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On the isomorphism problem for infinite group rings*

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1 Introduction

One of the central problems in the theory of group rings is the determination of the extent to which the group ring RH reflects the properties of the group H. More precisely, given a group H the following question is of great interest:

The Isomorphism Problem: *Is it true that the existence of an $R-$algebra isomorphism of the group rings RH and RH' implies the existence of a group isomorphism of H and H'?*

The answer strongly depends on the ring R. For example, when R is an algebraically closed field the classical theory of semisimple algebras tells us that for any finite abelian group H of order n prime to the characteristic of R, the $R-$algebras RH and $\prod_{i=1}^{n} R$ are isomorphic. The proper rings to consider seem to be commutative integral domains of characteristic 0 in which no element of $\pi(H)$ is invertible, where $\pi(H)$ denotes the set of all primes p such that H has an element of order p. Following K. Roggenkamp [9] and T. Furukawa [1] we will call such rings *$H-$adapted.* Of primary importance is the case when R is the ring of algebraic integers in a number field, especially $R = \mathbb{Z}$. In the case of integral coefficients there is a hope that the answer to the

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isomorphism problem is "yes" for any finite group H (note however that this assertion is not commonly believed). Much progress has been made toward this conjecture in recent years. For example, it is known to be true for finite metabelian ([21]) and nilpotent ([10], [20]) groups (see [11] for an excellent discussion of the isomorphism problem for finite groups). On the other hand, in the case of infinite groups our knowledge is still marginal. For torsion free groups a positive answer follows immediately from the following conjecture, which however seems to be unreachable at present:

The Unit Conjecture: *If H is a torsion free group then for any domain R the unit group of the ring RH consists only of trivial units, i.e. equals $R^{\times} \cdot H$ where $R^{\times}$ denotes the group of invertible elements of R.*

This conjecture has been established when H is a so called u.p. group (u.p. stands for unique product). By definition, this means that for any two nonempty finite subsets A, B of H there exists an element $h \in H$ which can be uniquely expressed in the form ab with $a \in A$ and $b \in B$. Left or right ordered groups are the basic examples of u.p. groups. The question whether there exist torsion free groups which are not u.p. was open for a long time. However D. S. Promislow ([7]) has proved that the following group Γ, called the Passman group, is not a u.p. group:

$$\Gamma = \langle x,\ y | x^{-1}y^2x = y^{-2},\ y^{-1}x^2y = x^{-2} \rangle$$

The group Γ is a metabelian torsion free group for which we do not know if the unit conjecture holds. As far as the author knows the isomorphism problem for this group remains unanswered too. For more details concerning u.p. groups we refer to [3].

A positive solution to the isomorphism problem for any abelian group H and $H-$adapted rings has been obtained by May in [4]. The same answer for matabelian groups with torsion or torsion free abelianization and $R = \mathbb{Z}$ follows from Whitcomb's proof for finite metabelian groups (see [3] p. 81 for details). For nilpotent groups we know that the answer is "yes" in the following cases:

— H finitely generated, nilpotent of class 2, $R = \mathbb{Z}$ (Ritter and Sehgal [8]);

— H a circle group of a nilpotent R–algebra A (i.e. $H = A$ with the multiplication $\circ$ defined as $a \circ b = a + b + ab$), R any integral domain of characteristic 0 (Sandling [16], Röhl [14] and Furukawa [1]).

No other results are known to the author (except the one obtained in this article).

Let C_∞ be the infinite cyclic group with generator t. For any group G and any automorphism $\phi \in \mathrm{Aut}G$ we construct the semidirect product $G \times_\varphi C_\infty$, where t^n acts as φ^n on G. More precisely, $G \times_\varphi C_\infty$ coincides with $G \times C_\infty$ as a set and the multiplication is given by the formula:

$$(g, t^n) \cdot (h, t^m) = (g\varphi^n(h), t^{m+n}).$$

The aim of this note is to study the isomorphism problem for groups of the type $G \times_\varphi C_\infty$ where G is finite and $\varphi \in \mathrm{Aut}G$. In particular we prove the following theorem:

Theorem 1. *Let G be a finite group, R a G–adapted ring and let W be any group. Then the R–algebras $R[G \times C_\infty]$ and RW are isomorphic if and only if $W = H \times_\varphi C_\infty$ where:*

— $H \subseteq W$ is a finite subgroup such that the R–algebras RG and RH are isomorphic,

— φ is an automorphism of H induced by a conjugation with a unit $x \in RH$ which normalizes H.

