pullback square. To do this, start by observing that if  $G$  is an abelian group, then there is a short exact sequence

$$0 \longrightarrow \bigoplus_{k=1}^{\infty} G \xrightarrow{f - \text{id}} \bigoplus_{k=1}^{\infty} G \xrightarrow{\Sigma} G \longrightarrow 0$$

where  $f$  is the map that shifts everything one slot to the right:

$$f(g_1, g_2, \dots) = (0, g_1, \dots)$$

Therefore we can add to the above pullback square

$$\begin{array}{ccc} X & \longrightarrow & \bigvee_{k=1}^{\infty} \Sigma_+^{\infty} BC_{p^k} \\ \downarrow & & \downarrow \\ \Sigma^{-1} \bigvee_{k=1}^{\infty} \Sigma_+^{\infty} S_{C_{p^k}}^1 & \xrightarrow{\Delta_p - \text{id}} & \Sigma^{-1} \bigvee_{n=1}^{\infty} \Sigma_+^{\infty} S_{C_{p^k}}^1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma^{-1} \Sigma_+^{\infty} S^1 \end{array}$$

and conclude

$$X \simeq \text{hofib} \left\{ \bigvee_{k=1}^{\infty} \Sigma_+^{\infty} BC_{p^k} \longrightarrow \Sigma^{-1} \Sigma_+^{\infty} S^1 \right\}$$

Finally,

$$TC(D_+S^1)_p^\wedge \simeq \mathbb{S} \vee \Sigma \mathbb{C}P_{-1}^\infty \vee \bigvee^\infty X$$

We will proceed to repeat this calculation by a second method, which starts with  $TF$  instead of  $TR$ . The motivation for this is that the above method will not generalize well to  $TC(D_+S^n)$ , but the below method (hopefully) will. It's a bit trickier though, so it helps to start on  $D_+S^1$  to ensure we get the same answer as in the above method.

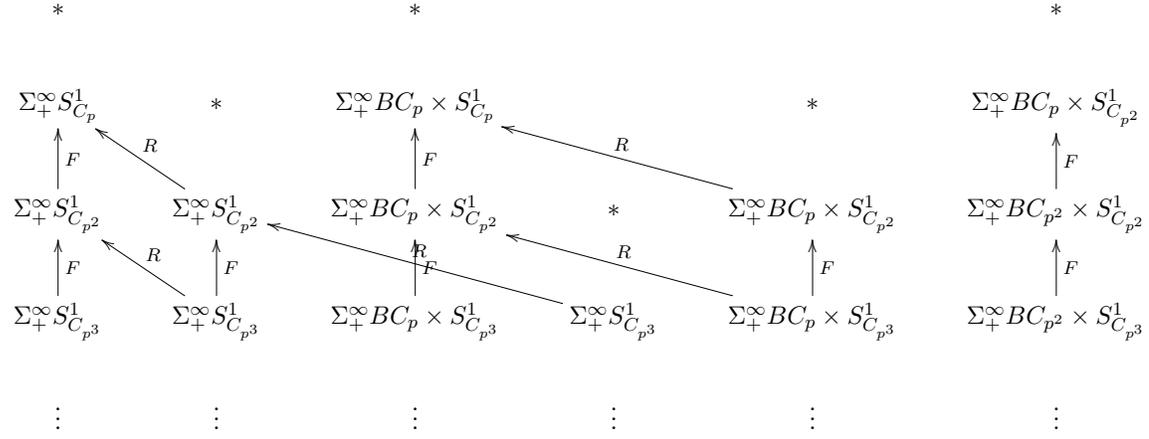
To begin, we define a fiber sequence of cyclotomic spectra whose fiber is

$$\begin{array}{cccccc}
 \Sigma_+^\infty S^1 & & \Sigma_+^\infty S_{C_p}^1 & & \Sigma_+^\infty S_{C_{p^2}}^1 & \\
 \swarrow R & & \nearrow F & & \swarrow R & & \nearrow F \\
 * & \Sigma_+^\infty S_{C_p}^1 & * & & \Sigma_+^\infty S_{C_{p^2}}^1 & * & \\
 * & * & * & & \Sigma_+^\infty S_{C_{p^2}}^1 & * & * \\
 * & * & * & & * & * & * \\
 \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots
 \end{array}$$

(we'll just draw the 1,  $p$ , and  $p^2$  summands), whose total space is

$$\begin{array}{cccccc}
 \Sigma_+^\infty S^1 & & \Sigma_+^\infty S_{C_p}^1 & & \Sigma_+^\infty S_{C_{p^2}}^1 & \\
 \uparrow F & \swarrow R & \nearrow F & \swarrow R & \nearrow F & \uparrow F \\
 \Sigma_+^\infty S_{C_p}^1 & \Sigma_+^\infty S_{C_p}^1 & \Sigma_+^\infty BC_p \times S_{C_p}^1 & \Sigma_+^\infty S_{C_{p^2}}^1 & \Sigma_+^\infty BC_p \times S_{C_{p^2}}^1 & \Sigma_+^\infty S_{C_{p^2}}^1 \\
 \uparrow F & \swarrow R & \uparrow F & \swarrow R & \nearrow F & \uparrow F \\
 \Sigma_+^\infty S_{C_{p^2}}^1 & \Sigma_+^\infty S_{C_{p^2}}^1 & \Sigma_+^\infty BC_p \times S_{C_{p^2}}^1 & \Sigma_+^\infty S_{C_{p^2}}^1 & \Sigma_+^\infty BC_p \times S_{C_{p^2}}^1 & \Sigma_+^\infty BC_{p^2} \times S_{C_{p^2}}^1 \\
 \uparrow F & \swarrow R & \uparrow F & \swarrow R & \nearrow F & \uparrow F \\
 \Sigma_+^\infty S_{C_{p^3}}^1 & \Sigma_+^\infty S_{C_{p^3}}^1 & \Sigma_+^\infty BC_p \times S_{C_{p^3}}^1 & \Sigma_+^\infty S_{C_{p^3}}^1 & \Sigma_+^\infty BC_p \times S_{C_{p^3}}^1 & \Sigma_+^\infty BC_{p^2} \times S_{C_{p^3}}^1 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
 \end{array}$$

and whose base is



Now  $TF$  of the base is a product of inverse limit systems, each of which consists of all columns that start at a specified horizontal level. Therefore the inverse limit is a product of the inverse limits:

$$TF = \prod_{k=0}^{\infty} \bigvee_{n \geq 1, p^k | n} \Sigma^{\infty} \Sigma_+ BC_{p^k}$$

Then  $R$  on this limit shifts the factors down by 1. To get the homotopy groups of the hofiber of  $R - \text{id}$ , note that if  $G$  is an abelian group, we have the short exact sequence

$$0 \longrightarrow G \xrightarrow{\Delta} \prod_{k=1}^{\infty} G \xrightarrow{f - \text{id}} \prod_{k=1}^{\infty} G \longrightarrow 0$$

where  $f$  is the map that shifts everything one slot to the left:

$$f(g_1, g_2, \dots) = (g_2, g_3, \dots)$$

Therefore the homotopy fiber of  $R - \text{id}$  is equivalent to

$$TC = \bigvee_{n \geq 1} \Sigma^{\infty} \Sigma_+ BC_{p^k}$$

where here  $k$  is the largest power of  $p$  dividing  $n$ . So much for the base; now we look at the fiber  $TC$ . Letting  $G$  take the place of homotopy groups of  $\Sigma_+^{\infty} S^1$ , the inverse system of

homotopy groups is

$$\dots \longrightarrow \bigoplus_{n=2}^{\infty} G \longrightarrow \bigoplus_{n=1}^{\infty} G \longrightarrow \bigoplus_{n=0}^{\infty} G$$

where the maps include summands. The inverse limit of this is zero, but it does not satisfy the Mittag-Leffler condition, so we have a  $\lim^1$ . To calculate  $\lim^1$  we do

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \bigoplus_{n=0}^{\infty} G & \longrightarrow & \prod_{n=0}^{\infty} G \longrightarrow \lim^1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \\ 0 & \longrightarrow & \bigoplus_{n=2}^{\infty} G & \longrightarrow & \bigoplus_{n=0}^{\infty} G & \longrightarrow & G \oplus G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bigoplus_{n=1}^{\infty} G & \longrightarrow & \bigoplus_{n=0}^{\infty} G & \longrightarrow & G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bigoplus_{n=0}^{\infty} G & \longrightarrow & \bigoplus_{n=0}^{\infty} G & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

This gives the homotopy groups of  $TF$ :

$$TF \simeq \Sigma^{-1} \text{cofib}(\bigvee_{+}^{\infty} \Sigma_{+}^{\infty} S^1 \longrightarrow \prod_{+}^{\infty} \Sigma_{+}^{\infty} S^1)$$

The  $\Sigma^{-1}$  is the grading shift from  $\lim^1$ , not the  $\Sigma^{-1}$  from  $THH$  that we are ignoring. The

next step is to take the fiber of  $R - \text{id}$ . We use the square

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & \bigoplus^{\infty} G & \xrightarrow{R-\text{id}} & \bigoplus^{\infty} G \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & G & \xrightarrow{\Delta} & \prod^{\infty} G & \xrightarrow{R-\text{id}} & \prod^{\infty} G \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & G & \longrightarrow & \lim^1 & \xrightarrow{R-\text{id}} & \lim^1 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

to see that  $TC \simeq \Sigma^{-1}S_+^1$ . Of course, we add in the extra  $\Sigma^{-1}$ s at the end to get the fiber sequence

$$\mathbb{S}^{-1} \vee \mathbb{S}^{-2} \xrightarrow{0} TC(D_+S^1)/TC(\mathbb{S}) \longrightarrow \bigvee_{n \geq 1} \Sigma_+^{\infty} BC_{p^k}$$

which agrees with the previous answer we got from “ $TR$  first.”

Now that we’ve finished both calculations of

$$\begin{aligned}
 TC(D_+S^1)_p^{\wedge} &\simeq \mathbb{S} \vee \Sigma \mathbb{C}\mathbb{P}_{-1}^{\infty} \vee \bigvee^{\infty} X \\
 X &\simeq \text{hofib}\{\bigvee_{k=1}^{\infty} \Sigma_+^{\infty} BC_{p^k} \longrightarrow \Sigma^{-1}\Sigma_+^{\infty} S^1\}
 \end{aligned}$$

take  $p \geq 7$  and examine the rational homotopy groups of the  $p$ -completion of our spectra.

spectrum	$\pi_{-2}^{\mathbb{Q}}$	$\pi_{-1}^{\mathbb{Q}}$	$\pi_0^{\mathbb{Q}}$	$\pi_1^{\mathbb{Q}}$	$\pi_2^{\mathbb{Q}}$	$\pi_3^{\mathbb{Q}}$	$\pi_4^{\mathbb{Q}}$	$\pi_5^{\mathbb{Q}}$	$\pi_6^{\mathbb{Q}}$
$K(\mathbb{S})_p^{\wedge}$	0	0	$\mathbb{Q}_p$	0	0	0	0	$\mathbb{Q}_p$	0
$(D_+S^1 \wedge K(\mathbb{S}))_p^{\wedge}$	0	$\mathbb{Q}_p$	$\mathbb{Q}_p$	0	0	0	$\mathbb{Q}_p$	$\mathbb{Q}_p$	0
$TC(\mathbb{S})_p^{\wedge}$	0	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	0	$\mathbb{Q}_p$	0	$\mathbb{Q}_p$	0
$(D_+S^1 \wedge TC(\mathbb{S}))_p^{\wedge}$	$\mathbb{Q}_p$	$\mathbb{Q}_p^2$	$\mathbb{Q}_p^2$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$
$X$	$\mathbb{Q}_p$	0	$\bigoplus^{\infty} \mathbb{Q}_p$	?	0	?	0	?	0
$TC(D_+S^1)_p^{\wedge}$	$\bigoplus^{\infty} \mathbb{Q}_p$	$\mathbb{Q}_p$	$\bigoplus^{\infty} \mathbb{Q}_p$	$\mathbb{Q}_p \oplus ?$	0	$\mathbb{Q}_p \oplus ?$	0	$\mathbb{Q}_p \oplus ?$	0

Table 3.2: Rational homotopy groups of  $TC(D_+S^1)$  and related spectra.

The question marks refer to the extra non-torsion that emerges in  $\pi_*(TC(D_+S^1)_p^{\wedge})$  as

a result in the infinite torsion in odd degrees in the homology of the infinite wedge

$$\bigvee_{k=1}^{\infty} \Sigma_+^{\infty} BC_{p^k}$$

and the assumption  $p \geq 5$  ensures that, in this range, the homotopy of our spectra with  $\mathbb{F}_p$  coefficients agrees with homology with  $\mathbb{F}_p$  coefficients. Now from [BHM93] the  $\mathbb{Q}_p$  summand in degree 4 in  $D_+S^1 \wedge K(\mathbb{S})$  maps to the corresponding  $\mathbb{Q}_p$  summand in  $D_+S^1 \wedge TC(\mathbb{S})$ . However, there is nothing in  $TC(D_+S^1)$  to hit this homotopy class. From this it is easy to deduce that

$$K(D_+S^1) \longrightarrow D_+S^1 \wedge K(\mathbb{S})$$

cannot be surjective on rational homotopy. So the dual  $K$ -theoretic Novikov conjecture is false for  $S^1$ .

### 3.6.2 Comparison with $TC(\Sigma_+^{\infty}\Omega S^1)$

Recall from [BHM93] that we have a homotopy pullback square

$$\begin{array}{ccc} TC(\Sigma_+^{\infty}\mathbb{Z})_p^{\wedge} & \longrightarrow & \Sigma^{\infty}\Sigma_+(LS^1)_{hS^1} \\ \downarrow & & \downarrow \\ \Sigma_+^{\infty}LS^1 & \xrightarrow{\Delta_p - \text{id}} & \Sigma_+^{\infty}LS^1 \end{array}$$

Under the equivalence  $LS^1 \simeq S^1 \times \mathbb{Z}$ , the  $S^1$ -action that rotates the loop (domain) coordinate acts on the component

$$S^1 \times \{n\}$$

by rotating the circle  $n$  times. Therefore

$$(LS^1)_{hS^1} \simeq \dots \amalg BC_2 \amalg * \amalg (S^1 \times \mathbb{C}\mathbb{P}^{\infty}) \amalg * \amalg BC_2 \amalg BC_3 \amalg \dots$$

and the pullback square becomes

$$\begin{array}{ccc} TC(\Sigma_+^{\infty}\mathbb{Z})_p^{\wedge} & \longrightarrow & \Sigma^{\infty}\Sigma_+(S^1 \times \mathbb{C}\mathbb{P}^{\infty}) \vee \bigvee_{n \neq 0} \Sigma^{\infty}\Sigma_+BC_{|n|} \\ \downarrow & & \downarrow \\ \Sigma_+^{\infty}S^1 \times \mathbb{Z} & \xrightarrow{\Delta_p - \text{id}} & \Sigma_+^{\infty}S^1 \times \mathbb{Z} \end{array}$$

Again, this splits into squares based on the equivalence class of  $n$  modulo multiplication by  $p$ . The simplest summand is  $n = 0$ , which is

$$\begin{array}{ccc} S_+^1 \wedge TC(\mathbb{S})_p^\wedge & \longrightarrow & S_+^1 \wedge \Sigma^\infty \Sigma_+ \mathbb{C}\mathbb{P}^\infty \\ \downarrow & & \downarrow \\ S_+^1 \wedge \mathbb{S} & \xrightarrow{0} & S_+^1 \wedge \mathbb{S} \end{array}$$

The others are all equivalent, and we simply name the pullback  $Y$ :

$$\begin{array}{ccc} Y & \longrightarrow & \bigvee_{k=1}^\infty \Sigma^\infty \Sigma_+ BC_{p^k} \\ \downarrow & & \downarrow \\ \bigvee_{k=1}^\infty \Sigma_+^\infty S^1 & \xrightarrow{\Delta_p - \text{id}} & \bigvee_{k=1}^\infty \Sigma_+^\infty S^1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma_+^\infty S^1 \end{array}$$

As above, the map  $\Delta_p$  forms an equivalence between each summand and the next, which justifies the lower half of the pullback square. We conclude

$$TC(\Sigma_+^\infty \Omega S^1)_p^\wedge \simeq S_+^1 \wedge (\mathbb{S} \vee \Sigma \mathbb{C}\mathbb{P}_{-1}^\infty) \vee \bigvee_{\mathbb{Z} \setminus \{0\}}^\infty Y$$

The similarity to  $TC(D_+ S^1)$  is striking. We put them side by side for comparison:

$$\begin{array}{ccc} X & \longrightarrow & \bigvee_{k=1}^\infty \Sigma_+^\infty BC_{p^k} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma^{-1} \Sigma_+^\infty S^1 \end{array} \quad \begin{array}{ccc} Y & \longrightarrow & \bigvee_{k=1}^\infty \Sigma^\infty \Sigma_+ BC_{p^k} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma_+^\infty S^1 \end{array}$$

Since both maps are transfers, it is easy to show that

$$Y \simeq \Sigma X$$

Now our comparison reads

$$TC(\Sigma_+^\infty \Omega S^1)_p^\wedge \simeq S_+^1 \wedge (\mathbb{S} \vee \Sigma \mathbb{C}\mathbb{P}_{-1}^\infty) \vee \bigvee_{\mathbb{Z} \setminus \{0\}} Y$$

$$TC(D_+S^1)_p^\wedge \simeq \mathbb{S} \vee \Sigma \mathbb{C}P_{-1}^\infty \vee \bigvee_{\mathbb{N}} X$$

There are curious parallels. Extending our previous table,

spectrum	$\pi_{-2}^{\mathbb{Q}}$	$\pi_{-1}^{\mathbb{Q}}$	$\pi_0^{\mathbb{Q}}$	$\pi_1^{\mathbb{Q}}$	$\pi_2^{\mathbb{Q}}$	$\pi_3^{\mathbb{Q}}$	$\pi_4^{\mathbb{Q}}$	$\pi_5^{\mathbb{Q}}$	$\pi_6^{\mathbb{Q}}$
$TC(\mathbb{S})_p^\wedge$	0	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	0	$\mathbb{Q}_p$	0	$\mathbb{Q}_p$	0
$X$	$\mathbb{Q}_p$	0	$\bigoplus_{\infty} \mathbb{Q}_p$	?	0	?	0	?	0
$Y$	0	$\mathbb{Q}_p$	0	$\bigoplus_{\infty} \mathbb{Q}_p$	?	0	?	0	?
$S_+^1 \wedge TC(\mathbb{S})_p^\wedge$	0	$\mathbb{Q}_p$	$\mathbb{Q}_p^2$	$\mathbb{Q}_p^2$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$
$TC(\Sigma_+^\infty \Omega S^1)_p^\wedge$	0	$\bigoplus_{\infty} \mathbb{Q}_p$	$\mathbb{Q}_p^2$	$\bigoplus_{\infty} \mathbb{Q}_p$	$\mathbb{Q}_p \oplus ?$	$\mathbb{Q}_p$	$\mathbb{Q}_p \oplus ?$	$\mathbb{Q}_p$	$\mathbb{Q}_p \oplus ?$
$TC(D_+S^1)_p^\wedge$	$\bigoplus_{\infty} \mathbb{Q}_p$	$\mathbb{Q}_p$	$\bigoplus_{\infty} \mathbb{Q}_p$	$\mathbb{Q}_p \oplus ?$	0	$\mathbb{Q}_p \oplus ?$	0	$\mathbb{Q}_p \oplus ?$	0
$(D_+S^1 \wedge TC(\mathbb{S}))_p^\wedge$	$\mathbb{Q}_p$	$\mathbb{Q}_p^2$	$\mathbb{Q}_p^2$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$	$\mathbb{Q}_p$

Table 3.3: Comparison to the rational homotopy groups of  $TC(\Sigma_+^\infty \Omega S^1)$ .

We conclude that the reduced  $TC$  of the ring  $\Sigma_+^\infty \Omega S^1$  is the suspension of  $TC$  of the Koszul dual  $D_+S^1$ .

### 3.6.3 Coassembly on $THH(D_+\Sigma X)$ and $TC(D_+\Sigma X)$

To determine the coassembly map for the functor  $X \rightarrow THH(D(X_+))$ , we compare to the assembly map for the dual

$$\Sigma_+^\infty X \rightarrow \Sigma_+^\infty LX$$

Inclusion of constant loops is an obvious choice, and since it defines a map from a linear functor into  $F$  that is an equivalence when  $X = *$ , it must be the assembly map. This gives a hint for what coassembly should do to homology, but we must be careful because there are many maps between spheres which give the same thing on homology (i.e. 0 for most of the summands) but which are nontrivial.

Fortunately it is possible to be very explicit. Each point in  $\Sigma X$  gives an *evaluation map*

$$\mathbb{S} \vee D(\Sigma X) \xrightarrow{\quad} D_+(\Sigma X) \xrightarrow{\text{ev}} \mathbb{S}$$

$\xrightarrow{\text{id} \vee \text{ev}}$

which is a map of ring spectra on both the left and middle terms. (The multiplication on the

left-hand spectrum is the one given by the diagonal, not zero.) This passes to a map from the above simplicial spectrum to a simpler one for  $\mathbb{S}^0$ . Assembling these together yields a map into the simplicial spectrum which at every level is  $D_+(\Sigma X)$ , with constant face and degeneracy maps:

$$\{(D_+\Sigma X)^{\wedge n}\}_n \longrightarrow \{(D_+\Sigma X) \wedge \mathbb{S}^{\wedge n}\}_n$$

This coassembly map is, at each level, just the dual of the diagonal map on  $\Sigma X$ . The simplices and  $\Lambda[n-1]$ s turn out to be irrelevant; the coassembly map squashes them down to a point. In total, the coassembly map

$$\mathbb{S} \vee \Sigma^{-1} \left( \bigvee_{n=1}^{\infty} D(X^{\wedge n}) \wedge_{C_n} S_+^1 \right) \longrightarrow \mathbb{S} \vee \Sigma^{-1} DX \simeq D_+(\Sigma X)$$

is the unit on the first summand, and on all the others squashes the circle and maps the rest in by the dual of the diagonal.

