Math 222 Calculus 2 Spring 2000 Practice Exam 1 Feingold

INSTRUCTIONS: Show all necessary work for each problem. Partial credit is possible if work shows some understanding. Numerical answers should be expressed as exact mathematical expressions rather than decimal approximations.

(1) (15 Points)

(a) Find $\int_0^1 5^{3x+2}\, dx$ (b) Find $\int_e^{e^2} \frac{dx}{x(\ln\ x)^3}$

(c) If $e^{(\ln\ x)^2} = 4$ then find $x$.

(2) (15 Points) Let $y = f(x) = x^5 + x^3 + 2x - 1$.

(a) Show that the inverse function $x = f^{-1}(y)$ exists for all real numbers $y$.

(b) Using the fact that $f(1) = 3$, find $(f^{-1})'(3)$.

(c) Find $\lim_{n\to\infty} \left(\frac{n+4}{n}\right)^n$

(3) (15 Points)

(a) Find $\int \frac{dx}{2x^2+3}$ (b) Find $\int_0^1 x^3 e^{2x^4+7}\, dx$

(c) If $f(x) = \ln(\ln(x^2 + 3))$ then find $f'(x)$.

(4) (15 Points)

(a) If $f(x) = \arcsin(5x^2)$ find $f'(x)$.

(b) If $f(x) = \log_3(\tan(x^2 + 1))$ then find $f'(x)$.

(c) Find $\int x^2 e^x\, dx$.

(5) (10 Points) Evaluate the following limits.

(a) $\lim_{x\to 0} \frac{\cos(x) - \cos(2x)}{x^2}$ (b) $\lim_{x\to\infty} \left(\frac{x+1}{x+2}\right)^x$

(6) (30 Points, 5 Points each) Evaluate the following integrals.

(a) $\int \tan(3x)\ \sec^3(3x)\ dx$ (b) $\int \cos^5(3x)\ dx$

(c) $\int \sin^2(4x)\ dx$ (d) $\int \frac{x+1}{\sqrt{9-x^2}}\, dx$

(e) $\int \frac{-x^3 + 8x^2 - 11x + 2}{(x-1)^2(x^2+1)}\, dx$ (f) $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx$

(1) (15 Points)

(a) Using $u = 3x+2$ we get $\int_0^1 5^{3x+2}dx = \int_2^5 5^u \frac{du}{3} = \frac{5^u}{3\ln\ 5}\Big]_{u=2}^{u=5} = \frac{5^5 - 5^2}{3\ln\ 5}$.

(b) Use the substitution $u = \ln\ x$, so $du = \frac{1}{x}dx$. The limits change to $1 \le u \le 2$ and we get $\int_e^{e^2} \frac{dx}{x(\ln\ x)^3} = \int_1^2 u^{-3}du = \frac{u^{-2}}{-2}\Big]_1^2 = \frac{3}{8}$.

(c) $(\ln(x))^2 = \ln(4)$ so $\ln(x) = \sqrt{\ln(4)}$ and then $x = e^{\sqrt{\ln(4)}}$.

(2) (15 Points)

(a) We have $f'(x) = 5x^4 + 3x^2 + 2 > 0$ for all real $x$, so $f(x)$ is strictly increasing on all of $\mathbb{R}$, which means it is one-to-one and has an inverse defined on its range. But as a fifth degree polynomial, $f(x)$ is continuous and $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. So Range(f) is all real numbers. Therefore, Domain($f^{-1}$) is all real numbers.

(b) $(f^{-1})'(3) = \frac{1}{f'(1)} = \frac{1}{10}$.

(c) Since $(n+4)/n = 1 + 4/n$, using $m = n/4 \to \infty$ as $n \to \infty$, we get

$$\lim_{n\to\infty}\left(\frac{n+4}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{4m} = \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^4 = e^4$$

(3) (15 Points)

(a) $\int \frac{dx}{2x^2+3} = \frac{1}{3}\int \frac{dx}{\frac{2x^2}{3}+1}$ and with $u = \sqrt{2}x/\sqrt{3}$ this equals

$$\frac{1}{\sqrt{6}}\int \frac{du}{u^2+1} = \frac{1}{\sqrt{6}}arctan(u) + C = \frac{1}{\sqrt{6}}arctan(\frac{\sqrt{2}x}{\sqrt{3}}) + C.$$

(b) Using the substitution $u = 2x^4 + 7$, so $du = 8x^3dx$, and the limits change to $7 \le u \le 9$, we get $\int_0^1 x^3e^{2x^4+7}dx = \int_7^9 \frac{e^u}{8}du = \frac{e^u}{8}\Big]_{u=7}^{u=9} = \frac{e^9 - e^7}{8}$.

(c) $f'(x) = \frac{1}{\ln(x^2+3)}\ \frac{2x}{(x^2+3)}$.

(4) (15 Points)

(a) $f'(x) = \frac{10x}{\sqrt{1-25x^4}}$ (b) $f'(x) = \frac{\sec^2(x^2+1)}{\tan(x^2+1)}\ \frac{2x}{\ln\ 3}$.