Using the fact that $G \times C_\infty \approx G \times_\psi C_\infty$ if and only if ψ is inner (see Corollary 1 below) we conclude that the isomorphism problem has positive solution for $G \times C_\infty$ if and only if it has positive solution for G and $\mathrm{Aut}_R G = \mathrm{Inn}G$, where $\mathrm{Aut}_R G$ is the group of automorphisms of G which are induced by conjugation by some unit of RG which normalizes G. This result sheds new light on the groups $\mathrm{Aut}_R G$ which were earlier considered in [2], [18] from a different point of view. When G has a normal Sylow 2−subgroup the equality $\mathrm{Aut}_{\mathbb{Z}} G = \mathrm{Inn} G$ has been proved by S. Jackowski and Z. Marciniak ([2]). Their result has been extended by the author ([5], [6]) to a wider class of rings and groups. Also in [6] it has been shown that in all known cases the equality is a consequence of properties of $\mathrm{Aut} G$ rather than group ring phenomena. Therefore one may suspect that the equality may not hold when the structure of the set of 2−subgroups of G is more complicated. In fact very recently K. W. Roggenkamp and A. Zimmermann ([12]) succeeded in constructing a family of groups for which $\mathrm{Aut}_R G \neq \mathrm{Inn} G$ for some ring of algebraic integers in a number field. It is very likely that one can take $R = \mathbb{Z}$ but there is still no proof for this. As a consequence they proved that $\mathrm{Aut}_{\mathbb{Z}_\pi} G \neq \mathrm{Inn} G$ for any semilocalization $\mathbb{Z}_\pi$ of $\mathbb{Z}$. Combining this result with Theorem 1 we get immediately a counterexample to the isomorphism problem. Since as far as the author knows no such result was known before, we state it as a theorem:

Theorem 2. *There exist a ring of algebraic integers R in a number field and a polycyclic group G for which the isomorphism problem has negative answer.*

For a more detailed discussion of this result we refer to [13], where it is also proved, using [12] and the author's methods.

In connection with the isomorphism problem for metabelian groups it would be very interesting to determine if the above equality holds for finite metabelian groups (the groups constructed in [12] are solvable of class 3). The author has some hope that this equality may remain true for groups with an abelian Sylow 2−subgroup but it is known only when the Sylow 2−subgroup is of order 2 ([6]).

2 The isomorphism problem

Lemma 1. *Let H be a finitely generated group such that the $R-$algebras RG and RH are isomorphic. Then G is also finitely generated.*

Proof: We can assume that $RG = RH$. Let $h_1, \dots, h_n$ be generators of H. Then $h_i = \sum a_{ij} g_{ij}$ for some $a_{ij} \in R$ and $g_{ij} \in G$. Let G_0 be the subgroup of G generated by all g_{ij}. Then $RG = RH \subseteq RG_0 \subseteq RG$, so $RG = RG_0$ and $G = G_0$. □

A group G is called an *$FC-$group* provided each element of G has a finite number of conjugates. For finitely generated groups this is equivalent to each of the following conditions:

— G is a finite extension of its center;

— the commutator subgroup of G is finite.

In particular the first of these properties implies that a finitely generated $FC-$group is polycyclic-by-finite. For more details concerning $FC-$groups see [19].

Lemma 2. *Let H_1 be a finitely generated $FC-$group such that the $R-$algebras RH_1 and RH_2 are isomorphic, where R is H_1-adapted. Then H_2 is also an $FC-$group.*

Proof: Since H_1 is polycyclic-by-finite then by Theorem VI.2.4 in [17] R is H_2-adapted. We will use the following theorem which can be deduced from [17], pp. 92-99.