But it doesn't stop there! This is a map of cyclotomic spectra, so it extends to a map on  $TR$  and  $TC$ . In the case  $\Sigma X = S^1$ ,  $TR$  is a product, and we can write the map explicitly on the factors (where for simplicity we are only taking the wedge summands  $n = p^k$  from  $THH$ ):

$$\begin{array}{l} THH \\ THH_{hC_p} \\ THH_{hC_{p^2}} \\ \vdots \end{array} \left| \begin{array}{l} TR(D_+S^1) \longrightarrow D_+S^1 \wedge TR(\mathbb{S}) \\ \mathbb{S} \vee \Sigma^{-1} \bigvee_{k=0}^{\infty} (S_{C_{p^k}}^1)_+ \longrightarrow \mathbb{S} \vee \Sigma^{-1} \mathbb{S} \\ \Sigma_+^{\infty} BC_p \vee \Sigma^{-1} \bigvee_{k=0}^{\infty} (BC_p \times S_{C_{p^k}}^1)_+ \longrightarrow \Sigma_+^{\infty} BC_p \vee \Sigma^{-1} \Sigma_+^{\infty} BC_p \\ \Sigma_+^{\infty} BC_{p^2} \vee \Sigma^{-1} \bigvee_{k=0}^{\infty} (BC_{p^2} \times S_{C_{p^k}}^1)_+ \longrightarrow \Sigma_+^{\infty} BC_{p^2} \vee \Sigma^{-1} \Sigma_+^{\infty} BC_{p^2} \\ \vdots \end{array} \right.$$

The maps respect the wedge sum of two things. The left wedge summand gets mapped by the identity to the left wedge summand. The right wedge summand gets mapped by the obvious “extremely surjective” map

$$\bigvee_{k=0}^{\infty} (S_{C_{p^k}}^1)_+ \longrightarrow (*)_+$$

times the identity on  $BC_{p^k}$ . Now we want to take the hofiber of  $F - \text{id}$  to get  $TC$ . As

above, we separate out what happens on the first row:

$$\begin{array}{ccccc}
 \Sigma^{-1} \bigvee_{k=0}^{\infty} S^1_+ & \xrightarrow{F-\text{id}} & \Sigma^{-1} \bigvee_{k=0}^{\infty} S^1_+ & \xrightarrow{\vee^{\infty} \text{id}} & \Sigma^{-1} S^1_+ \\
 \downarrow \vee^{\infty} \text{collapse} & & \downarrow \vee^{\infty} \text{collapse} & \swarrow \text{dotted} & \downarrow \text{not surjective!} \\
 S^0 & \xrightarrow{0} & S^0 & \xrightarrow{\text{dotted}} & S^0 \vee S^1
 \end{array}$$

The dotted arrow forms two commuting triangles. Using this fact we see that this map is not surjective on the third term, which is a bad sign because applying  $\Sigma^{-1}$  once more gives the coassembly map on the hofiber of  $F - \text{id}$ .

Now we turn our attention to the rest of the system and take the hofiber of  $F - \text{id}$ . We get the inverse limit of the  $hC_{p^k}$  orbits under the transfer, which is the  $hS^1$  orbits:

$$\begin{aligned}
 \Sigma_+(* )_{hS^1} \vee \bigvee_{k=0}^{\infty} ((S^1_{C_{p^k}})_{hS^1})_+ &\longrightarrow \Sigma_+(* )_{hS^1} \vee ((*)_{hS^1})_+ \\
 \Sigma_+ \mathbb{C}P^{\infty} \vee \bigvee_{k=0}^{\infty} (BC_{p^k})_+ &\longrightarrow \Sigma_+ \mathbb{C}P^{\infty} \vee \mathbb{C}P^{\infty}_+
 \end{aligned}$$

The first summands are connected by the identity map, so we remove them from consideration. The rest is, on  $\mathbb{Z}/p$  homology,

$$\begin{array}{ccc}
 H_*(\bigvee_{k=0}^{\infty} \Sigma_+^{\infty} BC_{p^k}; \mathbb{Z}/p) & & H_*(\Sigma_+^{\infty} \mathbb{C}P^{\infty}; \mathbb{Z}/p) \\
 0 & \bigoplus_{k=0}^{\infty} \mathbb{Z}/p \xrightarrow{(1,1,1,\dots)} & \mathbb{Z}/p \\
 1 & \bigoplus_{k=0}^{\infty} \mathbb{Z}/p & 0 \\
 2 & \bigoplus_{k=0}^{\infty} \mathbb{Z}/p \xrightarrow{(1,1,1,\dots)} & \mathbb{Z}/p \\
 3 & \bigoplus_{k=0}^{\infty} \mathbb{Z}/p & 0 \\
 4 & \bigoplus_{k=0}^{\infty} \mathbb{Z}/p \xrightarrow{(1,1,1,\dots)} & \mathbb{Z}/p \\
 \vdots & \vdots & \vdots
 \end{array}$$

Combine this with the above calculation

$$\begin{array}{ccc}
 H_*(S^{-1} \vee S^0; \mathbb{Z}/p) & & H_*(S^{-1} \vee S^0; \mathbb{Z}/p) \\
 -1 & \mathbb{Z}/p \xrightarrow{1} & \mathbb{Z}/p \\
 0 & \mathbb{Z}/p \xrightarrow{0} & \mathbb{Z}/p
 \end{array}$$

and take homotopy fibers. The homotopy fibers of the sources give a spectrum we call  $X$ ,

to be consistent with previous sections. We calculate its  $\mathbb{Z}/p$  homology as follows:

$$\begin{array}{rcccc}
 & H_*(X; \mathbb{Z}/p) & H_*(\bigvee_{k=0}^{\infty} \Sigma_+^{\infty} BC_{p^k}; \mathbb{Z}/p) & H_*(\Sigma^{-1} \Sigma_+^{\infty} S^1; \mathbb{Z}/p) \\
 -2 & \mathbb{Z}/p & 0 & 0 \\
 -1 & 0 & 0 & \mathbb{Z}/p \\
 0 & \bigoplus_{k=1}^{\infty} \mathbb{Z}/p & \longrightarrow \bigoplus_{k=0}^{\infty} \mathbb{Z}/p & \xrightarrow{(1,0,0,\dots)} \mathbb{Z}/p \\
 1 & \bigoplus_{k=0}^{\infty} \mathbb{Z}/p & \xrightarrow{\text{id}} \bigoplus_{k=0}^{\infty} \mathbb{Z}/p & 0 \\
 2 & \bigoplus_{k=0}^{\infty} \mathbb{Z}/p & \xrightarrow{\text{id}} \bigoplus_{k=0}^{\infty} \mathbb{Z}/p & 0
 \end{array}$$

The  $(1,0,0,\dots)$  is a transfer map. We calculated it by looking at the row  $y = 1$  in the  $E^2$ -page of the Serre SS for the fiber bundle

$$S^1 \longrightarrow S_{C_{p^k}}^1 \times ES^1 \longrightarrow BC_{p^k}$$

Moving on, the homotopy fibers of the targets give

$$\Sigma^{-1}TC(\mathbb{S}) = \Sigma^{-1}(\mathbb{S} \vee \Sigma\mathbb{C}\mathbb{P}_{-1}^{\infty})$$

Using all of this, we deduce that coassembly on  $X$  on  $\mathbb{Z}/p$  homology is

$$\begin{array}{rccc}
 & H_*(X; \mathbb{Z}/p) & H_*(\Sigma^{-1}(\mathbb{S} \vee \Sigma\mathbb{C}\mathbb{P}_{-1}^{\infty}); \mathbb{Z}/p) \\
 -2 & \mathbb{Z}/p & \xrightarrow{1} \mathbb{Z}/p \\
 -1 & 0 & \mathbb{Z}/p \\
 0 & \bigoplus_{k=1}^{\infty} \mathbb{Z}/p & \xrightarrow{(1,1,1,\dots)} \mathbb{Z}/p \\
 1 & \bigoplus_{k=0}^{\infty} \mathbb{Z}/p & 0 \\
 2 & \bigoplus_{k=0}^{\infty} \mathbb{Z}/p & \xrightarrow{(1,1,1,\dots)} \mathbb{Z}/p \\
 3 & \bigoplus_{k=0}^{\infty} \mathbb{Z}/p & 0 \\
 \vdots & \vdots & \vdots
 \end{array}$$

The  $TC$  of  $D_+S^1$ , once we remove  $TC(\mathbb{S})$  is actually an infinite wedge sum of spectra equivalent to  $X$ , but the map on each one is the same. So coassembly is not surjective on  $\mathbb{Z}/p$ -homology. Passing to connective covers does not help, since the missing  $\mathbb{Z}/p$  in dimension  $-1$  simply becomes a missing  $\mathbb{Z}/p$  in a higher dimension.

On rational homology/homotopy we also do not get a surjective map:

$$\begin{array}{ccccc}
 & H_*(X; \mathbb{Q}) & H_*(\bigvee_{k=0}^{\infty} \Sigma_+^{\infty} BC_{p^k}; \mathbb{Q}) & & H_*(\Sigma^{-1} \Sigma_+^{\infty} S^1; \mathbb{Q}) \\
 -2 & \mathbb{Q}_p & 0 & & 0 \\
 -1 & 0 & 0 & & \mathbb{Q}_p \\
 0 & \bigoplus_{k=1}^{\infty} \mathbb{Q}_p & \longrightarrow \bigoplus_{k=0}^{\infty} \mathbb{Q}_p & \xrightarrow{(1, p, p^2, \dots)} & \mathbb{Q}_p \\
 1 & 0 & 0 & & 0 \\
 2 & 0 & 0 & & 0 \\
 3 & 0 & 0 & & 0 \\
 4 & 0 & 0 & & 0
 \end{array}$$

$$\begin{array}{ccc}
 H_*(X; \mathbb{Q}) & H_*(\Sigma^{-1}(\mathbb{S} \vee \Sigma \mathbb{C}P_{-1}^{\infty}); \mathbb{Q}) \\
 -2 & \mathbb{Q}_p \xrightarrow{1} \mathbb{Q}_p \\
 -1 & 0 \qquad \qquad \mathbb{Q}_p \\
 0 & \bigoplus_{k=1}^{\infty} \mathbb{Q}_p \xrightarrow{(1, 1, 1, \dots)} \mathbb{Q}_p \\
 1 & 0 \qquad \qquad 0 \\
 2 & 0 \qquad \qquad \mathbb{Q}_p \\
 3 & 0 \qquad \qquad 0 \\
 4 & 0 \qquad \qquad \mathbb{Q}_p \\
 \vdots & \vdots \qquad \qquad \vdots
 \end{array}$$

Here passing to connective covers does indeed help with degree  $-1$ , but the map is still not surjective in any even degree. One should also be careful because  $H(-; \mathbb{Q})$  changes after we  $p$ -complete. However in the above coassembly table, the groups on the right do not change, and the groups on the left only change in odd positive degrees (when the degree is less than about  $2p$ ). So the map is still not surjective (except perhaps for a few small values of  $p$ ).

### 3.7 THH of finite spectra with a $G$ -action

Here we prove the following statements, which are reminiscent of the  $K$ -theory Novikov conjecture:

**Theorem 3.7.1.** *If  $G$  is a finite  $p$ -group, then the assembly and coassembly maps*

$$\begin{aligned} BG_+ \wedge THH(*) &\longrightarrow THH(BG) \\ THH(\mathcal{P}'(\Sigma_+^\infty G)) &\longrightarrow F(BG_+, THH(*)) \end{aligned}$$

*are split injective and split surjective maps of spectra, respectively, after  $p$ -completion.*

**Theorem 3.7.2.** *The composite of assembly, inclusion, and coassembly*

$$BG_+ \wedge A(*) \longrightarrow A(BG) \longrightarrow \forall(BG) \longrightarrow F(BG_+, A(*))$$

*is up to homotopy the transfer  $BG_+ \longrightarrow F(BG_+, \mathbb{S})$  smashed with the identity on  $A(*)$ .*

These both follow quickly from our main technical result:

**Theorem 3.7.3.** *The composite of assembly, inclusion, and coassembly*

$$BG_+ \wedge THH(*) \longrightarrow THH(BG) \longrightarrow THH(\mathcal{P}'(\Sigma_+^\infty G)) \longrightarrow F(BG_+, THH(*))$$

*is up to homotopy the transfer  $BG_+ \longrightarrow F(BG_+, \mathbb{S})$  along the bundle over  $BG \times BG$  with fiber  $G$  and monodromy given by left and right multiplication of  $G$  on itself. There is a similar composite*

$$BWH_+ \wedge THH(*) \longrightarrow THH(BWH) \longrightarrow THH(\mathcal{P}'(\Sigma_+^\infty G)) \longrightarrow F(BG_+, THH(*))$$

*which is up to homotopy the transfer  $BWH_+ \longrightarrow F(BG_+, \mathbb{S})$  along the bundle over  $BG \times BWH$  with fiber  $G/H$  and monodromy given by the left action of  $G$  and the right action of  $WH \cong \text{Aut}_G(G/H)^{\text{op}}$ .*

In this first section, we will prove the special case of  $G = \mathbb{Z}/2$ ,  $BG = \mathbb{R}\mathbb{P}^\infty$ . We prove the more general case in the next section. We will sometimes adopt the notation

$$THH(\Sigma_+^\infty \mathbb{Z}/2, \text{fn}) = THH(\mathcal{P}'(\Sigma_+^\infty \mathbb{Z}/2))$$

We link assembly and coassembly together in the following way:

$$\mathbb{R}P_+^\infty \wedge THH(\mathbb{S}) \xrightarrow{\alpha} THH(\Sigma_+^\infty \mathbb{Z}/2) \xrightarrow{THH(\iota)} THH(\mathcal{P}'(\Sigma_+^\infty \mathbb{Z}/2)) \xrightarrow{c\alpha} F(\mathbb{R}P_+^\infty, THH(\mathbb{S}))$$

Here  $\iota$  is simply an inclusion of Waldhausen categories: those modules over  $\Sigma_+^\infty \mathbb{Z}/2$  which are finite and  $\mathbb{Z}/2$ -free are contained inside those modules which are finite. We wish to prove that the composite, when summed with the map

$$THH(*) \longrightarrow THH(\mathbb{R}P^\infty; \text{fin}) \xrightarrow{c\alpha} F(\mathbb{R}P_+^\infty, THH(*))$$

gives an equivalence.

Before we dive into the *THH* result, we should begin by characterizing which maps

$$\mathbb{S} \vee \Sigma_+^\infty \mathbb{R}P^\infty \longrightarrow D(\mathbb{R}P_+^\infty)$$

are 2-adic equivalences. We know one such map in particular: it is the composite of the tom Dieck splitting and the usual inclusion of genuine fixed points into homotopy fixed points:

$$\mathbb{S} \vee \Sigma_+^\infty \mathbb{R}P^\infty \longrightarrow \mathbb{S}^{C_2} \longrightarrow \mathbb{S}^{hC_2} \cong D(\mathbb{R}P_+^\infty)$$

We can describe this composite fairly explicitly (here  $\sigma$  is the sign representation of  $C_2$ ):

$$\begin{array}{ccc}
 \mathbb{R}\mathbb{P}_+^{n-1} & \longrightarrow & (\Omega^{n\sigma} \mathbb{S}^{n\sigma})^{C_2} \\
 \downarrow & & \downarrow \\
 \mathbb{R}\mathbb{P}_+^\infty & \longrightarrow & (\operatorname{colim}_n \Omega^{n\sigma} \mathbb{S}^{n\sigma})^{C_2} \\
 & \searrow \text{dotted} & \downarrow \\
 & & (\operatorname{colim}_n \Omega^{n+n\sigma} \mathbb{S}^{n+n\sigma})^{C_2} \\
 & & \downarrow \\
 & & \operatorname{Map}_*(S(\infty)_+, \operatorname{colim}_n \Omega^{n+n\sigma} \mathbb{S}^{n+n\sigma})^{C_2} \\
 & & \uparrow \sim \\
 & & \operatorname{Map}_*(S(\infty)_+, \operatorname{colim}_n \Omega^n \mathbb{S}^n)^{C_2} \\
 & & \parallel \\
 & & \operatorname{Map}_*(\mathbb{R}\mathbb{P}_+^\infty, \Omega^\infty S^\infty)
 \end{array}$$

The dotted line lift-up-to-homotopy is formed by picking a  $\mathbb{Z}/2$ -equivariant homotopy between the two maps

$$\begin{array}{ccccccc}
 \mathbb{R}\mathbb{P}_+^\infty & \wedge & S(\infty)_+ & \wedge & S^\infty & \wedge & S^{\infty\sigma} & \longrightarrow & S_\epsilon^\infty & \wedge & S_\epsilon^{\infty\sigma} \\
 (x, & & y, & & z, & & w) & \mapsto & (z, & & w - \tilde{x}) \\
 (x, & & y, & & z, & & w) & \mapsto & (?, & & w)
 \end{array}$$

Here  $\tilde{x}$  is whichever lift of  $x$  to  $S(\infty)$  is closer to  $w$ ; as long as  $\epsilon < 1$  such a lift exists or it is irrelevant because the map goes to the basepoint. The  $\mathbb{Z}/2$ -action is negation on  $y$  and  $w$ , as well as the second slot of the output. All other coordinates are fixed. To fill in the (?) and pick a homotopy, we first pick a  $\mathbb{Z}/2 \times \mathbb{Z}/2$ -equivariant map  $g : S^\infty \times S^\infty \longrightarrow S^\infty$ , where the  $\mathbb{Z}/2 \times \mathbb{Z}/2$ -action on the right is trivial on the diagonal  $\mathbb{Z}/2$  and the usual action on the quotient by that diagonal. Not only does such a map  $g$  exist, but the space of all possible choices for  $g$  is weakly contractible. (This is because the source is a free  $\mathbb{Z}/2 \times \mathbb{Z}/2$ -CW complex and the target is contractible.) Now we take our homotopy to be

$$(x, y, z, w, t) \mapsto (z - (\sin t)g(\tilde{x}, y), w - (\cos t)\tilde{x})$$

Here again the map is well-defined because, though there may be two values of  $(\cos t)\tilde{x}$  that are within  $\epsilon$  of  $w$ , in that case only one of those values of  $\tilde{x}$  gives a value of  $(\sin t)g(\tilde{x}, y)$  that is within  $\epsilon$  of  $z$ . It is also straightforward to check it is equivariant. The end of the homotopy is

$$(x, y, z, w) \mapsto (z - g(\tilde{x}, y), w)$$

and then we can restrict to  $\text{Map}_*(\mathbb{R}\mathbb{P}_+^\infty, \Omega^\infty S^\infty)$  by forgetting the  $w$  coordinate:

$$(x, y, z) \mapsto z - g(\tilde{x}, y)$$

Therefore the composite is given in the following proposition. The map  $\bar{g}$  is the obvious map between the quotients

$$\mathbb{R}\mathbb{P}^\infty \times \mathbb{R}\mathbb{P}^\infty \xrightarrow{\bar{g}} \mathbb{R}\mathbb{P}^\infty$$

which is in fact uniquely characterized by the property that on  $\pi_1$  it sends the generator from each factor on the left to the generator on the right.

**Proposition 3.7.4.** *The Segal conjecture 2-adic equivalence*

$$\mathbb{S} \vee \Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty \longrightarrow D(\mathbb{R}\mathbb{P}_+^\infty)$$

*is the obvious collapse or unit map on the first  $\mathbb{S}$ . On the second term, it is adjoint to the composite*

$$\Sigma_+^\infty(\mathbb{R}\mathbb{P}^\infty \times \mathbb{R}\mathbb{P}^\infty) \xrightarrow{\bar{g}} \Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty \xrightarrow{\text{transfer}} \mathbb{S}$$

*Furthermore this map is an equivalence of rings, with multiplication on the left given by transfer and on the right by the dual of the diagonal.*

As a sanity check, we may also check that this map satisfies the condition given by Adams, that the functional  $\text{Sq}^4$  is nonzero. In fact, all functional  $\text{Sq}^i$ 's are nonzero. To see this, we first prove the same fact for the transfer

$$\Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty \longrightarrow \mathbb{S}$$

using that its cofiber is  $\mathbb{R}\mathbb{P}_{-1}^\infty$ , the Thom spectrum of the “real vector bundle” whose Stiefel-Whitney class is

$$(1 + a)^{-1} = 1 + a + a^2 + a^3 + \dots$$

Tracing through the definition of functional  $\text{Sq}^i$ , we see that the nonvanishing of the coefficient of  $a^i$  above is equivalent to the fact that functional  $\text{Sq}^i$  is nonzero for this map. Returning to the composite

$$\Sigma_+^\infty(\mathbb{R}\mathbb{P}^\infty \times \mathbb{R}\mathbb{P}^\infty) \xrightarrow{\bar{g}} \Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty \xrightarrow{\text{transfer}} \mathbb{S}$$

we prove the functional  $\text{Sq}^i$  is nonzero by an elementary diagram chase, using the relationships between the cofibers of the above two maps and the cofiber of their composition. (The only needed property of  $\bar{g}$  is that it is injective on mod 2 cohomology. Which is a bit disturbing, since the collapse of one copy of  $\mathbb{R}\mathbb{P}^\infty$  is also injective, but cannot give the Segal conjecture equivalence.) So the map we described above definitely gives the desired equivalence  $\mathbb{S} \vee \Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty \xrightarrow{\sim} D(\mathbb{R}\mathbb{P}_+^\infty)$ .

Now we will prove our splitting results on  $THH$  by building models for the assembly and coassembly maps, and comparing them to the above characterization of the Segal conjecture isomorphism. Consider the diagram

$$\mathbb{R}\mathbb{P}_+^\infty \wedge THH(\mathbb{S}) \xrightarrow{\alpha} THH(\Sigma_+^\infty \mathbb{Z}/2) \xrightarrow{THH(\iota)} THH(\mathcal{P}'(\Sigma_+^\infty \mathbb{Z}/2)) \xrightarrow{c\alpha} F(\mathbb{R}\mathbb{P}_+^\infty, THH(\mathbb{S}))$$

The assembly map on the left is simply inclusion of constant loops

$$\mathbb{R}\mathbb{P}_+^\infty \wedge \mathbb{S} \xrightarrow{\alpha} \Sigma_+^\infty L\mathbb{R}\mathbb{P}^\infty \simeq \Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty \amalg \mathbb{R}\mathbb{P}^\infty$$

which is an equivalence onto the first term in the disjoint union  $\mathbb{R}\mathbb{P}^\infty \amalg \mathbb{R}\mathbb{P}^\infty$ . This hits the component of  $THH(\Sigma_+^\infty \mathbb{Z}/2)$  composed of those simplices for which the product of all the elements of  $\mathbb{Z}/2$  is the identity.

Coassembly is trickier. We need to compare modules over  $\mathbb{S}[G]$  with parametrized spectra over  $BG$ . The comparison is straightforward:

- If  $X$  is a module over  $\mathbb{S}[G]$ , i.e. a naive  $G$ -spectrum, we form a parametrized spectrum  $X//G$  over  $BG = B(*, G, *)$  whose  $n$ th space is  $B(*, G, X_n)$ .
- If  $\mathcal{E}$  is a parametrized spectrum over  $BG$  then we take the fiber to recover our original spectrum with a  $G$ -action.
- The homotopy orbits  $X_{hG}$  are the quotient of  $X//G \rightarrow BG$  by the basepoint section. The homotopy fixed points  $X^{hG}$  are the sections  $\Gamma_{BG}(X//G)$ . These are both easily

proven by comparing adjoints of the “trivial  $G$ -action” and “constant bundle” functors from ordinary spectra.