(c) Use integration by parts, get $\int x^2e^xdx = x^2e^x - \int e^x 2xdx = x^2e^x - 2\int xe^xdx = x^2e^x - 2(xe^x - \int e^xdx) = x^2e^x - 2xe^x + 2e^x + C = (x^2 - 2x + 2)e^x + C$.

(5) (10 Points)

(a) $\lim_{x\to 0}\dfrac{\cos(x)-\cos(2x)}{x^2} = \lim_{x\to 0}\dfrac{-\sin(x)+2\sin(2x)}{2x} = \lim_{x\to 0}\dfrac{-\cos(x)+4\cos(2x)}{2} = \dfrac{-1+4}{2} = \dfrac{3}{2}$, each step by L'Hospital's Rule for a $\frac{0}{0}$ indeterminate form.

(b) Let $y = \left(\frac{x+1}{x+2}\right)^x$ so $\ln(y) = x\ln\left(\frac{x+1}{x+2}\right)$. Then $\lim_{x\to\infty}\ln(y) = \lim_{x\to\infty}\dfrac{\ln\left(\frac{x+1}{x+2}\right)}{x^{-1}}$ is a type $\frac{0}{0}$ indeterminate form. L'Hospital's Rule gives $\lim_{x\to\infty}\dfrac{(x+2)}{(x+1)}\dfrac{1}{(x+2)^2}(-x^2) = \lim_{x\to\infty}\dfrac{-x^2}{(x+1)(x+2)} = -1$ so $y = e^{-1}$.

(6) (30 Points)

(a) Write the integral as $\int \sec^2(3x)\tan(3x)\sec(3x)dx$ and substitute $u = \sec(3x)$ so $du = 3\tan(3x)\sec(3x)dx$, get $\displaystyle\int u^2\frac{du}{3} = \frac{u^3}{9} + C = \frac{\sec^3(3x)}{9} + C$.

(b) $\int \cos^5(3x)dx = \int(\cos^2(3x))^2\cos(3x)dx = \int(1-\sin^2(3x))^2\cos(3x)dx$. Use $u = \sin(3x)$ so $du = 3\cos(3x)dx$, get

$$\int(1-u^2)^2\frac{du}{3} = \frac{1}{3}\int(1-2u^2+u^4)du$$
$$= \frac{1}{3}\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right) + C = \frac{1}{3}\left(\sin(3x) - \frac{2\sin^3(3x)}{3} + \frac{\sin^5(3x)}{5}\right) + C.$$

(c)

$$\int \sin^2(4x)dx = \frac{1}{2}\int(1-\cos(8x)dx = \frac{x}{2} - \frac{1}{2}\int\cos(8x)dx$$
$$= \frac{x}{2} - \frac{1}{2}\frac{\sin(8x)}{8} + C = \frac{x}{2} - \frac{\sin(8x)}{16} + C.$$

(d) Break up the integral into the sum of the integrals $\displaystyle\int\frac{x}{\sqrt{9-x^2}}dx + \int\frac{1}{\sqrt{9-x^2}}dx$. In the first one substitute $u = 9-x^2$ so $du = -2xdx$ to get $\int\frac{1}{\sqrt{u}}\frac{du}{-2} = -\sqrt{u} + C = -\sqrt{9-x^2} + C$. In the second one substitute $x = 3\sin(\theta)$ so $dx = 3\cos(\theta)d\theta$ to get $\displaystyle\int\frac{3\cos(\theta)d\theta}{3\cos(\theta)} = \int d\theta = \theta + D = \sin^{-1}(x/3) + D$. The final answer is then $-\sqrt{9-x^2} + \sin^{-1}(x/3) + E$.

(e) Using partial fractions we can write

$$\frac{-x^3+8x^2-11x+2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

Then

$$\begin{aligned} &-x^3+8x^2-11x+2 = A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2 \\ &= (A+C)x^3+(-A+B-2C+D)x^2+(A+C-2D)x+(-A+B+D) \end{aligned}$$

which gives the equations $A+C=-1$, $-A+B-2C+D=8$, $A+C-2D=-11$, and $-A+B+D=2$, so $A=2$, $B=-1$, $C=-3$, $D=5$. So

$$\int \frac{-x^3+8x^2-11x+2}{(x-1)^2(x^2+1)}dx = \int \frac{2}{x-1}dx + \int \frac{-1}{(x-1)^2}dx + \int \frac{-3x}{x^2+1}dx + \int \frac{5}{x^2+1}dx$$
$$= 2\ln|x-1| + \frac{1}{x-1} - \frac{3}{2}\ln|x^2+1| + 5\tan^{-1}(x) + C.$$

(f) This integral can be solved by the rationalizing substitution $x = u^6$, so $dx = 6u^5 du$ and we get

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = \int \frac{6u^5}{u^3+u^2}\,du = 6\int \frac{u^3}{u+1}\,du = 6\int \left(u^2-u+1-\frac{1}{u+1}\right)\,du =$$
$$2u^3-3u^2+6u-6\ln|u+1|+C = 2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln|\sqrt[6]{x}+1|+C$$