Math 222 Calculus 2 Spring 2000 Practice Exam 2 Solutions Feingold

(1) (8 Points) $\int_1^9 \frac{1}{\sqrt[3]{x-9}}\,dx$ is improper since the denominator of the integrand is zero at $x = 9$. So the answer is

$$\lim_{t\to 9^-}\int_1^t (x-9)^{-1/3}dx = \lim_{t\to 9^-}\frac{3}{2}\left[(x-9)^{2/3}\right]_{x=1}^{x=t}$$
$$= \lim_{t\to 9^-}\frac{3}{2}\left[(t-9)^{2/3}-(1-9)^{2/3}\right] = \frac{3}{2}[0-4] = -6.$$

(2) (8 Points) This integral is improper since the upper bound is infinity. We find the indefinite integral, by the substitution $u = \ln(x)$, so $du = \frac{dx}{x}$ and we use $e^u = x$ so $x^2 = e^{2u}$. After the substitution, the integral can be done using integration by parts.

$$\int \frac{\ln(x)}{x^3}\,dx = \int u\,e^{-2u}du = \frac{ue^{-2u}}{-2} - \int \frac{e^{-2u}}{-2}du = \frac{ue^{-2u}}{-2} - \frac{1}{4}e^{-2u} = -x^{-2}(\frac{\ln(x)}{2}+\frac{1}{4}) + C$$

The improper integral we want is then

$$\lim_{t\to\infty}\left[-t^{-2}(\frac{\ln(t)}{2}+\frac{1}{4}) + 1^{-2}(\frac{\ln(1)}{2}+\frac{1}{4})\right] = \lim_{t\to\infty}\left[\frac{-\ln(t)}{2t^2} - \frac{1}{4t^2} + \frac{1}{4}\right] = \frac{1}{4}$$

using L'Hospital's Rule for the first term.

(3) (7 Points) Since $\frac{dx}{dy} = \frac{2-2y}{2\sqrt{2y-y^2}} = \frac{1-y}{\sqrt{2y-y^2}}$, the area of the surface of revolution about the y-axis is

$$\int_0^1 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 2\pi\sqrt{2y-y^2}\sqrt{1+\frac{(1-y)^2}{2y-y^2}}\,dy$$
$$= 2\pi\int_0^1 \sqrt{2y-y^2+(1-y)^2}\,dy = 2\pi\int_0^1\sqrt{2y-y^2+1-2y+y^2}\,dy$$
$$= 2\pi\int_0^1 \sqrt{1}\,dy = 2\pi\int_0^1 1\,dy = 2\pi y\Big]_0^1 = 2\pi$$

(4) (7 Points) Since $\frac{dy}{dx} = -e^{-x}$, the area of the surface of revolution about the x-axis is given by the integral

$$\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2\pi e^{-x}\sqrt{1+e^{-2x}}\,dx.$$

(5) (7 Points) $\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}$ so $1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2$ and the length is

$$\int_1^2\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 \frac{x^2}{2}+\frac{1}{2x^2}\,dx = \left(\frac{x^3}{6}-\frac{1}{2x}\right)\Big]_1^2 = \left(\frac{8}{6}-\frac{1}{4}\right)-\left(\frac{1}{6}-\frac{1}{2}\right) = \frac{17}{12}.$$

(6) (8 Points) Since $2 \le x$ we have $2 < 4 \le x^2$ so $x^2+2 < x^2+x^2 = 2x^2$ and then $\sqrt{x^2+2} < \sqrt{2x^2} = x\sqrt{2}$. It then follows that

$$0 < \frac{1}{x\sqrt{2}} < \frac{1}{\sqrt{x^2+2}}.$$

In the book and in class we proved that the integral $\int_2^\infty \frac{1}{x}dx$ diverges, as does any nonzero multiple of it. So by the Comparison Theorem, the integral $\int_2^\infty \frac{1}{\sqrt{x^2+2}}dx$ diverges.

(7) (8 Points) Using separation of variables, we get the integral $\int y\,dy = \int \frac{1+x}{x}\,dx = \int(\frac{1}{x}+1)\,dx$ which gives $\frac{y^2}{2} = \ln(x)+x+C$. Since $y(1) = -4$ we get $8 = 1+C$ so $7 = C$ and finally $\frac{y^2}{2} = \ln(x)+x+7$ for $x > 0$.