So if  $\mathcal{S}p_{\mathbb{R}P^\infty}$  denotes the category of parametrized spectra over  $\mathbb{R}P^\infty$  with derived mapping spectra, then the above diagram of  $THH$ s can be rewritten

$$\begin{aligned} \mathbb{R}P_+^\infty \wedge THH(\mathbb{S}) &\xrightarrow{\alpha} THH(\text{End}_{\mathcal{S}p_{\mathbb{R}P^\infty}}(\Sigma_{+\mathbb{R}P^\infty}^\infty S^\infty)) \\ \xrightarrow{THH(\iota)} THH(\mathcal{S}p_{\mathbb{R}P^\infty}, \text{fin}) &\xrightarrow{c\alpha} F(\mathbb{R}P_+^\infty, THH(\mathbb{S})) \end{aligned}$$

Here we have recognized that the  $\mathbb{S}[\mathbb{Z}/2]$ -module

$$\mathbb{S}[\mathbb{Z}/2] \cong \Sigma_+^\infty \mathbb{Z}/2 \cong \mathbb{S} \vee \mathbb{S} \simeq \mathbb{S} \times \mathbb{S}$$

becomes  $\Sigma_{+\mathbb{R}P^\infty}^\infty S^\infty$  as a parametrized spectrum over  $\mathbb{R}P^\infty$ . The endomorphisms of this module are

$$\left(\prod_2 \prod_2 \mathbb{S}\right)^{C_2} \simeq M_{2 \times 2}(\mathbb{S})^{C_2} \cong \left(\prod_2 \bigvee_2 \mathbb{S}\right)^{C_2}$$

There are a priori lots of ways to interpret the  $C_2$ -fixed points, but they all give equivalent results because the  $C_2$ -action is homotopically free. Specifically, we can take the genuine fixed points, the homotopy orbits (via the transfer), the homotopy fixed points, and even the naive fixed points (because the product is there and so they are nontrivial!). This was actually fortunate because the definition is technically the naive fixed points. Anyway, to prove our claim that everyone is right, we use Wirthmuller to rewrite our genuine  $C_2$ -spectrum as

$$\bigvee_2 \bigvee_2 \mathbb{S} \cong \Sigma_+^\infty (C_2 \amalg C_2)$$

and then the geometric fixed points vanish, whence all the other constructions give the same answer. This answer is equivalent to

$$\Sigma_+^\infty \mathbb{Z}/2 \cong \mathbb{S}[x]/(x^2 - 1)$$

where the identity  $\mathbb{S}$  corresponds to maps  $\mathbb{S} \vee \mathbb{S} \rightarrow \mathbb{S} \vee \mathbb{S}$  that preserve the summands and are identical on the two summands, while the  $\mathbb{S}\langle x \rangle$  corresponds to maps which switch the two summands but which otherwise also do the same thing to both summands.

As we said above, this module becomes the parametrized spectrum

$$\mathcal{E} = (\mathbb{S} \vee \mathbb{S}) \times_{C_2} S^\infty \longrightarrow \mathbb{R}\mathbb{P}^\infty$$

or more simply

$$\Sigma_{+\mathbb{R}\mathbb{P}^\infty}^\infty S^\infty$$

We showed above that its endomorphism ring

$$\Gamma_{\mathbb{R}\mathbb{P}^\infty}(\text{End}_{\mathbb{R}\mathbb{P}^\infty}(\mathcal{E}))$$

is equivalent to  $\mathbb{S}[x]/(x^2 - 1)$ , the  $C_2$ -homotopy fixed points of  $\mathbb{S} \vee \mathbb{S}$ . Note that the fibers of this parametrized spectrum are all  $\mathbb{S} \vee \mathbb{S}$ , so their endomorphisms are  $M_2(\mathbb{S}) \simeq \mathbb{S}^{\vee 4}$ . It is not, however, a trivial bundle: the monodromy of the fiber swaps the diagonal elements and swaps the off-diagonal elements of  $M_2(\mathbb{S})$ . The equivalence of ring spectra

$$\mathbb{S}[x]/(x^2 - 1) \xrightarrow{\sim} \Gamma_{\mathbb{R}\mathbb{P}^\infty}(\text{End}_{\mathbb{R}\mathbb{P}^\infty}(\mathcal{E}))$$

sends the first factor to the obvious fiberwise endomorphism that preserves summands, and the second factor to the obvious fiberwise endomorphism that swaps the summands.

Now to describe coassembly, we have to choose a contravariant functor on spaces over  $\mathbb{R}\mathbb{P}^\infty$  to approximate by an excisive functor. Given  $X \longrightarrow \mathbb{R}\mathbb{P}^\infty$ , define

$$F(X) = THH(\Gamma_X(\text{End}_{\mathbb{R}\mathbb{P}^\infty}(\mathcal{E})))$$

Technically this is not the same functor as

$$F'(X) = THH(\mathcal{S}_{\mathbb{R}\mathbb{P}^\infty}|_X, \text{fin}) \simeq THH(\Sigma_+^\infty \Omega X, \text{fin})$$

but we have a map  $F \longrightarrow F'$ , and we will describe the composite of this map with coassembly on  $F'$ , which is the same as coassembly on  $F$ :

$$\begin{array}{ccc} F & \longrightarrow & F' \\ \downarrow & & \downarrow \\ P_1 F & \xrightarrow{\sim} & P_1 F' \end{array}$$

When  $X = \mathbb{R}\mathbb{P}^\infty$ ,  $F(X)$  gives me the  $THH$  of the endomorphisms of  $\mathbb{S} \vee \mathbb{S}$ , which is what we want. When  $X = *$  this gives

$$THH(M_2(\mathbb{S})) \xrightarrow{\sim} THH(\mathbb{S}) \simeq \mathbb{S}$$

where the equivalence is the trace map, defined more precisely by the zig-zag

$$\begin{aligned} |THH_\bullet(M_2(\mathbb{S}))| &\xleftarrow{\sim} \|THH_\bullet(M_2(\mathbb{S}))\| \xleftarrow{\sim} \|THH_\bullet(W_2(\mathbb{S}))\| \xrightarrow{\text{tr}_\bullet^{(2)}} \|THH_\bullet(\mathbb{S})\| \\ &\xrightarrow{\sim} |THH_\bullet(\mathbb{S})| \end{aligned}$$

where  $W_2(\mathbb{S})$  is like matrices but the product has been replaced by a sum.

This trace equivalence motivates us to find a “fiberwise trace map” on the entire functor  $F$  which gives the above equivalence on  $F(*)$ . This can be defined in many equivalent ways:

$$\begin{array}{ccccc} |\Gamma_{\mathbb{R}\mathbb{P}^\infty}(\tilde{M}_2(\mathbb{S}))^{\wedge n+1}| & & & & \\ \uparrow \sim & & & & \\ \|\Gamma_{\mathbb{R}\mathbb{P}^\infty}(\tilde{M}_2(\mathbb{S}))^{\wedge n+1}\| & \xrightarrow{\text{interchange}} & \|\Gamma_{\mathbb{R}\mathbb{P}^\infty}(\tilde{M}_2(\mathbb{S})_{\mathbb{R}\mathbb{P}^\infty}^{\wedge n+1})\| & & \\ \uparrow \sim & & \uparrow \sim & & \\ \|\Gamma_{\mathbb{R}\mathbb{P}^\infty}(\tilde{W}_2(\mathbb{S}))^{\wedge n+1}\| & \xrightarrow{\text{interchange}} & \|\Gamma_{\mathbb{R}\mathbb{P}^\infty}(\tilde{W}_2(\mathbb{S})_{\mathbb{R}\mathbb{P}^\infty}^{\wedge n+1})\| & \xrightarrow{\text{interchange}} & \Gamma_{\mathbb{R}\mathbb{P}^\infty}(\|\tilde{W}_2(\mathbb{S})_{\mathbb{R}\mathbb{P}^\infty}^{\wedge n+1}\|) \\ \downarrow \text{trace} & & \downarrow \text{trace} & & \sim \downarrow \text{trace} \\ \Gamma_{\mathbb{R}\mathbb{P}^\infty}(\|\mathbb{S}_{\mathbb{R}\mathbb{P}^\infty}\|) & \xlongequal{\quad} & \Gamma_{\mathbb{R}\mathbb{P}^\infty}(\|\mathbb{S}_{\mathbb{R}\mathbb{P}^\infty}\|) & \xlongequal{\quad} & \Gamma_{\mathbb{R}\mathbb{P}^\infty}(\|\mathbb{S}_{\mathbb{R}\mathbb{P}^\infty}\|) \\ \downarrow \sim & & \downarrow \sim & & \\ \Gamma_{\mathbb{R}\mathbb{P}^\infty}(\mathbb{S}_{\mathbb{R}\mathbb{P}^\infty}) & \xlongequal{\quad} & F(\mathbb{R}\mathbb{P}_+^\infty, \mathbb{S}) & & \end{array}$$

Only the trace maps really need any explanation, but they are quite simple. Each point in  $W_2(\mathbb{S})$  corresponds to a  $2 \times 2$  matrix that has at most one nonzero entry, and an  $(n+1)$ -tuple of such matrices has product either 0, or has one nonzero entry. We define the trace on this point to be zero if the product of the matrices is 0 or off the main diagonal; otherwise we map into  $\mathbb{S}$  by smashing together the spheres in each of our  $(n+1)$  matrices. Now we know that the above diagram is easily extended to a diagram of functors, and every single map gives an equivalence on the linear approximations. The bottom functors are actually linear. Therefore coassembly on  $F(\mathbb{R}\mathbb{P}^\infty)$  is given by any route in the above diagram from

top to bottom. The top is equivalent to

$$LRP^\infty \simeq \mathbb{R}P^\infty \amalg \mathbb{R}P^\infty$$

and our task is now to trace the first copy of  $\mathbb{R}P^\infty$  through the coassembly map, and compare it with the Segal conjecture equivalence:

$$\begin{array}{ccc}
 |*_+ \wedge (\mathbb{Z}/2)_+^n| & \longrightarrow & |\Gamma_{\mathbb{R}P^\infty}(\tilde{M}_2(\mathbb{S}))^{\wedge n+1}| \\
 \downarrow \sim & & \uparrow \sim \\
 \|*_+ \wedge (\mathbb{Z}/2)_+^n\| & \longrightarrow & \|\Gamma_{\mathbb{R}P^\infty}(\tilde{M}_2(\mathbb{S}))^{\wedge n+1}\| \\
 \parallel & & \downarrow \\
 \|*_+ \wedge (\mathbb{Z}/2)_+^n\| & \longrightarrow & \|\Gamma_{\mathbb{R}P^\infty}(\tilde{M}_2(\mathbb{S})_{\mathbb{R}P^\infty}^{\wedge n+1})\| \\
 \uparrow \sim & & \uparrow \sim \\
 \|(E_{n+1})_+ \wedge *_+ \wedge (\mathbb{Z}/2)_+^n\| & \longrightarrow & \|\Gamma_{\mathbb{R}P^\infty}(\tilde{W}_2(\mathbb{S})_{\mathbb{R}P^\infty}^{\wedge n+1})\| \\
 & & \downarrow \text{trace} \\
 & & \Gamma_{\mathbb{R}P^\infty}(\|\mathbb{S}_{\mathbb{R}P^\infty}\|) \\
 & & \downarrow \sim \\
 & & F(\mathbb{R}P_+^\infty, \mathbb{S})
 \end{array}$$

This diagram commutes up to homotopy. The  $*_+$  is set to be one of the two elements of  $\mathbb{Z}/2$  so that the total product is always the identity. (In these complexes we will call the identity element “1” and the non-identity element “ $x$ .”) So the things on the left are all equivalent to  $\mathbb{R}P^\infty$ . The space  $E_n$  is defined to be a space of fiberwise embeddings:

$$E_n := \text{Emb}_{\mathbb{R}P^\infty}(S_{\mathbb{R}P^\infty}^\infty \times_{\mathbb{R}P^\infty} S_{\mathbb{R}P^\infty}^\infty \times_{\mathbb{R}P^\infty} \dots \times_{\mathbb{R}P^\infty} S_{\mathbb{R}P^\infty}^\infty, \mathbb{R}^\infty \times \mathbb{R}P^\infty) \simeq *$$

Now we explain the horizontal maps in the above commuting diagram. The first sends 1 to the section which at each fiber is the diagonal matrix, and  $x$  goes to the off-diagonal matrix. This also explains the second and third horizontal maps. The final horizontal map

is a Pontryagin-Thom collapse. At simplicial level  $n$  it is

$$\begin{aligned} (E_{n+1})_+ \wedge *_{+} \wedge (\mathbb{Z}/2)_+^n &\longrightarrow \Gamma_{\mathbb{R}\mathbb{P}^\infty}(\tilde{W}_2(\mathbb{S})_{\mathbb{R}\mathbb{P}^\infty}^{\wedge n+1}) \\ &\cong \Gamma_{\mathbb{R}\mathbb{P}^\infty}(\Sigma_{+\mathbb{R}\mathbb{P}^\infty}^\infty(S_1^\infty \amalg S_x^\infty)_{\mathbb{R}\mathbb{P}^\infty}^{\times n+1}) \end{aligned}$$

For each of our  $(n+1)$  slots on the right, we look at whether I have a 1 or an  $x$  in the corresponding slot in  $*_{+} \wedge (\mathbb{Z}/2)_+^n$ . From this, we pick either  $S_1^\infty$  or  $S_x^\infty$ . Once the selection is made, we now have a fiberwise product of  $(n+1)$  copies of  $S^\infty$  on the right; we use the chosen embedding from  $E_{n+1}$  and I Pontryagin-Thom collapse onto a small neighborhood of this embedding. This describes the desired section on the right. (To be precise, we must make the suspension spectrum an  $\epsilon$ -suspension spectrum and take the fibrant replacement via  $\text{colim}_k \Omega^k X_{n+k}$  before taking the section.)

It is now fairly clear that we can define the face maps on  $E_{n+1}$  on the left (depending on the string of 1s and  $x$ s) so that this is a map of semi-simplicial complexes. Specifically, the  $k$ th face on the right pairs the  $k$ th and  $(k+1)$ st copy (with wrap-around for the last face) of  $(S_1^\infty \amalg S_x^\infty)$  by

$$\begin{aligned} (S_1^\infty \amalg S_x^\infty)_+ \wedge (S_1^\infty \amalg S_x^\infty)_+ &\longrightarrow (S_1^\infty \amalg S_x^\infty)_+ \\ (p_1, p_1) &\mapsto p_1 \quad \text{when } p_1 \in S_1^\infty \\ (-p_1, p_1) &\mapsto * \\ (p_x, p_x) &\mapsto * \quad \text{when } p_x \in S_x^\infty \\ (-p_x, p_x) &\mapsto p_1 \\ (p_1, p_x) &\mapsto * \\ (-p_1, p_x) &\mapsto p_x \\ (p_x, p_1) &\mapsto p_x \\ (-p_x, p_1) &\mapsto * \end{aligned}$$

(These rules come directly from matrix multiplication in  $W_2(\mathbb{S})$ :

$$\begin{pmatrix} p_1 & -p_x \\ p_x & -p_1 \end{pmatrix}$$

Only 1 entry in each matrix is nonzero. This gives 16 possible matrix multiplications, but by  $\mathbb{Z}/2$ -equivariance they are determined by the above 8.) Therefore the  $k$ th face on the left must restrict the embedding along a diagonal  $S^\infty \rightarrow S^\infty \times S^\infty$  that sends  $p$  to  $(p, p)$  if the  $(k + 1)$ st slot is 1 and  $(-p, p)$  if the  $(k + 1)$ st slot is  $x$ . (Again, this is with wrap-around if  $k = n$ .)

Tracing through the trace, we get a map

$$\|(E_{n+1})_+ \wedge *_{+} \wedge (\mathbb{Z}/2)_+^n\| \wedge \mathbb{R}\mathbb{P}_+^\infty \rightarrow \Omega^\infty \mathbb{S}'$$

which for each pair of points on the left Pontryagin-Thom collapses onto a pair of points in  $\mathbb{R}^\infty$ . So it is indeed a composite

$$\|(E_{n+1})_+ \wedge *_{+} \wedge (\mathbb{Z}/2)_+^n\| \wedge \mathbb{R}\mathbb{P}_+^\infty \rightarrow \mathbb{R}\mathbb{P}^\infty \rightarrow \Omega^\infty \mathbb{S}'$$

where the second map is a Becker-Gottlieb transfer. To show it is the Segal conjecture isomorphism, it now suffices to calculate this first map on  $\pi_1$ . The second generator goes to the generator, since as we rove over the base  $\mathbb{R}\mathbb{P}^\infty$  the choice of which point in  $S^\infty$  is the privileged one gets switched. Similarly, we may build a closed loop in  $\|(E_{n+1})_+ \wedge *_{+} \wedge (\mathbb{Z}/2)_+^n\|$  which projects down to the generator in  $|\wedge *_{+} \wedge (\mathbb{Z}/2)_+^n|$ , by picking some fiberwise embedding

$$S^\infty \times_{\mathbb{R}\mathbb{P}^\infty} S^\infty \rightarrow \mathbb{R}^\infty \times \mathbb{R}\mathbb{P}^\infty$$

and a homotopy to its antipode. This gives a path through simplicial level 1. Taking the geometric realization and the diagonal, we get a path whose two endpoints coincide (the above multiplication rules tell us to restrict along  $p \mapsto (-p, p)$  at one end and  $p \mapsto (p, -p)$  at the other end). Projecting down to  $|\wedge *_{+} \wedge (\mathbb{Z}/2)_+^n|$ , we obviously get a generator. But this path also switches the choice of privileged point as we rove around it. Therefore it goes to the generator of  $\mathbb{R}\mathbb{P}^\infty$  and the proof is complete.

### 3.7.1 The general case of finite $p$ -groups

We begin by recalling a result from [LMM82]:

**Proposition 3.7.5.** *The composite of tom Dieck splitting and the Segal conjecture  $p$ -adic*

equivalence

$$\bigvee_{(H) \leq G} BWH_+ \xrightarrow{\sim} \mathbb{S}^G \xrightarrow{\sim} \mathbb{S}^{hG}$$

is on each summand a map

$$BWH_+ \longrightarrow \text{Map}(BG, \mathbb{S})$$

adjoint to

$$(BG \times BWH)_+ \longrightarrow \Omega^\infty S^\infty$$

which is a transfer along the bundle over  $BG \times BWH$  whose fiber is  $G/H$  and whose  $G \times WH$ -monodromy is given by  $G$  multiplying on the left and  $WH$  multiplying on the right (turned into a left action by inverting).

When  $G$  is abelian, this simplifies to

$$(B(G/H) \times BG)_+ \longrightarrow B(G/H)_+ \longrightarrow \Omega^\infty S^\infty$$

where the first map sends  $(g_1 + H, g_2) \mapsto (g_1 - g_2)$  and the second map is the  $G/H$ -transfer. For completeness we re-prove the result from the more standard formulations of the Segal conjecture.

*Proof.* The tom Dieck map

$$BWH_+ \longrightarrow \mathbb{S}^G$$

is a transfer along the  $G/H$ -bundle

$$G \times_{NH} EWH \longrightarrow BWH$$

followed by a collapse of the total space to a point. To define the transfer, we pick a  $G$ -representation  $V$  containing  $G/H$ , we embed the bundle into  $BWH \times V^{\oplus \infty}$ , and we pick an  $\epsilon > 0$  such that the  $\epsilon$ -balls about the various points of  $G/H \hookrightarrow V^{\oplus \infty}$  in each fiber are disjoint.

Now the above composite is given by

$$\begin{array}{ccc}
 BWH_+^{(n)} & \longrightarrow & (\Omega^{f(n)V} S_\epsilon^{f(n)V})^G \\
 \downarrow & & \downarrow \\
 BWH_+ & \longrightarrow & (\operatorname{colim}_n \Omega^{f(n)V} S_\epsilon^{f(n)V})^G \\
 & \searrow \text{dotted} & \downarrow \\
 & & (\operatorname{colim}_n \Omega^{n+f(n)V} S_\epsilon^{n+f(n)V})^G \\
 & & \downarrow \\
 & & \operatorname{Map}_*(EG_+, \operatorname{colim}_n \Omega^{n+f(n)V} S_\epsilon^{n+f(n)V})^G \\
 & \searrow \text{dotted} & \uparrow \sim \\
 & & \operatorname{Map}_*(EG_+, \operatorname{colim}_n \Omega^n S_\epsilon^n)^G \\
 & & \parallel \\
 & & \operatorname{Map}_*(BG_+, \Omega^\infty S_\epsilon^\infty)
 \end{array}$$

where  $f(n)$  is any sufficiently large increasing function of  $n$ . The dotted line lift-up-to-homotopy is formed by picking a  $G$ -equivariant homotopy between the two maps

$$\begin{array}{ccccccc}
 BWH_+ & \wedge & EG_+ & \wedge & S^\infty & \wedge & S^{\infty V} & \longrightarrow & S_\epsilon^\infty & \wedge & S_\epsilon^{\infty V} \\
 (x, & & y, & & z, & & w) & \mapsto & (z, & & w - \tilde{x}) \\
 (x, & & y, & & z, & & w) & \mapsto & (? , & & w)
 \end{array}$$

Here  $\tilde{x}$  is whichever lift of  $x$  to  $G \times_{NH} EWH$  is closer to  $w$ ; as long as  $\epsilon < 1$  such a lift exists or it is irrelevant because the map goes to the basepoint. The group  $G$  acts normally on  $y$  and  $w$ , as well as the second slot of the output. All other coordinates are fixed. To fill in the (?) and pick a homotopy, we first pick a fiberwise embedding

$$\left[ (G \times_{NH} EWH) \times EG \right] / G \hookrightarrow \mathbb{R}^\infty \times BWH \times BG$$

Not only does such a map exist, but the space of all possible choices for  $g$  is weakly contractible. Then take  $g$  to be the composite

$$(G \times_{NH} EWH) \times EG \longrightarrow \left[ (G \times_{NH} EWH) \times EG \right] / G \hookrightarrow \mathbb{R}^\infty \times BWH \times BG \xrightarrow{\pi_1} \mathbb{R}^\infty$$

and take our homotopy to be

$$(x, y, z, w, t) \mapsto (z - (\sin t)g(\tilde{x}, y), w - (\cos t)\tilde{x})$$

Here again the map is well-defined because, though there may be multiple values of  $(\cos t)\tilde{x}$  that are within  $\epsilon$  of  $w$ , in that case only one of those values of  $\tilde{x}$  gives a value of  $(\sin t)g(\tilde{x}, y)$  that is within  $\epsilon$  of  $z$ . It is also straightforward to check it is equivariant. The end of the homotopy is

$$(x, y, z, w) \mapsto (z - g(\tilde{x}, y), w)$$

and then we can restrict to  $\text{Map}_*(BG_+, \Omega^\infty S^\infty)$  by forgetting the  $w$  coordinate:

$$(x, y, z) \mapsto z - g(\tilde{x}, y)$$

Therefore the composite is as above. □

Now we need to describe how to pair assembly and coassembly together for  $A$ -theory and  $THH$ . This involves a few more steps than before:

$$\begin{array}{ccc} \bigvee_{(H) \leq G} BWH_+ \wedge K(\mathbb{S}) & \xrightarrow{\alpha} & \bigvee_{(H) \leq G} K(\Sigma_+^\infty WH) \\ & \xrightarrow{K(\iota)} & \bigvee_{(H) \leq G} K(\Sigma_+^\infty WH, \text{fin}) \\ & \xrightarrow{K(\text{trivial action})} & K(\Sigma_+^\infty NH, \text{fin}) \\ & \xrightarrow{K(G \times_{NH} -)} & K(\Sigma_+^\infty G, \text{fin}) \\ & \xrightarrow{c\alpha} & F(BG_+, K(\mathbb{S})) \end{array}$$

Put another way, for each conjugacy class of  $H \leq G$  we follow the diagram

$$\begin{array}{ccccc}
 A(BG) & \xrightarrow{K(\iota)} & \mathbb{V}(BG) & \xrightarrow{c\alpha} & F(BG_+, \mathbb{V}(*)) \\
 \uparrow \times_{NH} G & & \uparrow \times_{NH} G & & \\
 A(BNH) & \xrightarrow{K(\iota)} & \mathbb{V}(BNH) & & \\
 \uparrow \text{pullback} & & \uparrow & & \\
 BWH_+ \wedge A(*) & \xrightarrow{\alpha} & A(BWH) & \xrightarrow{K(\iota)} & \mathbb{V}(BWH)
 \end{array}$$

There is no obvious map  $A(BWH) \rightarrow A(BNH)$  making this commute; pullback doesn't work because  $BNH \rightarrow BWH$  has fiber  $BH$ , which is not finite. In both of these diagrams, we may replace  $K$ -theory with  $THH$ .