Theorem A. *Let G be polycyclic-by-finite and let R be $G-$adapted. Suppose that the $R-$algebras RG and RH are isomorphic and let Φ be a function from the set of finite normal subgroups of G to the set of finite normal subgroups of H defined as follows:*

$$\text{for } N \triangleleft G \text{ let } \Phi(N) = \{h \in H : h - 1 \in \Delta(G, N)\} \subseteq H$$

Then Φ is a bijection which preserves inclusions and orders. Moreover for all normal subgroups N of G we have $\Delta(G, N) = \Delta(H, \Phi(N))$ and the $R-$algebras $R[G/N]$ and $R[H/\Phi(N)]$ are isomorphic.

Let $N = [H_1, H_1]$ be the commutator subgroup of H_1. Thus N is a finite normal subgroup of H_1. By Theorem A we get $R[H_1/N] \approx R[H_2/\Phi(N)]$. But $R[H_1/N]$ is commutative so $[H_2, H_2] \subseteq \Phi(N)$ (in fact $\Phi(N) = [H_2, H_2]$). Therefore $[H_2, H_2]$ is a finite group and by Lemma 1 we know that H_2 is finitely generated which implies that H_2 is an $FC-$group. □

It would be interesting to know if the above lemma remains true for all $FC-$groups (not necessary finitely generated) and a wider class of rings.

Lemma 3. *Let G be a group. If the commutator subgroup of G is torsion then the torsion elements of G form a subgroup $T(G)$.*

Proof: Since $[G, G]$ is a torsion group, an element of G is torsion if and only if its image in $G/[G, G]$ is. Thus $T(G)$ is the preimage of the torsion subgroup of $G/[G, G]$. □

For any group ring RG the map

$$\epsilon : RG \ni \sum a_g g \longmapsto \sum a_g \in R$$

is a ring homomorphism called *augmentation.* Any unit with augmentation 1 is called *normalized.* It is an easy observation that when the $R-$algebras RG_1 and RG_2 are isomorphic the isomorphism can be chosen to be augmentation preserving.

Lemma 4. *Let R be a domain of characteristic 0. If there exists an $R-$algebra epimorphism from RC_∞^n onto RC_∞^m then $n \geq m$.*

Proof: It is well known that the group of normalized units of RC_∞^n is equal to C_∞^n (it is just another way of saying that the Unit Conjecture holds for C_∞^n and this turns out to be true since it is an ordered group). We can assume that the epimorphism preserves augmentation. Therefore the image N of C_∞^n is a subgroup of C_∞^m such that $RN = RC_\infty^m$. Thus $N = C_\infty^m$ and therefore $n \geq m$. □

The following lemma is due to Saksonov ([15]). We include the proof for the convenience of the reader.

Lemma 5. *Let G be a finite group. If R is $G-$adapted and G_1 is a finite subgroup of normalized units of RG then:*

1^0*)* $|G_1| \mid |G|$

2^0*) If* $|G_1| = |G|$ *then* $RG = RG_1$.

Proof: Let K be the quotient field of R, $\bar{K}$ its algebraic closure. For $x \in \bar{K}G$ let $tr(x)$ be the trace of x considered as an endomorphism of $\bar{K}G$ via left multiplication. It is easy to see that $tr(1) = |G|$ and $tr(g) = 0$ for $1 \neq g \in G$. In particular, if $x \in RG$ then $tr(x)/|G| \in R$. Let $h \in G_1$ be of order n. Thus the eigenvalues of h are $n-$th roots of unity $\zeta_1, \dots, \zeta_{|G|}$. For any integer m we have $tr(h^m) = \zeta_1^m + \dots + \zeta_{|G|}^m$ and therefore