(8) (10 Points) (a) We have $\Delta x = (11-1)/10 = 1$ so the Simpson's Rule approximation is

$$\frac{1}{3}\Big[1\ln(1) + 4\ln(2) + 2\ln(3) + 4\ln(4) + 2\ln(5) + 4\ln(6) + 2\ln(7) +$$
$$4\ln(8) + 2\ln(9) + 4\ln(10) + 1\ln(11)\Big].$$

(b) Since $|f^{(4)}(x)| = |-6x^{-4}| = \frac{6}{x^4} \le 6$ for $1 \le x \le 11$, we can use $K = 6$. Since $(b-a) = 11 - 1 = 10$, we have

$$|E_S| \le \frac{6(10)^5}{180n^4} = \frac{(10)^4}{3n^4}$$

and we want the smallest n such that $\frac{(10)^4}{3n^4} < \frac{1}{(10)^4}$. This means $\frac{(10)^8}{3} < n^4$, which means $\frac{(10)^2}{\sqrt[4]{3}} < n$. We find that $\frac{(10)^2}{\sqrt[4]{3}}$ is approximately 75.98 so the least such n is $n = 76$.

(9) (5 Points for each part) (a) The sequence converges because

$$a_n = \sqrt{n+3}(\sqrt{n+5} - \sqrt{n})\frac{\sqrt{n+5}+\sqrt{n}}{\sqrt{n+5}+\sqrt{n}} = \sqrt{n+3}\frac{(n+5)-n}{\sqrt{n+5}+\sqrt{n}} = \frac{5\sqrt{n+3}}{\sqrt{n+5}+\sqrt{n}}$$
$$= \frac{5\sqrt{1+\frac{3}{n}}}{\sqrt{1+\frac{5}{n}}+1} \to \frac{5\sqrt{1}}{\sqrt{1}+1} = \frac{5}{2} \text{ as } n \to \infty.$$

(b) Let $f(x) = \frac{\ln(2x^2+1)}{\ln(3x^2+2)}$ so that $f(n) = a_n$, and use L'Hospital's rule for type $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{\ln(2x^2+1)}{\ln(3x^2+2)} = \lim_{x\to\infty}\frac{\frac{4x}{2x^2+1}}{\frac{6x}{3x^2+2}} = \lim_{x\to\infty}\frac{(4x)(3x^2+2)}{(6x)(2x^2+1)} = \lim_{x\to\infty}\frac{(4)(3+2/x^2)}{(6)(2+1/x^2)} = 1.$$

(10) (5 Points for each part) (a) This series diverges by the Test for Divergence because

$$\lim_{n\to\infty}\frac{n^2}{8(n+1)(n+2)} = \lim_{n\to\infty}\frac{1}{8(1+\frac{1}{n})(1+\frac{2}{n})} = \frac{1}{8} \ne 0.$$

(b) Since $\frac{4}{n(n-1)} = \frac{4}{n-1} - \frac{4}{n}$, this is a telescoping series whose partial sums are $s_k = \sum_{n=2}^{k} \frac{4}{n-1} - \frac{4}{n} = \frac{4}{1} - \frac{4}{k}$ so the limit $\lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{4}{1} - \frac{4}{k} = 4$ so the series converges and the sum is 4.

(c) A geometric series with ratio $\frac{e}{\pi} < 1$ and first term e^2, converges to $\frac{e^2}{1-\frac{e}{\pi}} = \frac{\pi e^2}{\pi - e}$.

(11) (12 Points) (a) We are given that $25 = 100\, e^{150k}$, so $k = \frac{-\ln\, 4}{150} = \frac{-\ln\, 2}{75}$ and $y(t) = 100\, e^{-\ln(2)t/75} = 100(1/2)^{t/75}$. Then the half-life is the t such that $50 = 100(1/2)^{t/75}$, which means $1/2 = (1/2)^{t/75}$ giving $t/75 = 1$, so $t = 75$ years.

(b) Let $u(t) = f(t) - 5$. Then $u'(t) = 9u(t)$ and $u(0) = f(0) - 5 = 3$ so $u(t) = 3e^{9t}$. Then $f(t) = 5 + 3e^{9t}$.