Next, we'll model the above maps of Waldhausen categories by using parametrized spectra. They all come about by simple pullback  $f^*$  and pushforward  $f_!$  functors. We use the two-sided bar construction to construct the following maps of covering spaces over  $BWH$ ,  $BNH$ , and  $BG$  underlying these functors:

$$\begin{array}{ccccccc}
 WH & \xleftarrow{=} & WH & \xrightarrow{=} & WH & \xrightarrow{\quad} & G/H \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 B(*, WH, WH) & \xleftarrow{\text{pullback}} & B(*, NH, WH) & \xrightarrow{\sim \text{pullback}} & B(*, G, G/H) & \xrightarrow{\sim \text{pushforward}} & B(*, G, G/H) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 B(*, WH, *) & \xleftarrow{\quad} & B(*, NH, *) & \xrightarrow{\sim} & B(*, G, G/NH) & \xrightarrow{\quad} & B(*, G, *) \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 BWH & & BNH & & BNH' & & BG
 \end{array}$$

By abuse of notation, we will refer to the bundle of fiberwise stable endomorphisms of the first three bundles by  $\widetilde{M}_{WH}(\mathbb{S})$ , and for the last bundle  $\widetilde{M}_{G/H}(\mathbb{S})$ . These have clear analogs  $\widetilde{W}_{WH}(\mathbb{S})$  and  $\widetilde{W}_{G/H}(\mathbb{S})$ , defined by replacing the products with sums. There is a homomorphism from  $WH$  into every one of these fiberwise endomorphism spectra, simply because all of our covering spaces above have a right  $WH$ -action which commutes with all the maps. Finally, as before we have

$$\Gamma_{BWH}(\widetilde{M}_{WH}) \simeq M_{WH}(\mathbb{S})^{hWH} \simeq \Sigma_+^\infty WH$$

coming from the  $WH$ -action we discussed above.

We build the following maps of Waldhausen categories:

$$\mathcal{E}x_{\text{fin}}(BWH) \longrightarrow \mathcal{E}x^{\text{fin}}(BWH) \xrightarrow{g^*} \mathcal{E}x^{\text{fin}}(BNH) \xleftarrow[\sim]{g^*} \mathcal{E}x^{\text{fin}}(BNH') \xrightarrow{g!} \mathcal{E}x^{\text{fin}}(BG)$$

These maps are all enhanced exact functors of enhanced Waldhausen categories, so they induce maps on  $K$ -theory and  $THH$  that commute with the cyclotomic trace. The arrows denoted  $\sim$  have the approximation property, so they induce an equivalence on  $K$ -theory and  $THH$ .

The  $THH$  of these categories is defined in two stages: first we form a spectral enrichment

$$C^{\mathcal{S}}(x, y) = \{C(x, \Sigma^n y)\}_{n=0}^{\infty}$$

These are naturally orthogonal spectra; by neglect of structure they are symmetric spectra. Then we form  $THH$  by the Hochschild-Mitchell cyclic nerve. We don't care about the cyclotomic structure here, so we will simply take the ordinary cyclic nerve and just make sure the terms are cofibrant spectra. This has a natural weak equivalence into the Hochschild-Mitchell nerve.

Now the covering spaces we defined above become objects in these Waldhausen categories which agree under the given maps:

$$\mathcal{E}x_{\text{fin}}(BWH) \longrightarrow \mathcal{E}x^{\text{fin}}(BWH) \xrightarrow{g^*} \mathcal{E}x^{\text{fin}}(BNH) \xleftarrow[\sim]{g^*} \mathcal{E}x^{\text{fin}}(BNH') \xrightarrow{g!} \mathcal{E}x^{\text{fin}}(BG)$$

Our spaces over these bases are all covering spaces, therefore  $h$ -fibrant. They become  $f$ -cofibrant when a disjoint basepoint section is added. This gives a privileged choice of object in each of the above Waldhausen categories, which are taken to each other by the chosen maps.

In the following diagram, the right-hand column is obtained by applying the cyclic nerve to the above maps of Waldhausen categories. The middle column is what we get by

restricting the cyclic nerve to the endomorphisms of our one privileged object.

$$\begin{array}{ccccc}
 |B_{\bullet}(*, WH, *)| & \xrightarrow{WH\text{-action}} & |N_{\bullet}^{\text{cyc}}\Gamma_{BWH}(\widetilde{M}_{WH}(\mathbb{S}))| & \longrightarrow & |N_{\bullet}^{\text{cyc}}\mathcal{E}x_{\text{fin}}(BWH)^{\mathcal{S}}| \\
 \parallel & & \parallel & & \downarrow \\
 |B_{\bullet}(*, WH, *)| & \xrightarrow{WH\text{-action}} & |N_{\bullet}^{\text{cyc}}\Gamma_{BWH}(\widetilde{M}_{WH}(\mathbb{S}))| & \longrightarrow & |N_{\bullet}^{\text{cyc}}\mathcal{E}x^{\text{fin}}(BWH)^{\mathcal{S}}| \\
 \parallel & & \downarrow \text{pullback} & & \downarrow \text{pullback} \\
 |B_{\bullet}(*, WH, *)| & \xrightarrow{WH\text{-action}} & |N_{\bullet}^{\text{cyc}}\Gamma_{BNH}(\widetilde{M}_{WH}(\mathbb{S}))| & \longrightarrow & |N_{\bullet}^{\text{cyc}}\mathcal{E}x^{\text{fin}}(BNH)^{\mathcal{S}}| \\
 \parallel & & \sim \uparrow \text{pullback} & & \sim \uparrow \text{pullback} \\
 |B_{\bullet}(*, WH, *)| & \xrightarrow{WH\text{-action}} & |N_{\bullet}^{\text{cyc}}\Gamma_{BNH'}(\widetilde{M}_{WH}(\mathbb{S}))| & \longrightarrow & |N_{\bullet}^{\text{cyc}}\mathcal{E}x^{\text{fin}}(BNH')^{\mathcal{S}}| \\
 \parallel & & \downarrow \text{pushforward} & & \downarrow \text{pushforward} \\
 |B_{\bullet}(*, WH, *)| & \xrightarrow{WH\text{-action}} & |N_{\bullet}^{\text{cyc}}\Gamma_{BG}(\widetilde{M}_{G/H}(\mathbb{S}))| & \longrightarrow & |N_{\bullet}^{\text{cyc}}\mathcal{E}x^{\text{fin}}(BG)^{\mathcal{S}}|
 \end{array}$$

Notice that our parametrized spectra are not made fibrant before taking sections! Spectrum level  $n$  is literally the sections of the  $n$ -fold fiberwise reduced suspension of a space.

We claim that the top composite is equivalent to the assembly map of the functor

$$X \mapsto |N_{\bullet}^{\text{cyc}}\mathcal{E}x_{\text{fin}}(X)^{\mathcal{S}}| \simeq |N_{\bullet}^{\text{cyc}}\mathcal{R}f_{\text{fin}}(X)^{\mathcal{S}}|$$

on spaces  $X \rightarrow BWH$ , evaluated at  $X = BWH$ . To see that the above map is indeed assembly, we use

$$\Gamma_{BWH}(\widetilde{M}_{WH}) \simeq M_{WH}(\mathbb{S})^{hWH} \simeq \Sigma_+^{\infty} WH$$

to rewrite the map as the composite

$$|B_{\bullet}WH| \hookrightarrow |N_{\bullet}^{\text{cyc}}\Sigma_+^{\infty} WH| \xrightarrow{\sim} |N_{\bullet}^{\text{cyc}}\Gamma_{BWH}(\widetilde{M}_{WH}(\mathbb{S}))|$$

where the first map of bar complexes

$$X = |B_{\bullet}WH|, \quad LX \simeq |N_{\bullet}^{\text{cyc}}WH|$$

is at level  $n$

$$(g_1, \dots, g_n) \mapsto (g_n^{-1} \dots g_1^{-1}; g_1, \dots, g_n)$$

and it is a standard fact that this models the inclusion of constant loops  $X \rightarrow LX$ , which

is visibly the assembly map for the functor  $F(X) = \Sigma_{\dagger}^{\infty} LX$ .

Now we turn to coassembly. The bottom map of the above diagram extends to a natural transformation of homotopy functors

$$F(X) = |N_{\bullet}^{\text{cyc}} \Gamma_X(\widetilde{M}_{G/H}(\mathbb{S}))| \longrightarrow |N_{\bullet}^{\text{cyc}} \mathcal{E}x^{\text{fin}}(X)^{\mathcal{S}}| = G(X)$$

for CW-complexes  $X \rightarrow BG$  up to homotopy equivalence. When  $X = *$  we get

$$F(*) = |N_{\bullet}^{\text{cyc}} M_{G/H}(\mathbb{S})| = |N_{\bullet}^{\text{cyc}} \mathcal{E}x^{\text{fin}}(*)^{\mathcal{S}}(G/H_+, G/H_+)| \longrightarrow |N_{\bullet}^{\text{cyc}} \mathcal{E}x^{\text{fin}}(*)^{\mathcal{S}}| = G(*)$$

We will now show that

- This map  $F(*) \rightarrow G(*)$  is an equivalence, so the natural transformation  $F \rightarrow G$  gives an equivalence of linear approximations  $P_1 F \xrightarrow{\sim} P_1 G$ .
- We can build a zig-zag of homotopy functors  $F \xleftarrow{\sim} F' \rightarrow F'' \xleftarrow{\sim} \dots \rightarrow F^{(n)}$  which all induce equivalences on linear approximations  $P_1 F^{(i)}$ .
- The last functor  $F^{(n)}$  in our zig-zag is linear, therefore the composite

$$BWH \wedge \mathbb{S} \rightarrow F(BG) \rightarrow G(BG) \rightarrow P_1 G(BG)$$

is isomorphic in the homotopy category to the composite

$$BWH \wedge \mathbb{S} \rightarrow F(BG) \xleftarrow{\sim} F'(BG) \rightarrow \dots \rightarrow F^{(n)}(BG)$$

For the first bullet point, we use the Morita-invariance statement from the Blumberg Mandell paper. For the second bullet point, we realize that the equivalence we just discussed is the top map in a homotopy-commuting square

$$\begin{array}{ccc} |N_{\bullet}^{\text{cyc}} M_{G/H}(\mathbb{S})| & \longrightarrow & |N_{\bullet}^{\text{cyc}} \mathcal{E}x^{\text{fin}}(*)^{\mathcal{S}}| \\ \uparrow \sim & & \uparrow \sim \\ |N_{\bullet}^{\text{cyc}} W_{G/H}(\mathbb{S})| & & |N_{\bullet}^{\text{cyc}} \mathcal{E}x^{\text{fin}}(*)^{\mathcal{S}}(*_+, *_+)| \\ \sim \downarrow \text{trace} & & \downarrow \cong \\ |N_{\bullet}^{\text{cyc}} \mathbb{S}| & \xleftarrow{\cong} & \mathbb{S} \end{array}$$

To define  $F'$  and  $F''$  we build maps of parametrized spectra that do  $M_{G/H}(\mathbb{S}) \xleftarrow{\sim} W_{G/H}(\mathbb{S}) \rightarrow \mathbb{S}$  on each fiber:

$$\begin{array}{ccc}
 |B_{\bullet}(*, WH, *)| & \xrightarrow{WH\text{-action}} & |N_{\bullet}^{\text{cyc}}\Gamma_{BG}(\widetilde{M}_{G/H}(\mathbb{S}))| \\
 \uparrow \sim & & \uparrow \sim \\
 \|B_{\bullet}(*, WH, *)\| & \xrightarrow{WH\text{-action}} & \|N_{\bullet}^{\text{cyc}}\Gamma_{BG}(\widetilde{M}_{G/H}(\mathbb{S}))\| \\
 \parallel & & \downarrow \text{interchange} \\
 & & \|\Gamma_{BG}(N_{\bullet}^{\text{cyc}}\widetilde{M}_{G/H}(\mathbb{S}))\| \\
 & & \downarrow \Gamma(R) \\
 & & \|\Gamma_{BG}(RN_{\bullet}^{\text{cyc}}\widetilde{M}_{G/H}(\mathbb{S}))\| \\
 & & \downarrow \sim \text{small spheres} \\
 \|B_{\bullet}(*, WH, *)\| & \xrightarrow{WH\text{-action}} & \|\Gamma_{BG}(RN_{\bullet}^{\text{cyc}}\widetilde{M}_{G/H}(\mathbb{S}'))\| \\
 \uparrow \sim & & \uparrow \sim \\
 \|E_{\bullet} \times B_{\bullet}(*, WH, *)\| & \xrightarrow{\Phi} & \|\Gamma_{BG}(RN_{\bullet}^{\text{cyc}}\widetilde{W}_{G/H}(\mathbb{S}'))\| \\
 & & \downarrow \text{trace} \\
 & & \|\Gamma_{BG}(RS')\| \xrightarrow{\sim} F(BG_+, RS')
 \end{array}$$

We will explain the extra notation and the commutativity of the lowermost square, but before we do that, notice that this fulfills both the second and third bullet point of our above list. The right-hand column gives our zig-zag of homotopy functors (replace  $\Gamma_{BG}$  with  $\Gamma_X$ ) over  $G$ , which

Now for the new notation. First,  $R$  denotes the fibrant replacement of (fiberwise) ordinary spectra given by

$$R\mathcal{E}_n = \text{colim}_k \Omega_B^k \mathcal{E}_{n+k}$$

In general we only want to do this when the levels of our spectra are already fibrations over  $B$ .

Next, we define  $E_n$  to be the space of fiberwise embeddings

$$E_n \subset \text{Emb}_{BG}(B(*, G, G/H)_{BG}^{\times n+1}, \mathbb{R}^{\infty} \times BG) \times \text{Map}(BG, (0, 1])^{n+1}$$

consisting of only those embeddings  $e$  and choices of positive reals  $(\epsilon_0, \dots, \epsilon_n)$  with the following properties. On each fiber  $F \rightarrow B(*, G, G/H) \rightarrow BG$ , the embedding  $e$  must extend to a map

$$\mathbb{R}\langle F \rangle^{\oplus n+1} \hookrightarrow \mathbb{R}^\infty$$

which is a composite of scaling by a positive real number  $r_i$  in each of the  $\mathbb{R}\langle F \rangle$ -coordinates, followed by an orthogonal map. Here we include  $F^{\times n+1} \hookrightarrow \mathbb{R}\langle F \rangle^{\oplus n+1}$  by sums of basis vectors,

$$(x_0, \dots, x_n) \longrightarrow [x_0] \oplus \dots \oplus [x_n]$$

Put another way, it is the  $(n+1)$ -fold Cartesian product of the map  $F \rightarrow \mathbb{R}\langle F \rangle$  that takes each point to its corresponding basis vector. Notice that  $\mathbb{R}\langle F \rangle$  is a  $WH$ -representation under the usual action on the basis vectors. Finally, the reals  $(\epsilon_0, \dots, \epsilon_n)$  must be chosen so that the  $(n+1)$ -fold product of discs with these radii gives a polydisc about each point of  $F^{n+1}$  in  $\mathbb{R}\langle F \rangle^{\oplus n+1}$ , and all these polydiscs are disjoint.

Notice that  $E_n$  is contractible. We will use these spaces to thicken  $\|B_\bullet(*, WH, *)\|$ , with the aim of building a map which strictly commutes with faces. This is the same technique used by Cohen in [Coh04] to build strictly associative multiplications on Thom spectra.

Next we use  $\mathbb{S}'$  to denote the “small spheres” spectrum which at level  $n$  is

$$\mathbb{S}'_n = \mathbb{R}^n \times (0, 1] / \{(x, \epsilon) : \|x\| \geq \epsilon\}$$

The multiplication on  $\mathbb{S}'$  is by concatenation of coordinates, and taking the minimum of the two values of  $\epsilon$ . Under this convention, the collapse map  $\mathbb{S} \rightarrow \mathbb{S}'$  sending  $x \mapsto (x, 1)$  is a map of strictly associative unital ring spectra. Even better, the smash  $\mathbb{S}' \wedge \mathbb{S}'$  is homeomorphic to

$$\mathbb{S}'' = \mathbb{S}^{[2]}, \quad \mathbb{S}''_n = \mathbb{R}^n \times (0, 1] \times (0, 1] / \{(x, \epsilon_1, \epsilon_2) : \|x\| \geq \min(\epsilon_1, \epsilon_2)\}$$

and so on for higher smash powers. In the above diagram, we do this construction in the obvious fiberwise way. It is worth pointing out that when we build  $\widetilde{W}_{G/H}(\mathbb{S}')$  we think of our matrix entries as maps  $\mathbb{S} \rightarrow \mathbb{S}'$ , so that the fibers are homeomorphic to

$$W_{G/H}(\mathbb{S}') \cong \bigvee^{(G/H)^2} \mathbb{S}'$$

and similarly the fibers of  $\widetilde{W}_{G/H}(\mathbb{S}')^{\wedge_{BG} n+1}$  are

$$W_{G/H}(\mathbb{S}')^{\wedge n+1} \cong \bigvee^{(G/H)^{2(n+1)}} \mathbb{S}^{[n+1]}$$

Next we define a map  $\Phi$  of semi-simplicial spectra which on level  $n$  is the composite

$$\begin{aligned} (E_n)_+ \wedge *_{+} \wedge (WH)_+^n &\longrightarrow \Gamma_{BG} \left( R \left( \mathbb{S}_{BG}^{[n+1]} \wedge_{BG} (WH \times B(*, G, G/H))_{+BG}^{\times n+1} \right) \right) \\ &\longrightarrow \Gamma_{BG} \left( R \widetilde{W}_{G/H}(\mathbb{S}')^{\wedge_{BG} n+1} \right) \end{aligned}$$

The first map is easy: a point in the left-hand side gives us  $(n+1)$  elements of  $WH$  of the form

$$(w_n^{-1} \dots w_2^{-1} w_1^{-1}), w_1, w_2, \dots, w_n$$

and we use them to select the  $(n+1)$  elements of  $WH$  on the right; then the remaining map is a Pontryagin-Thom collapse onto the discrete set of points given by the embedding in  $E_n$ :

$$v \mapsto (v - e(x_0, \dots, x_n), x_0, \dots, x_n)$$

The  $R$  outside the  $N^{\text{cyc}}$  gives us the sphere coordinates needed to make this collapse map canonically and explicitly; the  $\epsilon_i$ s give us the coordinates in  $\mathbb{S}^{[n+1]}$ , though only their minimum matters when giving the sphere coordinate. In total, the first map of our composite is

$$\begin{aligned} &(e, \epsilon_0, \dots, \epsilon_n, w_n^{-1} \dots w_1^{-1}, w_1, \dots, w_n) \\ &\longrightarrow [(v, x) \mapsto (v - e(x_0, \dots, x_n), (w_n^{-1} \dots w_1^{-1}, x_0), \dots, (w_n, x_n))] \end{aligned}$$

The second map is also straightforward: it comes from the map of fiberwise spectra

$$\begin{aligned} \Sigma_{+BG}^{\infty}(WH \times B(*, G, G/H)) &\longrightarrow \Sigma_{+BG}^{\infty} \left( B(*, G, G/H) \times_{BG} B(*, G, G/H) \right) = \widetilde{W}_{G/H}(\mathbb{S}) \\ &(w, x) \mapsto (w \cdot x, x) \end{aligned}$$

where  $w \cdot x := xw^{-1}$  is the right action of  $WH$  on  $B(*, G, G/H)$  turned into a left action

by inversion. The total composite  $\Phi$  is now

$$(e, \epsilon_0, \dots, \epsilon_n, w_n^{-1} \dots w_1^{-1}, w_1, \dots, w_n) \\ \longrightarrow [(v, x) \mapsto (v - e(x_0, \dots, x_n), (w_n^{-1} \dots w_1^{-1} \cdot x_0, x_0), \dots, (w_n \cdot x_n, x_n))]$$

where  $(x_0, \dots, x_n)$  is the unique point in the fiber such that  $v - e(x_0, \dots, x_n) < \min(\epsilon_0, \dots, \epsilon_n)$ .

Next we describe the  $k$ th face map

$$d_k : (E_n)_+ \wedge *_+ \wedge (WH)_+^n \longrightarrow (E_{n-1})_+ \wedge *_+ \wedge (WH)_+^{n-1} \\ (e, \epsilon_0, \dots, \epsilon_n, w_n^{-1} \dots w_1^{-1}, w_1, \dots, w_n) \mapsto \\ (e \circ \Delta_{k, w_{k+1}}, \epsilon_0, \dots, \min(\epsilon_k, \epsilon_{k+1}), \dots, \epsilon_n, w_n^{-1} \dots w_1^{-1}, w_1, \dots, w_k w_{k+1}, \dots, w_n)$$

$$\Delta_{k, w} : B(*, G, G/H)_{BG}^{\times n} \longrightarrow B(*, G, G/H)_{BG}^{\times n+1} \\ (\dots, x_{k-1}, x_k, x_{k+1}, \dots) \mapsto (\dots, x_{k-1}, w \cdot x_k, x_k, x_{k+1}, \dots)$$

It's straightforward to check these face maps are associative and so define a semi-simplicial space. We carefully constructed  $E_n$  above so that the properties we required of the embedding would be preserved under restricting to a diagonal in this way. ( $\Delta_{k, w}$  is a composition of a scaling in one of the  $n$  coordinates and an orthogonal map. The new scaling factor  $r'_k$  is derived from the old ones by  $\sqrt{r_k^2 + r_{k+1}^2}$ .) We did not simply take the space of all embeddings, because the orthogonality assumptions are needed below to make the square above  $\Phi$  commute up to explicit homotopy.