$$u_m = \frac{\zeta_1^m + \dots + \zeta_{|G|}^m}{|G|} \in R.$$

All u_m are in $L = \mathbb{Q}(\xi_n)$ where ξ_n is a primitive $n-$th root of unity. Let S be the set of all $\varphi(n)$ integers k prime to n, $1 \leq k \leq n$. The norm $u = N_{L/\mathbb{Q}}(u_1) = \prod_{k \in S} u_k$ of u_1 is a rational number. But $u \in R$ and $|G|^{\varphi(n)}u$ is an integer (being the norm of the algebraic integer $tr(h)$) and since R is $G-$adapted we get $u \in \mathbb{Z}$. On the other hand u is a sum of $|G|^{\varphi(n)}$ roots of unity divided by $|G|^{\varphi(n)}$ and the only way it can be an integer is when all these roots of unity are equal or $u = 0$. In the first case $\zeta_1 = \ldots = \zeta_{|G|}$ so $h = \zeta_1$ and consequently $h = 1$ (since the augmentation of h is 1). The second case implies $tr(h) = 0$. We have thus proved that $tr(h) = 0$ for $1 \neq h \in G_1$. In particular, for $E = |G_1|^{-1} \sum_{g \in G_1} g$ we have $tr(E) = |G|/|G_1|$. On the other hand, since $E^2 = E$, we have $tr(E) \in \mathbb{N}$ which proves 1^0.

Suppose now that $\sum_{g \in G_1} a_g g = 0$. Thus for all $h \in G_1$ we have $0 = tr(\sum_{g \in G_1} a_g g h^{-1}) = a_h |G|$. Therefore $a_g = 0$ for all $g \in G_1$ so the elements of G_1 are $R-$linearly independent. Since $|G_1| = |G|$ for every $h \in G$ there exists an $a \in R$ such that $ah = \sum_{g \in G_1} a_g g$ for some $a_g \in R$. Thus for every $b \in G_1$ we have $ahb^{-1} = \sum_{g \in G_1} a_g g b^{-1}$ and taking traces we get $ahb^{-1}(1) = a_b$, i.e. $a | a_b$. Therefore $h = a^{-1} \sum a_g g \in RG_1$ and $RG = RG_1$. □

Theorem 3. *Let G be a finite group and let R be any $G-$adapted ring. If the $R-$algebras $R[G \times_\chi C_\infty]$ and RW are isomorphic then there exists a finite group H of the same order as G such that $W = H \times_\eta C_\infty$ for some automorphism η of H.*

Proof: Let $\psi : R[G \times_\chi C_\infty] \longrightarrow RW$ be an augmentation preserving $R-$algebra isomorphism. Since $G \times_\chi C_\infty$ is a finitely generated $FC-$group thus (by Lemmas 1 and 2) so is W. In particular $[W, W]$ is a finite group. Thus by Lemma 3 torsion elements of W form a subgroup $T(W) \supseteq [W, W]$. Since $T(W)/[W, W]$ is finite (being a torsion subgroup of a finitely generated abelian group) it follows that $T(W)$ is a finite normal subgroup of W. Moreover $W_1 = W/T(W)$ is a finitely generated, torsion free abelian group, i.e. $W_1 \approx C_\infty^n$ for some integer n. Let $p : RW \longrightarrow RW_1$ be the natural projection and consider the composition $\phi = p\psi$. Since ϕ preserves augmentation and RW_1 has only trivial units we get $\phi(G) = \{1\}$. Therefore ϕ induces an $R-$algebra epimorphism from RC_∞ onto RW_1. By Lemma 5 we get $n = 1$. Therefore we obtain an exact sequence:

$$0 \longrightarrow T(W) \longrightarrow W \longrightarrow C_\infty \longrightarrow 0$$

This sequence splits since C_∞ is free and therefore $W = T(W) \times_\eta C_\infty$ for some $\eta \in \mathrm{Aut}T(W)$. By the subgroup correspondence described in Theorem A the groups G and $T(W)$ have the same order. □

The following simple lemma is an observation of Z. Marciniak. It was the remark which initiated our work on the present article.

Lemma 6. *Let G be a finite group and $\phi, \psi \in AutG$ be such that $\eta = \phi\psi^{-1} \in Aut_R G$. Then the $R-$algebras $R[G \times_\phi C_\infty]$ and $R[G \times_\psi C_\infty]$ are isomorphic.*

Proof: Let $x \in RG$ be a unit inducing an automorphism $\eta \in \mathrm{Aut}_R G$. The isomorphism is given by $G \times_\phi C_\infty \ni ht^n \longmapsto h(xt)^n \in G \times_\psi C_\infty$. □

Remark. It is tempting to replace C_∞ by the cyclic group C_n of order n, where n is the order of ϕ, in order to get an isomorphism $R[G \times C_n] \approx R[G \times_\phi C_n]$. A moment's reflection tells us however that there is no hope for doing that. The reason is that the unit x inducing the automorphism ϕ has to be of infinite order (this can be easily derived from Lemma 5). Also, as pointed out by the referee, it is known that for finite groups G_1, G_2 with isomorphic integral group rings, if G_1 decomposes into direct product of two groups of orders m and n say, then so does G_2.

Lemma 7. *Let G be a group which admits no epimorphisms onto the infinite cyclic group. Then the groups $G \times_\phi C_\infty$ and $G \times_\psi C_\infty$ are isomorphic if and only if ϕ and ψ^ϵ are conjugate in OutG, where $\epsilon = 1$ or $\epsilon = -1$.*

Proof: Let $\Phi : G \times_\phi C_\infty \longrightarrow G \times_\psi C_\infty$ be any isomorphism. Composing Φ with the natural projection $p : G \times_\psi C_\infty \longrightarrow C_\infty$ we conclude that $\Phi(G) \subseteq \ker p = G$. The same argument gives $\Phi^{-1}(G) \subseteq G$ so $\Phi(G) = G$, i.e. $\Phi|_G \in AutG$. Let $\Phi(t) = (g, t^\epsilon)$. Thus $\epsilon = \pm 1$ since Φ is an isomorphism. Also, $\Phi((0,t)(h,0)) = \Phi((0,t))\Phi((h,0)) = (g,t^\epsilon)(\Phi(h),0) = (g\psi^\epsilon \circ \Phi(h), t^\epsilon)$. On the other hand, $(0,t)(h,0) = (\phi(h),t)$ and $\Phi((\phi(h),t)) = (\Phi \circ \phi(h)g, t^\epsilon)$. Thus we get $g\psi^\epsilon \circ \Phi(h) = \Phi \circ \phi(h)g$ for all $h \in G$ and therefore $\Phi^{-1} \circ \psi^\epsilon \circ \Phi = \phi$ in Out$G$.

Conversely, if ϕ and ψ^ϵ are conjugate in Out$G$ then $\phi(h) = q\Phi^{-1} \circ \psi^\epsilon \circ \Phi(h)q^{-1}$ for some $q \in G$, $\Phi \in AutG$ and all $h \in G$. It is easy to verify that the map $(h, t^m) \longmapsto (\Phi(hq), t^{\epsilon m})$ gives an isomorphism between $G \times_\phi C_\infty$ and $G \times_\psi C_\infty$. □

Corollary 1. *If G is a finite group then the groups $G \times C_\infty$ and $G \times_\psi C_\infty$ are isomorphic if and only if ψ is inner.*

Remark. The condition that G has no epimorphisms onto C_∞ is rather indispensable. Let $G = C_n \times C_\infty$, $W = AutC_n$. Thus W is an abelian group of order $\varphi(n)$. The group $AutG$ can be described as the set of triples (ϵ, a, α), where $\epsilon = \pm 1$, $a \in C_n$ and $\alpha \in W$, with multiplication given by:

$$(\epsilon_1, a_1, \alpha_1)(\epsilon_2, a_2, \alpha_2) = (\epsilon_1\epsilon_2, a_1^{\epsilon_2}\alpha_1(a_2), \alpha_1\alpha_2).$$