Finally, our definition of  $\Phi$  respects the face maps because of how matrices in  $W_{G/H}(\mathbb{S})$  multiply: the multiplication is on each fiber just a map of wedges of spheres

$$\bigvee \mathbb{S} \longrightarrow \bigvee \mathbb{S}$$

which either preserves each sphere or collapses it to 0, based on whether our two adjacent matrices (with one nonzero entry each) can multiply to give something nonzero. Post-composing a transfer with such a collapse map gives another transfer, in which we have struck out some of the points from our embedded  $(G/H)^{n+1}$ , leaving behind an embedded  $(G/H)^n$ . The points that remain are those whose  $k$ th and  $(k+1)$ st coordinates are of the

form

$$\begin{aligned} (\dots, w_{k+1} \cdot x, x, \dots) &\mapsto (\dots, (w_k w_{k+1} \cdot x, w_{k+1} \cdot x), (w_{k+1} \cdot x, x), \dots) \\ &\in \Sigma_{+BG}^\infty(B(*, G, G/H) \times_{BG} B(*, G, G/H))^{\times n+1} \end{aligned}$$

But this is exactly the image of  $\Delta_{k, w_{k+1}}$ , so our map of semi-simplicial spectra does indeed commute with faces.

Now we will show that the square above  $\Phi$  commutes up to homotopy. The main idea, and the motivation behind the above definition of  $\Phi$ , is that the fiberwise pretransfer

$$\mathbb{S}_{BG} \longrightarrow \Sigma_{+BG}^\infty EG \cong \bigvee^G \tilde{\mathbb{S}}$$

lifts the diagonal map  $\mathbb{S} \longrightarrow \prod^G \tilde{\mathbb{S}}$  up to homotopy. The homotopy collapses our embedding down to the origin, since in the product, the neighborhoods of the various points are allowed to overlap.

Now let's make this idea precise in our setting. Consider the square from the master diagram

$$\begin{array}{ccc} \|B_\bullet(*, WH, *)\| & \xrightarrow{WH\text{-action}} & \|\Gamma_{BG}(RN_\bullet^{\text{cyc}} \widetilde{M}_{G/H}(S'))\| \\ \uparrow \sim & & \uparrow \sim \\ \|E_\bullet \times B_\bullet(*, WH, *)\| & \xrightarrow{\Phi} & \|\Gamma_{BG}(RN_\bullet^{\text{cyc}} \widetilde{W}_{G/H}(S'))\| \end{array}$$

Observe that the composite along the bottom is given by

$$\begin{aligned} &(E_n)_+ \wedge *_{+} \wedge (WH)_+^n \\ \xrightarrow{\Phi} &\Gamma_{BG} \left( R \left( \mathbb{S}_{BG}^{[n+1]} \wedge_{BG} (WH \times B(*, G, G/H))_{+BG}^{\times n+1} \right) \right) \\ \longrightarrow &\Gamma_{BG} \left( R \left( \mathbb{S}_{BG}^{[n+1]} \wedge_{BG} \text{Map}_{BG}(B(*, G, G/H), B(*, G, G/H)_{+BG})^{\wedge n+1} \right) \right) \\ \longrightarrow &\Gamma_{BG} \left( R \left( \text{Map}_{BG} \left( B(*, G, G/H), \mathbb{S}'_{BG} \wedge_{BG} B(*, G, G/H)_{+BG} \right)^{\wedge n+1} \right) \right) \end{aligned}$$

Our map of bundles is, on each fiber, an  $(n+1)$ -fold smash of

$$\begin{aligned} \mathbb{S}' \wedge (WH \times G/H)_+ &\longrightarrow \mathbb{S}' \wedge \text{Map}_*(G/H_+, G/H_+) \longrightarrow \text{Map}_*(G/H_+, \mathbb{S}' \wedge G/H_+) \\ (v, \epsilon, w, g) &\mapsto [g \mapsto (v, \epsilon, w \cdot g), \quad \text{all others} \mapsto *] \end{aligned}$$

In other words, each pair  $(w, g)$  goes to the partially-defined map  $G/H \rightarrow G/H$  defined only on  $g$  by  $(g \mapsto w \cdot g)$ .

In total, the composite is

$$\begin{aligned} & (e, (\epsilon_i)_{i=0}^n, w_n^{-1} \dots w_1^{-1}, w_1, \dots, w_n) \\ \mapsto & [(v, x) \mapsto (v - e(x_0, \dots, x_n), (\epsilon_i)_{i=0}^n, (x_0 \mapsto w_n^{-1} \dots w_1^{-1} \cdot x_0), \dots, (x_n \mapsto w_n \cdot x_n))] \\ \mapsto & [(v, x) \mapsto (\rho_x, \tilde{v}) \wedge (x_0 \mapsto (v_0 - e_0(x_0), \epsilon_0, w_0 \cdot x_0)) \wedge \dots] \end{aligned}$$

Here  $\rho_x \in O(N)$ ,  $N \gg |G/H|^{n+1}$ , is any orthogonal map taking the first  $|G/H|^{n+1}$  coordinates of  $\mathbb{R}^N$  to the image of  $\mathbb{R}\langle F \rangle^{\oplus n+1}$  under  $e|_x$ . Pulling back  $v \in \mathbb{R}^\infty$  along  $\rho_x$ , we get coordinates  $(v_0, v_1, \dots, v_n, \tilde{v})$ . Though  $\rho_x$  is not unique, its behavior on the first  $|G/H|^{n+1}$  coordinates is unique and so are  $v_0$  through  $v_n$ . (It may also be chosen in a way that varies continuously with  $x$ .) The non-uniqueness changes the leftovers  $\tilde{v}$ , though this gives the same point in the smash product of orthogonal spectra. As before,  $(x_0, \dots, x_n)$  is a point in the fiber over  $x$ ; it is for now the unique point in the fiber such that  $v - e(x_0, \dots, x_n) < \min(\epsilon_0, \dots, \epsilon_n)$ , but it will soon be non-unique and the above partially-defined maps will become fully-defined maps. Define the homotopy  $\Phi_t$  to take the above input to

$$[(v, x) \mapsto (\rho, \tilde{v}) \wedge (x_0 \mapsto (v_0 - te_0(x_0), (1-t) + t\epsilon_0, w_0 \cdot x_0)) \wedge \dots]$$

When  $t = 1$  this gives the same as before. When  $t = 0$  it simplifies to

$$[(v, x) \mapsto (\rho, \tilde{v}) \wedge (x_0 \mapsto (v_0, 1, w_0 \cdot x_0)) \wedge \dots]$$

which is the same in the smash product of orthogonal spectra as

$$[(v, x) \mapsto (\text{id}, v) \wedge (x_0 \mapsto (*, 1, w_0 \cdot x_0)) \wedge \dots]$$

which is exactly the top route of our square, the  $WH$ -action. Of course, this is a homotopy on each simplicial level; it agrees with the face maps and so passes to a homotopy on the thick geometric realization.

This finishes commutativity of the master diagram. Every map either

- is an equivalence

- comes from one of our original maps of Waldhausen categories (i.e. the pullback and pushforward maps)
- gives an equivalence on linear approximations for the functor that replaces the “ $BG$ ” with  $X$  (i.e. the interchange and fiberwise trace maps)

Therefore our maps of Waldhausen categories, when paired with assembly and coassembly, result in the composite

$$\|E_{\bullet} \times B_{\bullet}(*, WH, *)\| \xrightarrow{\Phi} \|\Gamma_{BG}(RN_{\bullet}^{\text{cyc}} \widetilde{W}_{G/H}(\mathbb{S}'))\| \xrightarrow{\text{trace}} \|\Gamma_{BG}(RS')\| \xrightarrow{\sim} F(BG_+, RS')$$

We’re almost there! This is adjoint to a transfer

$$\|E_{\bullet} \times B_{\bullet}(*, WH, *)\| \times BG \longrightarrow \Omega^{\infty} \mathbb{S}'$$

and we just need to check that the  $G/H$ -bundle we are transferring along has the correct  $G \times WH$ -monodromy. The bundle is built by taking

$$\begin{array}{c} B(*, G, G/H)_{BG}^{\times n+1} \\ \downarrow \\ \Delta^n \times E_n \times B_n(*, WH, *) \times BG \end{array}$$

and restricting to those points of the form

$$\begin{array}{c} (w_1 \dots w_n \cdot \tilde{x}, \dots, w_{n-1} w_n \cdot \tilde{x}, w_n \cdot \tilde{x}, \tilde{x}), \quad (\tilde{x} \in B(*, G, G/H)) \mapsto (x \in BG) \\ \downarrow \\ (t, e, \epsilon, w_1, \dots, w_n, x) \end{array}$$

This gives a set of size  $|G/H|$  over every point of the base, respecting the face maps that glue simplices together, so that they glue together to a  $G/H$ -bundle. To calculate the monodromy under  $G$ , we restrict to simplicial level 0. The closed loop in  $\pi_1(BG)$  corresponding to  $g \in G$  reshuffles the fiber of  $B(*, G, G/H)$  by the  $G$ -action, as expected. To calculate the monodromy under  $WH$ , we look at simplicial level 1. Let  $e$  be a fiberwise embedding

$$B(*, G, G/H)_{BG}^{\times 2} \longrightarrow \mathbb{R}^{\infty} \times BG$$

and let  $e_t$  be a homotopy from  $e$  to  $e \circ (w \cdot -)$  for some fixed  $w \in WH$ . Ignoring the choice of  $\epsilon$  because it is irrelevant, we pick a closed loop through  $\|E_\bullet \times B_\bullet(*, WH, *)\|$  which traverses simplicial level 1 by

$$(e_t, \epsilon, w^{-1}, w, * \in BG)$$

The 0th face is

$$(e_0 \circ \Delta_{0,w}, \epsilon, 1, *)$$

and the 1st face is

$$(e_1 \circ \Delta_{1,w^{-1}}, \epsilon, 1, *)$$

These agree because

$$\begin{aligned} e_0(\Delta_{0,w}(y)) &= e(w \cdot y, y) \\ e_1(\Delta_{1,w^{-1}}(y)) &= e(w \cdot (y, w^{-1} \cdot y)) \\ &= e(w \cdot y, y) \end{aligned}$$

So we get a closed loop. It is clear that we can define the monodromy along this loop using the homotopy  $e_t$ . This gives the expected action of  $WH$  on the fiber, and the proof is complete.

### 3.7.2 Lifting the theorem to $TC$

We would like to also prove that the composite along this top row is an equivalence:

$$\begin{array}{ccccccc} \mathbb{R}P_+^\infty \wedge TC(*) & \xrightarrow{\alpha} & TC(\mathbb{R}P^\infty) & \xrightarrow{TC(\iota)} & TC(\mathcal{P}'(\Sigma_+^\infty \mathbb{Z}/2)) & \xrightarrow{c\alpha} & F(\mathbb{R}P_+^\infty, TC(*)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}P_+^\infty \wedge THH(\mathbb{S}) & \xrightarrow{\alpha} & THH(\Sigma_+^\infty \mathbb{Z}/2) & \xrightarrow{THH(\iota)} & THH(\mathcal{P}'(\Sigma_+^\infty \mathbb{Z}/2)) & \xrightarrow{c\alpha} & F(\mathbb{R}P_+^\infty, THH(\mathbb{S})) \end{array}$$

Although the left-hand map on the top row is easily checked to be split, we cannot verify and indeed suspect it is not the case that the right-hand map is split after  $p$ -completion.

This arises from the fact that  $F(\mathbb{RP}_+^\infty, THH(\mathbb{S}))$  is not a cyclotomic spectrum:

$$\begin{aligned} F(\mathbb{RP}_+^\infty, THH(\mathbb{S}))^{C_2} &\cong F(\mathbb{RP}_+^\infty, THH(\mathbb{S})^{C_2}) \\ &\simeq F(\mathbb{RP}_+^\infty, \mathbb{S} \vee \Sigma_+^\infty \mathbb{RP}^\infty) \\ &\simeq F(\mathbb{RP}_+^\infty, \mathbb{S}) \vee F(\mathbb{RP}_+^\infty, \Sigma_+^\infty \mathbb{RP}^\infty) \end{aligned}$$

which by work of May et al. on stable maps between classifying spaces cannot fit into a cofiber sequence with  $F(\mathbb{RP}_+^\infty, THH(\mathbb{S}))_{hC_2}$  and  $F(\mathbb{RP}_+^\infty, THH(\mathbb{S}))$ .

One may define a notion of universal approximation of the pre-cyclotomic spectrum  $F(\mathbb{RP}_+^\infty, THH(\mathbb{S}))$  by a cyclotomic spectrum  $T$ , essentially because homotopy colimits commute with geometric fixed points and the change-of-group isomorphism. We conjecture that the resulting map

$$THH(\mathcal{P}'(\Sigma_+^\infty \mathbb{Z}/2)) \longrightarrow T$$

is split surjective as a map of cyclotomic spectra, which implies that it gives a split surjection on  $TC$ .

## Appendix A

# A treatment of twisted Poincaré duality

This appendix represents work done in 2011 on giving a geometrically flavored write-up of twisted Poincaré duality. We prove Poincaré duality for a noncompact manifold  $M$  with boundary, with coefficients given by a parametrized spectrum over  $M$ . The equivalence is given by a scanning map and the method of proof is by “local to global,” both ideas that are known around this time but difficult to find explicitly in the literature. In the process, we carefully define the notion of cohomology of  $M$  with compact supports when the coefficients are taken in spectra. We remark on some pathologies that can arise if one is too careless with the way this space is defined.

## A.1 Definitions and statement of the theorem

We begin with definitions and notation. By “spaces” we always mean  $k$ -spaces.

**Definition A.1.1.** • We will usually take the  $n$ -sphere  $S^n$  to be the one-point compactification of  $\mathbb{R}^n$ . The suspension  $\Sigma^n X$  is defined to be  $S^n \wedge X$ . Fix a homeomorphism  $(0, 1) \rightarrow \mathbb{R}$ ; we will sometimes use this homeomorphism to identify  $(0, 1)^n$  with  $\mathbb{R}^n$ , and by extension  $I^n/\partial I^n$  with  $S^n$ .

- We say that  $E$  is a *retractive space* or *ex-space* over  $B$  if there are two maps  $B \xrightarrow{s} E \xrightarrow{p} B$  giving  $B$  as a retract of  $E$ . Equivalently, we think of  $E$  as a space over  $B$  with a chosen section. This is a parametrized notion of a based space, since every fiber  $E_b = p^{-1}(b)$  has a basepoint given by  $s(b)$ . We need a few related concepts:
  - Write  $E/B$  as shorthand for  $E/s(B)$ . This is simply a based space, no longer parametrized over  $B$ . In the notation of [MS06] this space is denoted  $r_1 E$ .
  - Let  $\Gamma_B(E)$  be the space of sections of the map  $p$ , with basepoint  $s$ . Again, this is no longer parametrized over  $B$ . In the notation of [MS06] this space is denoted  $r_* E$ .
  - Given a section  $B \xrightarrow{\sigma} E$ , we say  $\sigma$  *vanishes* at  $b \in B$  if  $\sigma(b) = s(b)$ . The *support* of  $\sigma$  is then defined to be the closure of the complement of the vanishing set. If  $B$  is locally compact Hausdorff, then let  $\Gamma_B^c(E)$  be the set of sections with compact support in  $B$ , topologized as a subspace of  $\Gamma_K(E \cup_{s(B)} K)$ , where  $K$  is any compactification of  $B$ .
  - For any two ex-spaces  $X$  and  $Y$  over  $B$ , there is a notion of fiberwise product  $X \times_B Y$  and fiberwise smash  $X \wedge_B Y$ . The definitions are straightforward as long as we describe what happens on each fiber:

$$(X \times_B Y)_b \cong X_b \times Y_b$$

$$(X \wedge_B Y)_b \cong X_b \wedge Y_b$$

- The fiberwise suspension  $\Sigma_B^n E$  is defined to be the fiberwise smash product  $(S^n \times B) \wedge_B E$ . We will also write it as  $\Sigma^n E$  if no confusion will arise. If  $\xi \rightarrow B$  is a vector bundle, its one-point compactification is denoted  $B^\xi$ , and its fiberwise

one-point compactification is denoted  $S^\xi$ . The twisted suspension of  $E \rightarrow B$  by  $\xi$  is

$$\Sigma^\xi E = S^\xi \wedge_B E$$

- A *parametrized prespectrum*  $E \rightarrow B$  is a sequence of ex-spaces  $E_n \rightarrow B$ , together with maps  $\Sigma_B E_n \rightarrow E_{n+1}$  from the fiberwise suspension of each space into the next.
- Our theorem below will not be true if we do not assume some sort of uniform behavior in  $E$ . We therefore say that a parametrized prespectrum  $E \rightarrow B$  is *nice* if each projection  $E_n \rightarrow B$  is a Hurewicz fibration, and in addition each  $\Sigma_B E_n \rightarrow E_{n+1}$  is a cofibration in the fiberwise sense. That is, there is a fiberwise retract of  $E_{n+1} \times I$  onto  $\Sigma_B E_n \times I \cup E_{n+1} \times \{0\}$ . This implies that every level  $E_n$  is an *ex-fibration* as defined in the next section.
- Given a parametrized prespectrum  $E \rightarrow B$ , we have four ways to create ordinary prespectra:
  - Take the fibers  $E_b$ .
  - Take the quotient  $E/B$ .
  - Take the space of sections  $\Gamma_B(E)$ , or compactly supported sections  $\Gamma_B^c(E)$ .

Just apply one of these constructions to each of the levels  $E_n$ . This gives a sequence of based spaces  $\{X_n\}$ . Then the fiberwise structure maps  $\Sigma_B E_n \rightarrow E_{n+1}$  give non-fiberwise structure maps  $\Sigma X_n \rightarrow X_{n+1}$ .

- Following [CK09], we define *homology prespectrum of  $B$  with coefficients in  $E$*  to be

$$H_\bullet(B; E) = E/B$$

This can be naturally extended to homology of parametrized spaces or spectra over  $B$ . Similarly, define *cohomology* and *cohomology with compact supports* as

$$H^\bullet(B; E) = \Gamma_B(E) \quad H_c^\bullet(B; E) = \Gamma_B^c(E)$$

**Remark.** The prespectrum  $H_\bullet(B; E)$  contains more information than the usual homology

groups  $H_*(B)$ . In particular, if we take  $E = H\mathbb{Z} \times B$  to be a trivial fibered Eilenberg-MacLane spectrum, then

$$\pi_q(H_\bullet(B; E)) \cong \pi_q(B_+ \wedge H\mathbb{Z}) \cong H_q(B; \mathbb{Z})$$

Similarly for cohomology:

$$\pi_{-q}(H^\bullet(B; E)) \cong \pi_{-q}(\text{Map}_*(B_+, H\mathbb{Z})) \cong [B_+, K(\mathbb{Z}, q)] \cong H^q(B; \mathbb{Z})$$

Each bundle of groups with fiber  $\mathbb{Z}$  gives a twisted Eilenberg-MacLane spectrum  $E$ . The homotopy groups of  $H_\bullet(B; E)$  and  $H^\bullet(B; E)$  give the usual notion of twisted homology and cohomology, because they satisfy the same axioms. Philosophically, these prespectra are an intermediary between  $B$  and  $H_*(B)$ , so they play the same role that the chains  $C_*(B)$  play in singular homology. This framework is strictly more powerful than the more classical framework for twisted homology and cohomology, since the classical framework only allows flat bundles, whereas ours has no such restriction.

**Definition A.1.2.** • Let  $M$  be a closed  $m$ -manifold. Let  $e : M \hookrightarrow \mathbb{R}^n$  be an embedding with normal bundle  $\nu_n$ . Define the parametrized prespectrum  $S^{-TM} \rightarrow M$  as follows. When  $p < n$ ,  $S_p^{-TM} = M$ , which in each fiber is just a single point. When  $p = n$ ,  $S_p^{-TM}$  is the fiberwise one-point compactification of the normal bundle  $\nu_n$ . When  $p > n$ , embed  $\mathbb{R}^n$  into  $\mathbb{R}^p$  as the last  $n$  coordinates, then take  $S_p^{-TM}$  to be the fiberwise one-point compactification of the normal bundle  $\nu_p \cong \mathbb{R}^{p-n} \oplus \nu_n$ . The structure maps  $\Sigma_M S_p^{-TM} \rightarrow S_{p+1}^{-TM}$  come from the canonical map  $\mathbb{R} \oplus \nu_p \xrightarrow{\cong} \nu_{p+1}$ . The notation  $S^{-TM}$  suggests that each fiber is a sphere; if we quotient out by the basepoint section, we get the usual Thom prespectrum  $\Sigma^{-n} \Sigma^\infty M^{\nu_n}$ .

- For each sufficiently small  $\epsilon \in (0, 1)$ , let  $S_\epsilon^{-TM}$  be the  $\epsilon$ -disc bundle of the normal bundle of  $M$ , quotiented out fiberwise by its boundary. There is an obvious map  $S^{-TM} \rightarrow S_\epsilon^{-TM}$ , and on each fiber this map is a level homotopy equivalence of prespectra. We can also use the exponential map to identify  $S_\epsilon^{-TM}$  with the  $\epsilon$ -tubular neighborhood of  $M$ , quotiented out by its boundary. To deal with noncompact manifolds, we also allow continuous functions  $\epsilon : M \rightarrow (0, 1)$ , and construct  $S_\epsilon^{-TM}$  in the obvious way.
- If  $M$  has a choice of metric we let  $S(TM)$  denote the unit sphere bundle of the tangent

bundle of  $M$ .

Since  $H_\bullet(M)$  and  $H^\bullet(M)$  capture more information than the usual homology and cohomology of  $M$ , it is natural to ask whether they have a relationship that generalizes Poincaré duality. This is our main theorem:

**Theorem A.1.3** (Twisted Poincaré Duality). *Let  $M$  be any (smooth, second-countable) manifold with no boundary. If  $E$  is a parametrized prespectrum over  $M$  which is nice in the sense outlined above, then there is a (weak) stable homotopy equivalence of prespectra*

$$\begin{aligned} H_\bullet(M; \Sigma^{-TM} E) &\simeq H_c^\bullet(M; E) \\ (S^{-TM} \wedge_M E)/M &\simeq \Gamma_M^c(E) \end{aligned}$$

*The equivalence is a zig-zag of  $\pi_*$ -isomorphisms, so it corresponds to an actual map in the stable homotopy category of prespectra as described in [MMSS01].*

**Remark.** This result is not new; it appears in [CK09] and is proven in ([MS06], Ch. 18). Our proof here is relatively self-contained and follows the notation of [CK09]. It appears that this proof is significantly different in at least two ways. First, we work with cohomology with compact supports, which allows us to closely follow the classical proof of Poincaré duality, and easily extends the known result to noncompact manifolds. Second, we use an explicit formula for the Alexander map  $M^\nu \wedge M_+ \rightarrow S^n$ , which is well-suited for proving multiplicative versions of Poincaré duality relating intersection products in homology to cup products in cohomology.

**Remark.** The theorem does not assume that  $M$  is oriented or even orientable; this comes at the cost of the  $\Sigma^{-TM}$  on the left-hand side. To get rid of that term, we must choose an “orientation” of  $M$  with respect to  $E$ . The theorem does not assume that  $M$  is compact, though if it is then the right-hand side is equal to  $H^\bullet(M; E)$ .

**Corollary A.1.4** (Atiyah Duality). *If  $M$  is compact then the Alexander map gives a stable homotopy equivalence of prespectra  $\Sigma^{-n} M^{\nu_n} \simeq F(M_+, \mathbb{S})$ .*

*Proof.* Take  $E = \mathbb{S} \times M$  in main theorem. Tracing the map through, it is given by

$$\begin{aligned} M^{\nu_n} \wedge M_+ &\longrightarrow \mathbb{S}'_n \\ (z, y) &\mapsto (\exp(z) - e(y), \epsilon) \end{aligned}$$

when  $\|z\| \leq \epsilon$ , and is constant for  $\|z\| \geq \epsilon$ . This is the Alexander map as described in [Coh04].  $\square$

**Remark.** Since  $M_+$  and  $M^\nu$  are finite CW complexes, the two spectra  $\Sigma^\infty M_+$  and  $M^{-TM}$  are strongly dualizable. So the simple form of Atiyah duality given above implies that  $\Sigma^\infty M_+ \simeq F(M^{-TM}, \mathbb{S})$ . It also implies that if  $h$  is any prespectrum, then  $M^{-TM} \wedge h \simeq F(M_+, h)$ . This also follows from the main theorem by taking  $E = M \times h$ .

**Definition A.1.5.** • Each prespectrum  $h$  defines a (reduced) homology and cohomology theory on based spaces:

$$\begin{aligned}\tilde{h}_q(X) &= \pi_q(X \wedge h) \\ \tilde{h}^q(X) &= \pi_{-q}(\text{Map}_*(X, h)) = [X, h]_q\end{aligned}$$

We get unreduced homology theories adding a disjoint basepoint to an unbased space, and using the above definition.

- An  $E$ -orientation of an  $n$ -dimensional real vector bundle  $\xi \rightarrow M$  is a choice of isomorphism of parametrized spectra  $\Sigma^\xi E \rightarrow \Sigma^n E$ .
- An  $h$ -orientation is an  $E$ -orientation where  $E = h \times M$ .

We leave it to the reader that each  $H\mathbb{Z}$ -orientation of the normal bundle  $\nu$  corresponds to an orientation of the manifold  $M$ .

**Corollary A.1.6** (Oriented Poincaré Duality). *An  $h$ -orientation of the normal bundle of  $M$  yields an isomorphism*

$$h_{m-q}(M) \xrightarrow{\cong} h^q(M)$$

**Corollary A.1.7** (Classical Poincaré Duality). *If  $M$  is oriented,*

$$H_{m-q}(M; \mathbb{Z}) \xrightarrow{\cong} H^q(M; \mathbb{Z})$$

*If  $M$  not necessarily oriented,*

$$H_{m-q}(M; \mathbb{Z}/2) \xrightarrow{\cong} H^q(M; \mathbb{Z}/2)$$

## A.2 Excision lemmas for parametrized spaces

In this section we dig a bit deeper into the point-set topology of the spectra  $E/B$ ,  $\Gamma_B(E)$ , and  $\Gamma_B^c(E)$  and prove that they have the correct excisive behavior, so that we can prove Poincaré duality by an inductive argument along the open subsets of  $M$ . In particular, we have to be careful with our choice of how to topologize the sections with compact supports  $\Gamma_B^c(E)$ . The obvious subspace topology  $\Gamma_B^c(E) \subset \Gamma_B(E)$  has pathological behavior when  $B$  is not compact, because one may “push” nontrivial sections of  $E$  off the boundary of  $B$  and make them disappear in a continuous way.

We also adopt some additional definitions for parametrized spaces following [MS06], Def. 8.1.1:

- Definition A.2.1.**
- $E$  is *well-sectioned* if it satisfies the parametrized homotopy extension property over  $B$ : A map  $E \rightarrow X$  over  $B$  and a homotopy over  $B$  of that map on  $s(B)$  extends to a homotopy over  $B$  of the map on  $E$ . Equivalently, there is a retraction of  $E \times I$  onto the mapping cylinder of  $s$ ,  $E \cup_B (B \times I)$ , that agrees with the map into  $B$ .
  - $E$  is *well-fibered* if it satisfies the parametrized homotopy lifting property over  $B$ : Any family of paths  $K \rightarrow B^I$  equipped with a family of lifts of the starting point  $K \rightarrow E$  extends to a family of lifts  $K \rightarrow E^I$ , subject to the additional constraint that a lift starting in  $s(B)$  must extend to the canonical lift that stays in  $s(B)$ . Equivalently, there is a path-lifting function  $E \times_B B^I \rightarrow E^I$  over  $B$  and under  $B^I$ .
  - $E$  is an *ex-fibration* if it is well-sectioned and well-fibered.

From [MS06] section 8.3, every space is  $h$ -equivalent to an ex-fibration. (An  $h$ -equivalence is a non-fiberwise homotopy equivalence, i.e. a map  $E \rightarrow X$  over  $B$  with reverse  $X \rightarrow E$  not over  $B$  whose compositions are homotopic to the identity.)

Next we elaborate a bit on our definition of “cohomology with compact supports” from the previous section. If  $B$  is locally compact Hausdorff, let  $\Gamma_B^c(E)$  be the set of sections over  $B$  with compact support. As usual, the support is the closure of the complement of the vanishing set. To topologize  $\Gamma_B^c(E)$ , we let  $K$  be any compact Hausdorff space equipped with an open inclusion  $i : B \hookrightarrow K$ . (Note that the usual one-point compactification is one valid choice for  $K$ , since  $B$  is Hausdorff and so its compact subsets are closed.) Let  $E \cup_B K$  be the pushout of  $B \xrightarrow{s} E$  along  $i$ . Then every section of  $E$  over  $B$  with compact support

extends by zero to a section of  $E \cup_B K$  over  $K$ . This extended section is continuous on  $K$  because it is continuous on  $B$  and the complement of its support in  $B$ , both of which are open sets that cover  $K$ . Moreover, the support of this new section in  $K$  is equal to its original support in  $B$ . Going the other way, any section over  $K$  whose support is contained in  $B$  comes from a section over  $B$  with compact support. So give  $\Gamma_K(E \cup_B K)$  the compact-open topology, and identify  $\Gamma_B^c(E)$  with the subset of all sections whose support in  $K$  is actually contained in  $B$ ; give it the compactly-generated subspace topology. (Note that we take the subspace topology, then apply  $k$ . Since the subspace is not open or closed, if we applied  $k$  first and then took the subspace topology, we would need to apply  $k$  again; however this gratuitous use of  $k$  would still result in the same topology.)

Finally, if  $B$  is locally compact Hausdorff and  $A$  is a closed subspace then define  $\Gamma_{(B,A)}^c(E)$  to be the subspace of  $\Gamma_B^c(E)$  consisting of (compactly supported) sections that vanish on  $A$ .

**Lemma A.2.2.** *The topology defined on  $\Gamma_B^c(E)$  does not depend on the choice of  $K$ .*

*Proof.* It suffices to show the subspace topology coming from  $\Gamma_K(E \cup_B K)$  does not depend on  $K$ , before we apply  $k$ . So let  $K_1$  and  $K_2$  be two Hausdorff compactifications of  $B$ . In the topology coming from  $K_1$ , a basic open set has the form

$$W(K, U) \cap \Gamma_B^c(E)$$

where  $K \subset K_1$  is compact and  $U \subset E \cup_B K_1$  is open. This is equal to the basic open set coming from  $K_2$

$$W(\overline{K \cap B}, U \cap E) \cap \Gamma_B^c(E)$$

where the closure of  $K \cap B$  is taken in  $K_2$ . (Note that this closure contains no points in  $B$  other than  $K \cap B$ . Note also that we need the maps  $B \rightarrow K_1$  and  $B \rightarrow K_2$  to be open inclusions to conclude that  $U \cap E$  is open.) An element either set is a section  $B \rightarrow E$  with compact support contained in  $B$ , sending  $K \cap B$  into  $U \cap E$ . By symmetry, every basic open set coming from  $K_2$  is equal to a basic open set coming from  $K_1$ , so the two topologies coincide.  $\square$

**Proposition A.2.3.** *Each open inclusion  $f : A \rightarrow B$  (of locally compact Hausdorff spaces) induces a continuous “extension by zero” map  $\Gamma_A^c(f^*E) \rightarrow \Gamma_B^c(E)$ .*

*Proof.* Let  $B \rightarrow K$  be an open inclusion of  $B$  into a compact Hausdorff space; by the last lemma, we can compose this with  $A \rightarrow B$  to get an open inclusion  $A \rightarrow K$  giving the right topology on  $\Gamma_A^c(f^*E)$ . Note that  $f^*E \rightarrow E$  is an open inclusion. Before applying  $k$ , a basic open set in  $\Gamma_B^c(E)$  is

$$W(C, U) \cap \Gamma_B^c(E)$$

where  $C \subset K$  is compact and  $U \subset E \cup_B K$  is open. Therefore  $U \cap E$  and  $U \cap K$  are open. Then  $U \cap f^*E$  is an open subset of  $f^*E$ , so  $(U \cap f^*E) \cup (U \cap K)$  is an open subset of  $f^*E \cup_A K$ . So the preimage of this basic open set in  $\Gamma_A^c(f^*E)$  is the basic open set

$$W(C, (U \cap f^*E) \cup (U \cap K)) \cap \Gamma_A^c(f^*E)$$

and the constructed map is continuous.  $\square$

**Proposition A.2.4.** *If  $B$  is compact Hausdorff,  $E$  is well-fibered, and  $A \hookrightarrow B$  is a closed subspace with a mapping cylinder neighborhood, then  $\Gamma_{B-A}^c(E|_{B-A}) \rightarrow \Gamma_{(B,A)}(E)$  is a homotopy equivalence.*

*Proof.* Note that the map is the above “extension by zero” map  $\Gamma_{B-A}^c(E|_{B-A}) \rightarrow \Gamma_B^c(E)$ , which happens to land in the subspace  $\Gamma_{(B,A)}^c(E)$ . We know that  $B$  contains a mapping cylinder neighborhood  $N = C \times [0, 1)$ , where  $C \times 0$  is collapsed to  $A$ . Take the rescaling map

$$C \times [0, 1] \rightarrow C \times [1/2, 1] \subset C \times [0, 1]$$

This is clearly homotopic to the identity. Over this homotopy, we can define a (basepoint-section-preserving) homotopy  $H_t : E|_{C \times [0, 1]} \rightarrow E|_{C \times [0, 1]}$ . So  $H_0$  is the identity, and  $H_1$  sends a point over  $(c, t)$  to a point over  $(c, (1+t)/2)$ . Now given a section  $g : B \rightarrow E$  that vanishes at  $A$ , create a new section  $\tilde{g}$  that agrees with  $g$  outside  $N = C \times [0, 1)$ , and on  $C \times [0, 1]$  it is given by

$$\tilde{g}(c, t) \mapsto \begin{cases} s(c, t) & t \leq 1/2 \\ H_1(g(c, 2t - 1)) & t \geq 1/2 \end{cases}$$

This is clearly continuous and homotopic to  $g$  in a natural way. Moreover, it has compact support away from  $A$ , the set where  $t = 0$ . This gives the desired homotopy equivalence.  $\square$

**Proposition A.2.5.** *If  $E$  is well-sectioned, then  $E_b$ ,  $E/B$ , and  $\Gamma_B^c(E)$  are all well-based*

(nondegenerate basepoints). If  $B$  is compact then  $\Gamma_B(E)$  is well-based as well.

*Proof.* The retract  $E \times I \rightarrow E \cup (B \times I)$  restricts to a retract  $E_b \times I \rightarrow E_b \cup (* \times I)$ , which is the condition that  $E_b$  be well-based. It also descends to a retract  $(E/B) \times I \rightarrow (E/B) \cup (B/B \times I)$ , which is the condition that  $E/B$  be well-based.

For  $\Gamma_B(E)$  the argument is more involved. Let  $i$  be the composite

$$E \times I \rightarrow (E \times 0) \cup (B \times I) \rightarrow I$$

and let  $j$  be the composite

$$E \times I \rightarrow (E \times 0) \cup (B \times I) \rightarrow E$$

Then define the self-map

$$\begin{aligned} \Gamma_B(E) \times I &\rightarrow \Gamma_B(E) \times I \\ (f, t) &\mapsto [j(f, t), \min_{b \in B} i(f(b), t)] \end{aligned}$$

Note that  $j(f, t)$  is a section of  $E \rightarrow B$  because  $j$  is a map over  $B$ . Also, note that the second coordinate is not in general continuous or even well-defined; we alleviate this problem by assuming that  $B$  is compact, or by restricting to sections with compact support. When  $\min_{b \in B} i(f(b), t)$  is positive, the  $j(f, t)$  is a constant section; therefore this map factors through the subspace  $(\Gamma_B(E) \times 0) \cup (* \times I)$ . This gives the desired retraction

$$\Gamma_B(E) \times I \rightarrow \Gamma_B(E) \cup (* \times I)$$

so  $\Gamma_B(E)$  has a nondegenerate basepoint when  $B$  is compact.

If  $B$  is locally compact Hausdorff and  $K$  is a compactification of  $B$ , then  $i$  and  $j$  extend continuously to  $E \cup_B K$ , giving a retraction

$$\Gamma_K(E \cup_B K) \times I \rightarrow \Gamma_K(E \cup_B K) \cup (* \times I)$$

Note that if  $f$  vanishes at  $b$ , then  $j(f, t)$  also vanishes at  $b$ . So if  $f$  has compact support, then  $j(f, t)$  has support that is closed and contained in the support of  $f$ , therefore compact as well. So the above retraction restricts to a retraction

$$\Gamma_B^c(E) \times I \rightarrow \Gamma_B^c(E) \cup (* \times I)$$

so  $\Gamma_B^c(E)$  has a nondegenerate basepoint for all (locally-compact Hausdorff)  $B$ .  $\square$

**Proposition A.2.6.** *If  $E$  is well-sectioned, and  $U_1 \subset U_2 \subset \dots$  is a sequence of inclusions of open sets in  $B$  with union  $U \subset B$ , then  $(E|_U)/B$  is a homotopy colimit of  $(E|_{U_i})/B$ . If in addition  $B$  is locally compact Hausdorff then  $\Gamma_U^c(E)$  is a homotopy colimit of  $\Gamma_{U_i}^c(E)$ .*

*Proof.* Clearly the colimit of a sequence of open inclusions of ordinary spaces gives a homotopy colimit, as compactness of  $S^k$  forces

$$\pi_k(U) \cong \operatorname{colim}_n \pi_k(U_n)$$

For the parametrized version,  $(E|_{U_i})/B$  is not quite an open subspace of  $(E|_U)/B$ . To fix this, let  $i$  be the composite

$$E \times I \longrightarrow (E \times 0) \cup (B \times I) \longrightarrow I$$

Then  $V = i^{-1}((0, 1]) \cap (E \times 1) \subset E$  is an open subspace of  $E$ , containing  $B$ , with a fiberwise deformation retraction onto  $B$  provided by  $j$ :

$$E \times I \longrightarrow (E \times 0) \cup (B \times I) \longrightarrow E$$

Clearly then  $(E|_{U_i \cup V})/B$  is a sequence of open subspaces of  $E/B$  with union  $(E|_U \cup V)/B$ , so  $(E|_U \cup V)/B$  is a homotopy colimit of  $(E|_{U_i \cup V})/B$ . To get back to our original sequence of spaces, use the retract on  $E \times I$  to establish a homotopy inverse to the inclusion maps

$$(E|_{U_i \cup V})/B \leftarrow j(E|_{U_i \times 1})/B \hookrightarrow (E|_{U_i})/B$$

Therefore these inclusions are homotopy equivalences, so  $(E|_{U_i})/B \hookrightarrow (E|_{U_i \cup V})/B$  is a weak equivalence. Therefore  $(E|_U)/B$  is a homotopy colimit of  $(E|_{U_i})/B$ .

For the sections, as usual, we pick a compactification  $K$  of  $B$ . Define  $\tilde{\Gamma}_{U_i}^c(E)$  to be the sections of  $E \cup_B K$  over  $K$  which have compact “support” in  $U_i$ , where a section “vanishes” if it lands in the open subspace  $V \cup_B K$ . Clearly there is an inclusion map  $\Gamma_{U_i}^c(E) \subset \tilde{\Gamma}_{U_i}^c(E)$ , and the above techniques show that this map is a weak equivalence. So it suffices to check that  $\tilde{\Gamma}_U^c(E)$  is a homotopy colimit of  $\tilde{\Gamma}_{U_i}^c(E)$ . This will follow if we can show that  $\tilde{\Gamma}_{U_i}^c(E) \subset \tilde{\Gamma}_U^c(E)$  is an open subspace.

Take a point in this subspace; that’s a section of  $E \cup_B K$  which lands in  $V \cup_B K$  outside

of a compact set  $C \subset U_i$ . Since  $B$  is locally compact Hausdorff, standard point-set topology tells us that we can find an open set  $O \subset U_i$  containing  $C$  such that  $\overline{O} \subset U_i$ . Then consider  $W(K - O, V \cup_B K)$ , the basic open subset of  $\tilde{\Gamma}_U^c(E)$  of all sections that send the complement of  $O$  into  $V$ . This open subset contains our original point because  $C \subset O$ , and is contained in the subspace  $\tilde{\Gamma}_{U_i}^c(E)$  because any section that vanishes outside  $O$  has compact support in  $\overline{O} \subset U_i$ . Therefore  $\tilde{\Gamma}_{U_i}^c(E) \subset \tilde{\Gamma}_U^c(E)$  is an open subspace and we are done.  $\square$

**Proposition A.2.7.** *Let  $X$  be a metric space containing two open sets  $U$  and  $V$ , and let  $E$  be an  $ex$ -space over  $X$ . Then this square is based co-Cartesian*

$$\begin{array}{ccc} (E|_U)/X & \longrightarrow & (E|_{U \cup V})/X \\ \uparrow & & \uparrow \\ (E|_{U \cap V})/X & \longrightarrow & (E|_V)/X \end{array}$$

and this square is Cartesian:

$$\begin{array}{ccc} \Gamma_U(E) & \longleftarrow & \Gamma_{U \cup V}(E) \\ \downarrow & & \downarrow \\ \Gamma_{U \cap V}(E) & \longleftarrow & \Gamma_V(E) \end{array}$$

If  $X$  is locally compact Hausdorff and  $E$  is well-sectioned, this square is also Cartesian:

$$\begin{array}{ccc} \Gamma_U^c(E) & \longrightarrow & \Gamma_{U \cup V}^c(E) \\ \uparrow & & \uparrow \\ \Gamma_{U \cap V}^c(E) & \longrightarrow & \Gamma_V^c(E) \end{array}$$

*Proof.* Construct a continuous function  $\phi : U \cup V \rightarrow [0, 1]$  that is 0 on  $U - V$  and 1 on  $V - U$ . For example, we can define  $\phi$  on  $U \cap V$  by  $\phi(x) = \frac{d(x, U - V)}{d(x, U - V) + d(x, V - U)}$ , rescaled from  $(0, \infty)$  to  $(0, 1)$ .

For the first square, the based homotopy pushout is the the reduced double mapping cylinder  $(E|_U \vee (E|_{U \cap V} \wedge I_+) \vee E|_V)/X$ , which maps into the final space  $(E|_{U \cup V})/X$  by forgetting the cylinder coordinate and using an inclusion map. We map  $(E|_{U \cup V})/X$  back into the mapping cylinder, using  $\phi$  as the cylinder coordinate. This is a homotopy equivalence because the composition of these two maps in one direction is equal to the identity,

and in the other direction we can easily give an explicit homotopy to the identity. So the first square is based homotopy co-Cartesian.

For the second square, the homotopy pullback consists of sections over  $U$  and  $V$  that agree on  $U \cap V$  up to a chosen homotopy. Given a section over  $U \cup V$ , we restrict to  $U$  and to  $V$ , and use the constant homotopy. Going backwards, we can take two sections over  $U$  and  $V$  and patch them together over  $U \cap V$  using  $\phi$  and the given homotopy between the sections on  $U \cap V$ , yielding a continuous section on  $U \cup V$ . Again, these are clear based homotopy inverses, so the second square is homotopy Cartesian.