The triple (ϵ, a, α) corresponds to the automorphism Φ such that $\Phi|_{C_n} = \alpha$ and $\Phi(t) = at^{\epsilon}$. Let $\eta = (1,1,\alpha)$ for some $\alpha \in W$ of order m say, and let $H = G \times_\eta \langle t_1 \rangle$, where $\langle t_1 \rangle$ is the infinite cyclic group with generator t_1 acting on G as η. For any integers k, l, s such that $ks - lm = \pm 1$ the subgroup $G_k = C_n \times \langle t^k t_1^m \rangle$ of H is isomorphic to G and $H = G_k \times_\beta \langle t^l t_1^s \rangle$ where $\beta = (1,1,\alpha^s) \in AutG_k$. In particular, for any s prime to m the groups $G \times_\eta C_\infty$ and $G \times_{\eta^s} C_\infty$ are isomorphic. Notice that if the automorphisms $(\epsilon_1, a_1, \alpha_1)$ and $(\epsilon_2, a_2, \alpha_2)$ are conjugate in $OutG = AutG$ then $\alpha_1 = \alpha_2$. In particular, if $s \not\equiv 1 \mod m$ then η and η^s are not conjugate.

3 Proof of Theorem 1

Let $\psi : R[G \times C_\infty] \longrightarrow RW$ be the augmentation preserving R–algebra isomorphism. As in the proof of Theorem 3 we get $W = T(W) \times_\eta C_\infty$ for some $\eta \in \mathrm{Aut}T(W)$ where the groups G and $T(W)$ have the same order. Thus if $\psi^{-1}(T(W)) = H$ then by Lemma 4 we get $R[C_\infty]G = R[C_\infty]H$. Dividing by the ideal generated by $1 - C_\infty$ we get $RG = RH$. Moreover if $t_1 \in W = T(W) \times_\eta C_\infty$ is a generator of C_∞ and $t = \psi^{-1}(t_1)$ then the image of t in RH is a unit inducing on H the automorphism $\sigma = \psi^{-1}\eta\psi$ (by conjugation). Since W is isomorphic to $H \times_\sigma < t >$ this proves one implication of Theorem 1.

Conversely, if $x \in RH$ is a unit inducing an automorphism η on H and $RG \approx RH$ then $R[G \times C_\infty] \approx R[H \times C_\infty] \approx R[H \times_\eta C_\infty]$ where the last isomorphism holds by Lemma 6. This completes the proof. □

Corollary 2. *If G is a finite group and R is a G–adapted ring then the isomorphism problem for $R[G \times C_\infty]$ has a positive answer if and only if it has a positive answer for G and $Aut_R G = InnG$. In particular the isomorphism problem has positive solution for $R = \mathbb{Z}$ and groups $G \times C_\infty$ such that:*

1) G has a nilpotent commutator subgroup

2) the Sylow 2–subgroup of G is either normal or of order 2.

Proof: The first statement is an easy consequence of Theorem 1 and Corollary 1. If G satisfies 1) then the isomorphism problem for G and $R = \mathbb{Z}$ has a positive solution by [11] Theorem XII.1.6 (it would be interesting to see if this theorem holds for all G–adapted rings). 2) then implies that $\mathrm{Aut}_R G = \mathrm{Inn} G$ for R the ring of integers in a CN number field (i.e. a field such that complex conjugation is in the center of the Galois group of its normal closure) by [5]. □

4 Final remarks

It would be very interesting to answer the following questions:

1. Is it possible to obtain similar results for groups of the form $G \times C_\infty^n$ with G finite ?
2. Is it true that in the notation of Theorem 3 the group rings RG and RH are isomorphic as R–algebras?
3. Is it true that the R–algebras $R[G \times_\psi C_\infty]$ and $R[G \times_\phi C_\infty]$ are isomorphic if and only if ϕ and ψ are conjugate in $AutG/Aut_R G$?
4. Is it true that every finite subgroup of normalized units of $R[G \times C_\infty]$ is conjugate to some subgroup of RG ?

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