For the third square, the homotopy pullback consists of compactly-supported sections over  $U$  and  $V$  and a chosen homotopy between them over  $U \cup V$ . Given a section over  $U \cap V$ , we push forward to  $U$  and  $V$  and use the constant homotopy. Going backwards, suppose we have a homotopy of sections  $h(t, x) : I \times U \cup V \rightarrow E$  such that

- For each time  $t$ ,  $x \mapsto h(t, x)$  has compact support in  $U \cup V$
- $h(0, x)$  has support contained in  $V$
- $h(1, x)$  has support contained in  $U$

Since  $E$  is well-sectioned, we can use  $i$  and  $j$  from a previous lemma:

$$\begin{aligned} i : E \times I &\longrightarrow E \cup (B \times I) \longrightarrow I \\ j : E \times I &\longrightarrow E \cup (B \times I) \longrightarrow E \end{aligned}$$

Then we create this section over  $U \cap V$ :

$$f(x) = \begin{cases} s(x) & \phi(x) \leq 1/4 \\ j(h(2(\phi(x) - 1/4), x), 1) & 1/4 \leq \phi(x) \leq 3/4 \\ s(x) & \phi(x) \geq 3/4 \end{cases}$$

It's easy to create explicit homotopies making this an inverse homotopy equivalence:

$$H_1(x, u) = \begin{cases} s(x) & \phi(x) \leq u/4 \\ j(h(((4 - 2u)/4)\phi(x) - u/4, x), u) & u/4 \leq \phi(x) \leq (4 - u)/4 \\ s(x) & \phi(x) \geq (4 - u)/4 \end{cases}$$

$$H_2((x, t), u) = \begin{cases} s(x) & \phi(x) \leq u/4 \\ j(h(2u(\phi(x) - 1/4) + (1 - u)t), x), u) & u/4 \leq \phi(x) \leq (4 - u)/4 \\ s(x) & \phi(x) \geq (4 - u)/4 \end{cases}$$

so long as we can guarantee that any section of the form

$$g(x) = j(h(c_1\phi(x) + c_2, x), c_3), \quad c_3 > 0$$

actually has compact support contained in  $U \cup V$ . To prove that  $g$  has compact support, first note that  $\Gamma_{U \cup V}^c(E)$  is topologized so that  $h(t, x)$  must extend by 0 continuously to  $I \times K$ , where  $K$  is a compact Hausdorff space including  $U \cup V$ . If we consider  $i$  and  $j$  as functions on  $E \times \{c_3\}$ , then  $i$  is zero whenever  $j$  is nonvanishing, hence  $i(h(\dots), c_3)$  is zero whenever  $g = j(h(\dots), c_3)$  is nonvanishing. Now  $i$  extends continuously to  $(E \times \{c_3\}) \cup_{B \times \{c_3\}} K$  by  $c_3 \neq 0$ , so  $i(h(\dots), c_3)$  is a continuous function  $K \rightarrow [0, 1]$  that is constant and nonzero on  $K - (U \cup V)$ , and 0 anywhere  $g$  vanishes. Therefore the support of  $g$  in  $K$  is contained in the inverse image of 0, which is entirely in  $U \cup V$ . Therefore the support of  $g$  in  $U \cup V$  is compact. So our homotopy equivalence is well-defined, and the third square is Cartesian.  $\square$

### A.3 Proof of the theorem

Before we dive into the proof, here is an outline of the main argument. Motivated by [Coh04], we use the explicit formula for the Alexander map

$$\begin{aligned} \alpha : \mathbb{R}^n / (\mathbb{R}^n - \text{tube}_\epsilon(M)) \wedge M_+ &\longrightarrow \mathbb{R}^n / (\mathbb{R}^n - \text{ball}_\epsilon(0)) \\ \alpha(x, y) &= x - e(y) \end{aligned}$$

which we expect to give a  $\pi_*$ -isomorphism of prespectra

$$\Sigma^\infty \mathbb{R}^n / (\mathbb{R}^n - \text{tube}_\epsilon(M)) \longrightarrow \text{Map}_*(M_+, \Sigma^\infty \mathbb{R}^n / (\mathbb{R}^n - \text{ball}_\epsilon(0)))$$

We use the exponential map to relate the spectrum on the left to  $S^{-TM}$ , and we use a homotopy equivalence to relate the spectrum on the right to  $\text{Map}_*(M_+, \mathbb{S})$ . The bundle of prespectra  $E$  comes along for the ride, and we arrive at a well-defined map

$$\alpha : (S^{-TM} \wedge_M \mathcal{E}) / M \simeq \Gamma_M(\mathcal{E})$$

We prove that  $\alpha$  is a  $\pi_*$ -isomorphism in three steps.

First, we define a restriction  $\alpha|_U$  for each open subset  $U \subset M$ . This requires taking sections over  $U$  with compact support, denoted  $\Gamma_U^c(E)$ . We must change  $\alpha$  by a homotopy for the map to still be defined in this case. Second, we prove  $\alpha|_U$  is a  $\pi_*$ -isomorphism when  $U$  is just a ball. In this case, the map becomes

$$\Sigma^{-m}\mathbb{S} \wedge E_x \longrightarrow \Omega^m \Sigma^m(\Sigma^{-m}\mathbb{S} \wedge E_x)$$

We might expect that this map is the unit of the adjunction between  $\Sigma^m$  and  $\Omega^m$ ; this is very nearly the case, and it is enough to demonstrate that the map is a  $\pi_*$ -isomorphism. Third, we use the excision lemmas from the last section to “glue together” the isomorphisms from the second step, and show that  $\alpha$  is a  $\pi_*$ -isomorphism on all of  $M$ . This parallels the use of the Mayer-Vietoris sequence in the usual proof of Poincaré duality:

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_c^k(U \cap V) & \xrightarrow{(1,-1)} & H_c^k(U) \oplus H_c^k(V) & \longrightarrow & H_c^k(U \cup V) \longrightarrow \dots \\ & & \updownarrow & & \updownarrow & & \updownarrow \\ \dots & \longrightarrow & H_{n-k}(U \cap V) & \xrightarrow{(1,-1)} & H_{n-k}(U) \oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(U \cup V) \longrightarrow \dots \end{array}$$

*Proof.* Throughout we will use ordinary prespectra and handcrafted smash products; see §7, 8, 9, 11 of [MMSS01] for some useful basic properties of these objects. Define the “small spheres spectrum”  $\mathbb{S}'$  to be

$$\mathbb{S}'_n = (\mathbb{R}^n \times (0, 1)) / \{(x, \epsilon) : \|x\| \geq \epsilon\}$$

The map  $\mathbb{S}^1 \wedge \mathbb{S}'_n \longrightarrow \mathbb{S}'_{n+1}$  is just the quotient of the usual map  $\mathbb{R} \times \mathbb{R}^n \times (0, 1) \xrightarrow{\cong} \mathbb{R}^{n+1} \times (0, 1)$  that concatenates coordinates. The multiplication maps are also defined by concatenating coordinates in  $\mathbb{R}^n$ , while the value of  $\epsilon$  in the product is the minimum of the  $\epsilon$ s in the two factors. This defines an orthogonal commutative ring spectrum  $\mathbb{S}'$ . For each  $\epsilon \in (0, 1)$ , there is a map of orthogonal ring spectra  $\mathbb{S} \longrightarrow \mathbb{S}'$  by  $x \mapsto (x, \epsilon)$ . This map is a level homotopy equivalence. It follows that

$$\Gamma_M(\mathbb{S} \wedge_M E) \longrightarrow \Gamma_M(\mathbb{S}' \wedge_M E)$$

is also a level homotopy equivalence. (A homotopy inverse  $\mathbb{S}'_n \longrightarrow \mathbb{S}_n$  induces a homotopy

inverse of the above map at level  $n$ .) It's also straightforward to show that

$$\Gamma_M(\mathbb{S} \wedge_M E) \longrightarrow \Gamma_M(E)$$

is a stable homotopy equivalence. (Here we are taking handcrafted smash products of prespectra, so the map is not automatically an isomorphism. We prove that it's a stable homotopy equivalence using the same argument that when  $X$  is a prespectrum,  $\mathbb{S} \wedge X \longrightarrow X$  is a stable homotopy equivalence. We simply compare the colimit systems that define the homotopy groups on each side.)

Therefore it suffices to construct a  $\pi_*$ -isomorphism

$$\alpha : (S^{-TM} \wedge_M E)/M \xrightarrow{\sim} \Gamma_M(\mathbb{S}' \wedge_M E)$$

for then the final map is the zig-zag

$$(S^{-TM} \wedge_M E)/M \xrightarrow{\sim} \Gamma_M(\mathbb{S}' \wedge_M E) \xleftarrow{\sim} \Gamma_M(\mathbb{S} \wedge_M E) \xrightarrow{\sim} \Gamma_M(E)$$

Choose a proper embedding  $e : M \hookrightarrow \mathbb{R}^n$ . Choose a continuous function  $\epsilon : M \longrightarrow (0, 1)$  such that a closed  $\epsilon$ -neighborhood of  $e(M) \subset \mathbb{R}^n$  is a tubular neighborhood, every  $2\epsilon$ -ball in  $M$  is geodesically convex (using the metric induced by  $e : M \hookrightarrow \mathbb{R}^n$ ). Let  $\psi$  be a *local connection on  $E$* , by which we mean any lift in the square

$$\begin{array}{ccc} E \times_M S(TM) \times \{0\} & \xrightarrow{\quad} & E \\ \downarrow & \nearrow \psi & \downarrow \\ E \times_M S(TM) \times [0, 2\epsilon) & \xrightarrow{t \rightarrow \exp(t)} & M \end{array}$$

Such a  $\psi$  exists, by inductively showing that a lift exists in this square

$$\begin{array}{ccc} \Sigma_M E_{n-1} \times_M S(TM) \times [0, 2\epsilon) \cup E_n \times_M S(TM) \times \{0\} & \xrightarrow{\quad} & E \\ \downarrow & \nearrow & \downarrow \\ E_n \times_M S(TM) \times [0, 2\epsilon) & \xrightarrow{t \rightarrow \exp(t)} & M \end{array}$$

This lift exists because the right-hand vertical is a Hurewicz fibration and the left-hand vertical is a DR-pair. This fact and the excision lemmas above are the two places where we really use the assumption that  $E \longrightarrow B$  is *nice* as defined in the first section.

Given an open proper subset  $U \subset M$  and a point  $x \in U$ , let  $B$  be the open ball of radius  $\frac{1}{2}d(x, M - U)$ . Let  $\delta_U(x)$  to be the distance in  $\mathbb{R}^n$  from  $e(M - B)$  to the slice of the tubular neighborhood over  $x$ . (If  $U = M$  then set  $\delta_M(x) = +\infty$ .) Then  $\delta_U(x)$  is just a little bit less than  $\frac{1}{2}d(x, M - U)$ , but could be much less at a point where the embedding of  $M$  turns sharply. Still, it is continuous:

**Lemma A.3.1.** *Given an open proper subset  $U \subset M$  with compact closure and a point  $x \in U$ , let  $B_x \subset M$  be the open ball of radius  $\frac{1}{2}d(x, M - U)$ . Let  $D_x$  be the slice of the closed  $\epsilon$  tubular neighborhood over  $x$ . Let  $\delta_U(x)$  to be the distance in  $\mathbb{R}^n$  from  $e(M - B_x)$  to  $D_x$ . Then  $\delta_U(x)$  is a continuous positive function for  $x \in U$ .*

*Proof.* Since  $\bar{U}$  is compact, there is an upper bound on the second derivative of every unit-speed geodesic through  $\bar{U}$ . Given  $\epsilon' > 0$ , use this bound to fix  $\delta > 0$  such that  $d(x, y) < \delta$  implies that the furthest two points of the slice over  $x$  and the slice over  $y$  is less than  $\frac{1}{3}\epsilon'$ . If  $d(x, y)$  is smaller than  $\frac{1}{3}\epsilon'$ , then  $B_x$  and  $B_y$  have radii differing by less than  $\frac{1}{3}\epsilon'$ . So if we take a point  $w \in M - B_y$  and  $v \in D_y$  such that  $\|w - v\| = \delta_U(y)$ , then  $w$  is within  $\frac{2}{3}\epsilon'$  of a point  $w' \in M - B_x$ , and  $v$  is within  $\frac{1}{3}\epsilon'$  of a point in  $D_x$ , so  $\delta_U(x) < \delta_U(y) + \epsilon'$ . Similarly,  $\delta_U(y) < \delta_U(x) + \epsilon'$ . Therefore  $\|\delta_U(x) - \delta_U(y)\| < \epsilon'$  when  $d(x, y) < \min(\frac{1}{3}\epsilon', \delta)$ . Therefore  $\delta_U$  is continuous.  $\square$

Now given an open subset  $U \subset M$  (possibly all of  $M$ ), define the Alexander map

$$\alpha_{p,q}|_U : (S^{\nu_p} \wedge_M E_q)/M|_U \longrightarrow \Gamma_U^c(S'_p \wedge_M E_q)$$

$$\alpha(z_x, w_x) = \begin{cases} \{y \mapsto ((\exp(z_x) - e(y), \min(\delta_U(x), \epsilon)), \psi_{x \rightarrow y}(w_x))\} & \|z_x\| < \epsilon \\ \text{zero section} & \|z_x\| \geq \epsilon \end{cases}$$

Here the notation  $\psi_{x \rightarrow y}$  suggests that we choose the unique geodesic connecting  $x$  to  $y$  within the ball of radius  $2\epsilon(x)$  about  $x$ , and use the direction specified by that geodesic and  $\psi$  to map the fiber of  $E_q$  over  $x$  to the fiber of  $E_q$  over  $y$ . Note that if  $y$  is at least  $\frac{1}{2}d(x, M - U)$  away from  $x$ , and  $\|z_x\| < \epsilon$ , then  $\|\exp(z_x) - e(y)\|$  will be at least  $\delta_U(x)$ , and so  $y$  will be sent to the basepoint. So the image of  $(z_x, w_x)$  under  $\alpha_{p,q}|_U$  is a section supported in a closed ball about of radius  $\frac{1}{2}d(x, M - U)$  about  $x \in U$ . This justifies the claim that the image of  $\alpha_{p,q}|_U$  is sections with compact support in  $U$ . Similarly, if  $y$  is at least  $2\epsilon(x)$  away from  $x$  then the map goes to the basepoint, so it is unnecessary to define

$\psi_{x \rightarrow y}(w_x)$ . Note that when  $U = M$  the map becomes

$$\alpha_{p,q} : (S^{\nu_p} \wedge_M E_q)/M \longrightarrow \Gamma_M(\mathbb{S}'_p \wedge_M E_q)$$

$$\alpha(z_x, w_x) = \begin{cases} \{y \mapsto ((\exp(z_x) - e(y), \epsilon, \psi_{x \rightarrow y}(w_x))\} & \|z_x\| < \epsilon \\ \text{zero section} & \|z_x\| \geq \epsilon \end{cases}$$

It's easy to check that the suspension  $\Sigma \alpha_{p,q}|_U$  agrees with both  $\alpha_{p+1,q}|_U$  and  $\alpha_{p,q+1}|_U$ . Therefore the  $\alpha_{p,q}|_U$  determine a map of external smash products

$$(S^{-TM} \otimes_M E)/M|_U \longrightarrow \Gamma_U^c(\mathbb{S}' \otimes_M E)$$

and therefore they give a map  $\alpha|_U$  between the smash products of parametrized prespectra

$$\alpha|_U : (S^{-TM} \wedge_M E)/M|_U \longrightarrow \Gamma_U^c(\mathbb{S}' \wedge_M E)$$

so long as we pick the same sequence of values of  $(p, q)$  to construct the handcrafted smash product on the left side and on the right side.

Our goal is to prove that  $\alpha|_U$  is a  $\pi_*$ -isomorphism for every open subset  $U \subset M$ , in particular the subset  $U = M$ . We start with  $\alpha|_U$  where  $U \subset M$  is homeomorphic to an open ball  $B^m$ . Form an isotopy of  $e$  to a map that sends  $U$  to the unit ball  $B^m \subset \mathbb{R}^m \subset (\mathbb{R}^{p-m} \oplus \mathbb{R}^m)$ . Make  $\epsilon$  sufficiently small that at every point in the isotopy, the  $\epsilon$ -neighborhood of the image of  $M$  is a tubular neighborhood. This yields a homotopic map  $\alpha_{p,q}|_U$ , whose definition simplifies to

$$(((B_\epsilon^{p-m}/\partial B^{p-m}) \times B^m) \wedge_{B^m} E_q)/B^m \longrightarrow \Gamma_{B^m}^c((\mathbb{S}'_p \times B^m) \wedge_{B^m} E_q)$$

$$(z, x, w_x) \mapsto \{y \mapsto ((z, x - y), \min(\delta_U(x), \epsilon), \psi_{x \rightarrow y}(w_x))\}$$

The bundle  $E_q$  is homotopically trivial over  $U \cong B^m$ ; using the connection  $\psi$  to identify the fibers to the fiber  $F_q$  over  $0 \in B^m$  we get the homotopy equivalent map

$$(((B_\epsilon^{p-m}/\partial B^{p-m}) \times B^m) \wedge_{B^m} (F_q \times B^m))/B^m \longrightarrow \text{Map}(B^m/\partial, \mathbb{S}'_p \wedge F_q)$$

$$((B_\epsilon^{p-m}/\partial B^{p-m} \wedge F_q) \times B^m)/B^m \longrightarrow \text{Map}(B^m/\partial, \mathbb{S}'_p \wedge F_q)$$

$$(z, w, x) \mapsto \{y \mapsto ((z, x - y), \min(\delta_U(x), \epsilon), w)\}$$

These homotopy equivalences agree with suspension of the  $p$  or  $q$  factors, so  $\alpha|_U$  is a  $\pi_*$ -isomorphism if this map is as well.

These maps involve spheres of the form  $B/\partial B$  for some ball  $B$ . Consider instead the much simpler map involving ordinary spheres (one-point compactifications of  $\mathbb{R}^n$ )

$$\alpha : S^{p-m} \wedge F_q \longrightarrow \Omega^m(S^p \wedge F_q)$$

$$(z, w) \mapsto \{y \mapsto ((z, -y), w)\}$$

We claim that the following commutes up to homotopy, and the vertical maps are homotopy equivalences agreeing with suspension of the  $p$  and  $q$  factors up to homotopy.

$$\begin{array}{ccc} S^{p-m} \wedge F_q & \xrightarrow{\alpha} & \Omega^m(S^p \wedge F_q) \\ \downarrow \text{into the fiber over 0} & & \downarrow \text{precompose with } \rho \\ ((B_\epsilon^{p-m}/\partial B^{p-m} \wedge F_q) \times B^m)/B^m & \xrightarrow{\alpha|_U} & \text{Map}(B^m/\partial, \mathbb{S}'_p \wedge F_q) \end{array}$$

where  $\rho$  is a homeomorphism of the open ball into  $\mathbb{R}^m$  that preserves orientation. In fact, the image of  $(z, w)$  under the two branches is

$$\{y \mapsto ((z, -y), \min(1/2, \epsilon), w)\} \quad \{y \mapsto ((z, -\rho(y)), 1/2, w)\} \quad (y \in B^m)$$

and these maps are clearly homotopic through maps that send  $\partial B^m$  to the basepoint.

Therefore we just need to show that the greatly simplified map  $\alpha$  is a  $\pi_*$ -isomorphism. It looks very much like the unit of the adjunction:

$$\mu^m : S^{p-m} \wedge F_q \longrightarrow \Omega^m \Sigma^m(S^{p-m} \wedge F_q)$$

$$(z, w) \mapsto \{y \mapsto (y, (z, w))\}$$

In fact, the induced maps on  $\pi_*$  differ by multiplication by  $(-1)^m$ . Since the unit of the adjunction is a  $\pi_*$ -isomorphism ([MMSS01] 7.4(i')), our map  $\alpha$  is a  $\pi_*$ -isomorphism as well. Therefore  $\alpha|_U$  is a  $\pi_*$ -isomorphism for  $U \cong B^m$ .

Now that we have done the case of a small ball in  $M$ , we need to glue these results together to get the same result for larger sets. By Prop A.2.7, every pair of open sets

$U, V \subset M$ , gives two Cartesian squares (homotopy pullback squares) of prespectra

$$\begin{array}{ccc}
 (S^{-TM} \wedge_M E)/M|_U & \longrightarrow & (S^{-TM} \wedge_M E)/M|_{U \cup V} \\
 \uparrow & & \uparrow \\
 (S^{-TM} \wedge_M E)/M|_{U \cap V} & \longrightarrow & (S^{-TM} \wedge_M E)/M|_V \\
 \\ 
 \Gamma_U^c(\mathbb{S}' \wedge_M E) & \longrightarrow & \Gamma_{U \cup V}^c(\mathbb{S}' \wedge_M E) \\
 \uparrow & & \uparrow \\
 \Gamma_{U \cap V}^c(\mathbb{S}' \wedge_M E) & \longrightarrow & \Gamma_V^c(\mathbb{S}' \wedge_M E)
 \end{array}$$

The four maps  $\alpha|_{U \cap V}$ ,  $\alpha|_U$ ,  $\alpha|_V$ , and  $\alpha|_{U \cup V}$  give a map between these two Cartesian squares.

There is however a small technicality: the maps  $\alpha|_{\dots}$  commute except for the  $\epsilon$  coordinate. Clearly the space of choices for this coordinate is contractible. Therefore we can replace either of these two squares with a weakly equivalent square, and  $\alpha$  will define a strictly commuting map of squares. Specifically, we could replace each space in the first square with the homotopy colimit of everything over that vertex. Or, we could replace each space in the second square with the homotopy limit of everything under that vertex.

Once  $\alpha$  defines a strictly commuting map between Cartesian squares, we conclude that if three of the maps  $\alpha|_{U \cap V}$ ,  $\alpha|_U$ ,  $\alpha|_V$ , and  $\alpha|_{U \cup V}$  are equivalences, so is the fourth. This will give us the power to prove the theorem for compact manifolds. For the noncompact case we also need Prop A.2.6, which guarantees that if  $U_1 \subset U_2 \subset U_3 \subset \dots \subset \bigcup_i U_i = U$  is a sequence of inclusions of open sets with colimit  $U$ , and  $\alpha_{U_i}$  is an isomorphism for all  $i$ , then we get a commuting diagram in the stable homotopy category

$$\begin{array}{ccccc}
 H_c^\bullet(U_1; E) & \longrightarrow & H_c^\bullet(U_2; E) & \longrightarrow & \dots & H_c^\bullet(U; E) \\
 \uparrow \alpha|_{U_1} & & \uparrow \alpha|_{U_2} & & & \uparrow \alpha|_U \\
 H_\bullet(U_1; \Sigma^{-TM} E) & \longrightarrow & H_\bullet(U_2; \Sigma^{-TM} E) & \longrightarrow & \dots & H_\bullet(U; \Sigma^{-TM} E)
 \end{array}$$

The spectra at the right are equivalent to the homotopy colimit of the spectra on the left. Therefore  $\alpha_U$  induces isomorphisms on the homotopy groups.

Now we can finish the proof using a standard argument. Call an open subset  $U \subset M$  “good” if  $\alpha_U$  is a  $\pi_*$ -isomorphism of prespectra. The above tells us that if  $U$ ,  $V$ , and  $U \cap V$  are good, then  $U \cup V$  is good. Also, if  $U$  is a union of an increasing sequence  $U_1 \subset U_2 \subset \dots$

and each  $U_i$  is good, then  $U$  is good.

Our first goal is to show that each coordinate chart  $U \rightarrow \mathbb{R}^m$  is a good open subset of  $M$ . Fix one such chart; then we can cover its image by countably many open balls  $B_1, B_2, \dots$ . From before, each of these balls is good, and in fact each convex open subset of  $U$  is good. For each pair of convex open subsets, their intersection is convex open, so their union is good too. Inductively, if every  $(n - 1)$ -fold union of convex subsets is good, and we are given  $n$  open convex subsets  $U_{i_1}, \dots, U_{i_n}$ , then the union of the first  $(n - 1)$  sets is good,  $U_{i_n}$  is good, and the intersection is

$$\left( \bigcup_{j=1}^{n-1} U_{i_j} \right) \cap U_{i_n} = \bigcup_{j=1}^{n-1} (U_{i_j} \cap U_{i_n})$$

a union of  $(n - 1)$  open convex subsets. So the union of these  $n$  subsets is good. Therefore any finite union of convex subsets of  $U$  is good.

Now define  $V_n = \bigcup_{j=1}^n B_j$ . Clearly  $V_1$  is good. Inductively, if  $V_{n-1}$  is good, then  $B_n \cap V_{n-1}$  is a union of  $(n - 1)$  convex open subsets, which is good by the above induction, so  $V_n$  is good. Taking a sequential colimit, we get that  $U = \operatorname{colim}_n V_n$  is good. So every coordinate chart of  $M$  is good. Finally, we cover  $M$  by countably many coordinate charts. Every nonempty intersection of coordinate charts is a coordinate chart, so again we can use the same technique to show that any finite union of coordinate charts is good. Taking the sequential colimit again,  $M$  is good. Therefore  $\alpha_M$  gives a stable homotopy equivalence  $H_\bullet(M; \Sigma^{-TM} E) \simeq H_c^\bullet(M; E)$ .  $\square$

## A.4 Functoriality

Here are two naturality statements for the isomorphism we just proved. The above proof easily yields this:

**Corollary A.4.1** (Functoriality of Poincaré Duality, I). *If  $N \subset M$  is an open submanifold, the Poincaré duality equivalence for  $M$  and  $N$  fit into a commuting diagram*

$$\begin{array}{ccc} H_\bullet(M; \Sigma^{-TM} E) & \xrightarrow{\simeq} & H_c^\bullet(M; E) \\ \iota \uparrow & & \uparrow \iota \\ H_\bullet(N; \Sigma^{-TN} E) & \xrightarrow{\simeq} & H_c^\bullet(N; E) \end{array}$$

in the stable homotopy category of prespectra.

Now consider a sequence of proper embeddings of manifolds  $N \xrightarrow{f} M \xrightarrow{e} \mathbb{R}^n$ . Pick a continuous function  $\epsilon : M \rightarrow (0, 1)$  such that the  $\epsilon$ -neighborhood of  $M$  in  $\mathbb{R}^n$  is a tubular neighborhood, and the  $\epsilon|_N$ -neighborhood of  $N$  in  $M$  is a tubular neighborhood. (If  $N$  and  $M$  are compact, we can take  $\epsilon$  to be constant.) Define

$$\begin{aligned} f^* : H_c^\bullet(M; E) &\longrightarrow H_c^\bullet(N; f^*E) \\ \Gamma_M^c(E) &\longrightarrow \Gamma_N^c(f^*E) \end{aligned}$$

by pulling back each section  $a : M \rightarrow E_q$  to  $a \circ f : N \rightarrow E_q$ , which gives a section  $a \circ f : N \rightarrow f^*E_q$ . Here it is essential that  $f$  is proper; otherwise the pullback will not have compact support in general. On the homology side, define the zig-zag map

$$\begin{aligned} f^* : H_\bullet(M; \Sigma^{-TM}E) &\longrightarrow H_\bullet(N; \Sigma^{-TN}f^*E) \\ (S_p^{-TM} \wedge_M E_q)/M &\xrightarrow{\sim} (S_{\epsilon,p}^{-TM} \wedge_M E_q)/M \\ \longrightarrow (S_{\epsilon,p}^{-TN} \wedge_N f^*E_q)/N &\xleftarrow{\sim} (S_p^{-TN} \wedge_N f^*E_q)/N \end{aligned}$$

Using the exponential map, we identify  $S_{\epsilon,p}^{-TM}$  with the  $\epsilon$ -tubular neighborhood of  $M \xrightarrow{0 \oplus \epsilon} \mathbb{R}^{p-n} \oplus \mathbb{R}^n$ , and  $S_{\epsilon,p}^{-TN}$  with the  $\epsilon$ -tubular neighborhood of  $N \xrightarrow{f} M \xrightarrow{0 \oplus \epsilon} \mathbb{R}^{p-n} \oplus \mathbb{R}^n$ . Let  $\pi : \text{tube}_\epsilon(f(N)) \rightarrow N$  be the projection from each point in the  $\epsilon$ -tubular neighborhood of  $f(N)$  to the closest point in  $N$ . Then the middle map is given by the formula

$$\begin{aligned} (S_{\epsilon,p}^{-TM} \wedge_M E_q)/M &\longrightarrow (S_{\epsilon,p}^{-TN} \wedge_N f^*E_q)/N \\ (z_x, w_x) &\longrightarrow (z_{\pi(x)}, \psi_{x \rightarrow \pi(x)}(w_x)) \\ x \in M \quad z_x \in \mathbb{R}^p \quad w_x \in E_q \end{aligned}$$

when  $x$  is within  $\epsilon(\pi(x))$  of  $f(N)$ . The other points are sent to the basepoint.

**Theorem A.4.2** (Functoriality of Poincaré Duality, II). *If  $f : N \hookrightarrow M$  is a proper embedding of manifolds, the Poincaré duality equivalence for  $M$  and  $N$  fit into a commuting*

diagram

$$\begin{array}{ccc} H_{\bullet}(M; \Sigma^{-TM} E) & \xrightarrow{\simeq} & H_c^{\bullet}(M; E) \\ \downarrow f^* & & \downarrow f^* \\ H_{\bullet}(N; \Sigma^{-TN} f^* E) & \xrightarrow{\simeq} & H_c^{\bullet}(N; f^* E) \end{array}$$

in the stable homotopy category of prespectra.

*Proof.* We need to show that  $f^*$  commutes with each of these components of the Poincaré duality isomorphism:

$$(S^{-TM} \wedge_M E)/M \xrightarrow{\simeq} (S_{\epsilon}^{-TM} \wedge_M E)/M \xrightarrow{\alpha} \Gamma_M^c(\mathbb{S}' \wedge_M E) \xleftarrow{\simeq} \Gamma_M^c(\mathbb{S} \wedge_M E) \xrightarrow{\simeq} \Gamma_M^c(E)$$

But  $f^*$  commutes with the first map by definition, and it is easy to see that  $f^*$  commutes with the last two maps. For the second map, we just need to show that the following square commutes:

$$\begin{array}{ccc} (S_{\epsilon,p}^{-TM} \wedge_M E_q)/M & \xrightarrow{\alpha_{p,q}} & \Gamma_M^c(\mathbb{S}'_p \wedge_M E_q) \\ \downarrow f^* & & \downarrow f^* \\ (S_{\epsilon,p}^{-TN} \wedge_N f^* E_q)/N & \xrightarrow{\alpha_{p,q}} & \Gamma_N^c(\mathbb{S}'_p \wedge_N f^* E_q) \end{array}$$

In formulas, the bottom branch is

$$(z_x, w_x) \longrightarrow (z_{\pi(x)}, \psi_{x \rightarrow \pi(x)} w_x) \longrightarrow \{y \mapsto ((z - e(y), \epsilon(x), \psi_{x \rightarrow \pi(x) \rightarrow y}(w_x)))\}$$

which is homotopic to the top branch

$$(z_x, w_x) \longrightarrow \{y \mapsto ((z - e(y), \epsilon(x), \psi_{x \rightarrow y}(w_x)))\}$$

using the fact that any two choices of  $\psi$  (defined only on  $\text{tube}_{\epsilon}(f(N))$ ) are homotopic:

$$\begin{array}{ccc} (E_q \times_M S(TM) \times [0, 1] \times \{0, 1\}) \cup (E_q \times_M S(TM) \times \{0\} \times [0, 1]) & \xrightarrow{\psi_1, \psi_2} & E_q \\ \downarrow & \text{homotopy} & \downarrow \\ E_q \times_M S(TM) \times [0, 1] \times [0, 1] & \xrightarrow{\text{exp}} & M \end{array}$$

□

**Remark.** The map  $f^*$  on the homology side is an “umkehr map” in the sense of [CK09].

If  $M$  and  $N$  are compact and oriented, and if we take  $E = M \times H\mathbb{Z}$  and take the homotopy groups of both sides, we get the classical commuting diagram

$$\begin{array}{ccc} H_{\dim(M)-q}(M) & \xrightarrow{\cong} & H^q(M) \\ \downarrow f_! & & \downarrow f^* \\ H_{\dim(N)-q}(N) & \xrightarrow{\cong} & H^q(N) \end{array}$$

The shriek map  $f_!$  takes a chain represented by an oriented manifold transverse to  $N$  and intersects it with  $N$ . If we take  $f$  to be the diagonal map  $\Delta : M \hookrightarrow M \times M$ , we get an ingredient in the proof that classical Poincaré duality takes the intersection product on homology to the cup product on cohomology:

$$\begin{array}{ccc} H_{m-p}(M) \otimes H_{m-q}(M) & \xrightarrow{\cong} & H^p(M) \otimes H^q(M) \\ \downarrow \times & & \downarrow \times \\ H_{2m-p-q}(M \times M) & \xrightarrow{\cong} & H^{p+q}(M \times M) \\ \downarrow \Delta_! & & \downarrow \Delta^* \\ H_{m-p-q}(M) & \xrightarrow{\cong} & H^{p+q}(M) \end{array}$$

## A.5 The relative version

**Definition A.5.1.** • Given a vector bundle  $\nu \rightarrow M$  and a subspace  $A \subset M$  we define the *relative Thom space*  $(M, A)^\nu$  to be the quotient of the usual Thom space  $M^\nu$  by the subspace of all points lying in some fiber over  $A$ .

- For any subspace  $B \subset M$  we define  $\Gamma_{(M,B)}(E)$  to be the subspace of  $\Gamma_M(E)$  consisting of sections which vanish on  $B$ , and similarly for the sections with compact support  $\Gamma_{(M,B)}^c(E)$ .
- Let  $E$  be a nice parametrized prespectrum over  $M$  and let  $A$  be a subspace of  $M$ . In the spirit of [CK09] we define the *relative homology prespectrum of  $(M, A)$  with coefficients in  $E$*  to be

$$H_\bullet(M, A; E) = E/(M \cup E|_A)$$

Similarly, given a subspace  $B$  define *relative cohomology* and *relative cohomology with compact supports* as

$$H^\bullet(M, B; E) = \Gamma_{(M, B)}(E) \quad H_c^\bullet(M, B; E) = \Gamma_{(M, B)}^c(E)$$

Now suppose  $M$  is a possibly noncompact manifold with boundary  $\partial M$ . Let  $e : (M, \partial M) \hookrightarrow (\mathbb{R}_{\geq 0}^n, \mathbb{R}^{n-1})$  be an embedding with normal bundle  $\nu_n$ ; then we can define  $S^{-TM}$  as in the first section.

**Theorem A.5.2** (Twisted Lefschetz Duality). *If  $E \rightarrow M$  is a nice parametrized prespectrum then there is a stable homotopy equivalence of prespectra*

$$H_\bullet(M, \partial M; \Sigma^{-TM} E) \simeq H_c^\bullet(M; E)$$

$$(S^{-TM} \wedge_M E) / (M \cup (S^{-TM} \wedge_M E)|_{\partial M}) \simeq \Gamma_M^c(E)$$

*The equivalence is a zig-zag of  $\pi_*$ -isomorphisms, so it corresponds to an actual map in the stable homotopy category.*

*Proof.* This is almost the same as the absolute case, except that our map  $\alpha|_U$  needs to be further modified so as to vanish on the boundary  $\partial M$ . We assume our embedding  $e : (M, \partial M) \hookrightarrow (\mathbb{R}_{\geq 0}^n, \mathbb{R}^{n-1})$  sends a collar of  $\partial M$  to  $\partial M \times [0, 1]$ , and define a self-homotopy of this collar neighborhood from the identity to the map that stretches  $[0, 1]$  to  $[-1, 1]$ . Then in the definition of  $\alpha|_U$  we apply this stretched map to  $\exp(z_x)$  before subtracting  $e(y)$ . This new map  $\alpha|_U$  is clearly homotopic to the original one, so it has all the same properties, but in addition it now vanishes when  $x \in \partial M$  and so descends to a map on the quotient

$$(S^{-TM} \wedge_M E) / (M \cup (S^{-TM} \wedge_M E)|_{\partial M})$$

Now the induction runs as before, only there is one more base case to check. If  $U$  is an open half-ball intersecting the boundary of  $M$ , we check that  $\alpha|_U$  is a map defined between two contractible spectra, so it is trivially a stable equivalence. The same inductive argument finishes the proof.  $\square$

**Corollary A.5.3** (Twisted Relative Poincaré Duality). *If  $M$  is a compact  $n$ -manifold whose boundary  $\partial M$  is expressed as a union of two  $(n-1)$ -manifolds  $A$  and  $B$  along their common*

boundary  $A \cap B$ , there is a stable homotopy equivalence

$$H_{\bullet}(M, A; \Sigma^{-TM} E) \simeq H^{\bullet}(M, B; E)$$

*Proof.* Simply apply the previous theorem to the noncompact manifold  $M - B$  and apply Prop A.2.4 to conclude that

$$H_c^{\bullet}(M - B; E) \simeq H^{\bullet}(M, B; E)$$

□

# Bibliography

- [ABG<sup>+</sup>12] V. Angeltveit, A. J. Blumberg, T. Gerhardt, M. Hill, and T. Lawson, *Interpreting the Bökstedt smash product as the norm*, arXiv preprint arXiv:1206.4218 (2012).
- [ABG<sup>+</sup>14] V. Angeltveit, A. J. Blumberg, T. Gerhardt, M. Hill, T. Lawson, and M. Mandell, *Relative cyclotomic spectra and topological cyclic homology via the norm*, arXiv preprint arXiv:1401.5001 (2014).
- [AF] David Ayala and John Francis, *Poincaré/Koszul duality*, Work in preparation.
- [Aro99] G. Arone, *A generalization of Snaitth-type filtration*, Transactions of the American Mathematical Society **351** (1999), no. 3, 1123–1150.
- [Bar14] C. Barwick, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, arXiv preprint arXiv:1404.0108 (2014).
- [BCC<sup>+</sup>96] M. Bökstedt, G. Carlsson, R. L. Cohen, T. Goodwillie, W. C. Hsiang, and I. Madsen, *On the algebraic K-theory of simply connected spaces*, Duke Mathematical Journal **84** (1996), no. 3, 541–563.
- [BCS08] A. J. Blumberg, R. L. Cohen, and C. Schlichtkrull, *Topological Hochschild homology of Thom spectra and the free loop space*, arXiv preprint arXiv:0811.0553 (2008).
- [BHM93] M. Bökstedt, W. C. Hsiang, and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Inventiones mathematicae **111** (1993), no. 1, 465–539 (English).
- [BK87] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, vol. 304, Springer, 1987.

- [BM88] C. F. Bödigheimer and I. Madsen, *Homotopy quotients of mapping spaces and their stable splitting*, Quart. J. Math. Oxford **39** (1988), 401–409.
- [BM08] A. J. Blumberg and M. A. Mandell, *Localization theorems in topological Hochschild homology and topological cyclic homology*, arXiv preprint arXiv:0802.3938 (2008).
- [BM11a] C. Berger and I. Moerdijk, *On an extension of the notion of Reedy category*, Mathematische Zeitschrift **269** (2011), no. 3-4, 977–1004.
- [BM11b] A. J. Blumberg and M. A. Mandell, *Derived Koszul duality and involutions in the algebraic K-theory of spaces*, Journal of Topology **4** (2011), no. 2, 327–342.
- [BM11c] A. J. Blumberg and M. A. Mandell, *Localization for THH(ku) and the topological Hochschild and cyclic homology of Waldhausen categories*, ArXiv e-prints (2011).
- [BM13] A. J. Blumberg and M. A. Mandell, *The homotopy theory of cyclotomic spectra*, arXiv preprint arXiv:1303.1694 (2013).
- [Böd87] C. F. Bödigheimer, *Stable splittings of mapping spaces*, Algebraic Topology (1987), 174–187.
- [Boh14] A. M. Bohmann, *A comparison of norm maps*, Proceedings of the American Mathematical Society (2014).
- [Bök85] M. Bökstedt, *Topological hochschild homology*, preprint, Bielefeld (1985).
- [Bro69] W. Browder, *The Kervaire invariant of framed manifolds and its generalization*, The Annals of Mathematics **90** (1969), no. 1, 157–186.
- [Cam14] J. A. Campbell, *Derived Koszul Duality and Topological Hochschild Homology*, arXiv preprint arXiv:1401.5147 (2014).
- [CC87] GE Carlsson and RL Cohen, *The cyclic groups and the free loop space*, Commentarii Mathematici Helvetici **62** (1987), no. 1, 423–449.
- [CD06] B. Chorny and W. G. Dwyer, *Homotopy theory of small diagrams over large categories*, Arxiv preprint math/0607117 (2006).

- [Chi10] M. Ching, *A chain rule for Goodwillie derivatives of functors from spectra to spectra*, American Mathematical Society **362** (2010), no. 1, 399–426.
- [Cho07] B. Chorny, *Brown representability for space-valued functors*, Arxiv preprint arXiv:0707.0904 (2007).
- [Chu11] T. Church, *Homological stability for configuration spaces of manifolds*, Inventiones Mathematicae (2011), 1–40.
- [CJ02] R. L. Cohen and J. D. S. Jones, *A homotopy theoretic realization of string topology*, Mathematische Annalen **324** (2002), no. 4, 773–798.
- [CJ13] ———, *Gauge theory and string topology*, arXiv preprint arXiv:1304.0613 (2013).
- [CJY04] R. L. Cohen, J. D. S. Jones, and J. Yan, *The loop homology algebra of spheres and projective spaces*, Progress In Mathematics **215** (2004), 77–92.
- [CK09] R. L. Cohen and J. R. Klein, *Umkehr maps*, Homology, Homotopy and Applications **11** (2009), no. 1, 17–33.
- [CKS08] R. L. Cohen, J. R. Klein, and D. Sullivan, *The homotopy invariance of the string topology loop product and string bracket*, Journal of Topology **1** (2008), no. 2, 391–408.
- [Coh80] R. L. Cohen, *Stable proofs of stable splittings*, Math. Proc. Cambridge Philos. Soc **88** (1980), 149–151.
- [Coh87] ———, *A model for the free loop space of a suspension*, Algebraic topology, Springer, 1987, pp. 193–207.
- [Coh04] ———, *Multiplicative properties of Atiyah duality.*, Homology, Homotopy & Applications **6** (2004), no. 1.
- [Con83] A. Connes, *Cohomologie cyclique et foncteurs  $Ext^n$* , CR Acad. Sci. Paris Sér. I Math **296** (1983), no. 23, 953–958.
- [CS99] M. Chas and D. Sullivan, *String topology*, Arxiv preprint math/9911159 (1999).

- [dBW12] P. B. de Brito and M. S. Weiss, *Manifold calculus and homotopy sheaves*, arXiv preprint arXiv:1202.1305 (2012).
- [DHK85] W. G. Dwyer, M. J. Hopkins, and D. M. Kan, *The homotopy theory of cyclic sets*, Transactions of the American Mathematical Society **291** (1985), no. 1, 281–289.
- [DK80] W. G. Dwyer and D. M. Kan, *Function complexes in homotopical algebra*, Topology **19** (1980), no. 4, 427–440.
- [DM96] B. Dundas and R. McCarthy, *Topological Hochschild homology of ring functors and exact categories*, Journal of Pure and Applied Algebra **109** (1996), no. 3, 231 – 294.
- [Dwy96] W. G. Dwyer, *The centralizer decomposition of  $BG$* , Progress In Mathematics **136** (1996), 167–184.
- [FT04] Y. Félix and J. C. Thomas, *Monoid of self-equivalences and free loop spaces*, Proceedings of the American Mathematical Society **132** (2004), no. 1, 305.
- [Goo90] T. G. Goodwillie, *Calculus I: The first derivative of pseudoisotopy theory*, K-theory **4** (1990), no. 1, 1–27.
- [Goo91] ———, *Calculus II: analytic functors*, K-theory **5** (1991), no. 4, 295–332.
- [Goo03] ———, *Calculus III: Taylor series*, Geometry & Topology **7** (2003), 645–711.
- [GS08] K. Gruher and P. Salvatore, *Generalized string topology operations*, Proceedings of the London Mathematical Society **96** (2008), no. 1, 78–106.
- [GW99] T. G. Goodwillie and M. Weiss, *Embeddings from the point of view of immersion theory: Part II*, Geometry and Topology **3** (1999), no. 103, 118.
- [Hel08] R. D. Helmstutler, *Model category extensions of the Pirashvili-Słomińska theorems*, Arxiv preprint arXiv:0806.1540 (2008).
- [HHR09] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *On the non-existence of elements of Kervaire invariant one*, arXiv preprint arXiv:0908.3724 (2009).

- [Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, no. 63, American Mathematical Society, 1999.
- [Jon87] J. D. S. Jones, *Cyclic homology and equivariant homology*, *Inventiones mathematicae* **87** (1987), no. 2, 403–423.
- [LMM82] G. Lewis, J. P. May, and J. McClure, *Classifying  $G$ -spaces and the Segal conjecture*, *Current Trends in Algebraic Topology*, CMS Conference Proc. 2, 1982, pp. 165–179.
- [LMSM86] L. G. Lewis, J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, vol. 1213, Springer-Verlag Berlin-New York, 1986.
- [Mad95] I. Madsen, *Algebraic  $K$ -theory and traces*, *Current Developments in Mathematics*, International Press, 1995.
- [May75] J. P. May, *Classifying spaces and fibrations*, *Memoirs of the American Mathematical Society*, no. 155, American Mathematical Society, 1975.
- [McC01] R. McCarthy, *Dual calculus for functors to spectra*, *Contemporary Mathematics* **271** (2001), 183–216.
- [McD75] D. McDuff, *Configuration spaces of positive and negative particles*, *Topology* **14** (1975), no. 1, 91–107.
- [MM02] M. A. Mandell and J. P. May, *Equivariant orthogonal spectra and  $S$ -modules*, *Memoirs of the American Mathematical Society*, no. 755, American Mathematical Society, 2002.
- [MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, *Proceedings of the London Mathematical Society* **82** (2001), no. 02, 441–512.
- [Mor09] J. Morava, *A theory of base motives*, arXiv preprint arXiv:0908.3124 (2009).
- [MS06] J. P. May and J. Sigurdsson, *Parametrized homotopy theory*, *Mathematical Surveys and Monographs*, vol. 132, American Mathematical Society, 2006.
- [Pir00] T. Pirashvili, *Dold-Kan type theorem for  $\Gamma$ -groups*, *Mathematische Annalen* **318** (2000), no. 2, 277–298.

- [Sch09] C. Schlichtkrull, *The cyclotomic trace for symmetric ring spectra*, Arxiv preprint arXiv:0903.3495 (2009).
- [Shu06] M. Shulman, *Homotopy limits and colimits and enriched homotopy theory*, Arxiv preprint math/0610194 (2006).
- [Str09] N. P. Strickland, *The category of CGWH spaces*, Unpublished, notes (2009).
- [Wal85] F. Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology, Springer, 1985, pp. 318–419.
- [Wei99] M. Weiss, *Embeddings from the point of view of immersion theory: Part I*, Arxiv preprint math/9905202 (1999).
- [Wil00] B. Williams, *Bivariant Riemann Roch theorems*, Contemporary Mathematics **258** (2000), 377–393.
- [WW93] M. Weiss and B. Williams, *Assembly, Novikov conjectures, index theorems and rigidity* **2** (1993), 332–352.