# Solutions to Exam I

**Problem 1.** a) Let $f(x) = x + e^x$.

i) Since the derivative $f'(x) = 1 + e^x$ is clearly positive for all $x$, the function $f$ is increasing hence one-to-one. Consequently, it has an inverse function $g$. Since $f$ is continuous and $\lim_{x\to+\infty}(x + e^x) = +\infty$, $\lim_{x\to-\infty}(x + e^x) = -\infty$, the range of $f$ consists of all real numbers (by the Intermediate Value Theorem). Thus the domain of $g$ is $\mathbb{R}$.

ii) Since $f(0) = 1$, we see that $g(1) = 0$. From the formula $g'(x) = 1/f'(g(x))$ we get $g'(1) = 1/f'(0) = 1/(1 + e^0) = 1/2$.

iii) Since $f$ and $g$ are inverse functions, $f(g(x)) = x$ for all $x$ in the domain of $g$. Thus $f(g(5)) = 5$ and $f(g(f(g(5)))) = f(g(5)) = 5$.

b) In order to find the inverse function of $f(x) = e^{\tan x}$, $x \in (-\pi/2, \pi/2)$ we want to express $x$ in terms of $y$ from the equation $y = e^{\tan x}$. By taking logarithms of both sides we get $\ln y = \tan x$ so $x = \arctan(\ln y)$. Thus the inverse function of $f$ is $g(x) = \arctan(\ln x)$.

**Problem 2.** a) By the main properties of logarithms, we have

$$\log_3(x + 1) + \log_3(5 - x) = \log_3[(x + 1)(5 - x)]$$

and $2 = \log_3 3^2$. Thus our equation can be written as

$$\log_3[(x + 1)(5 - x)] = \log_3 3^2.$$

Since the logarithmic function is one-to-one, we conclude that $(x + 1)(5 - x) = 3^2$, i.e. $x^2 - 4x + 4 = 0$. Thus $x = 2$ is the only solution.

b) We use logarithmic differentiation: $f' = f(\ln f)'$. Since

$$\ln f(x) = \ln[\sqrt{x}e^{\sin x}(x^2 + 1)^{12}] = 1/2 \ln x + \sin x + 12 \ln(x^2 + 1)$$

we see that

$$(\ln f(x))' = \frac{1}{2x} + \cos x + \frac{24x}{x^2 + 1}$$

Thus,

$$f'(x) = \sqrt{x}e^{\sin x}(x^2 + 1)^{12}[\frac{1}{2x} + \cos x + \frac{24x}{x^2 + 1}]$$

**Problem 3.** a)The limit $\lim_{x\to\infty}(e^x+x)^{1/(x+1)}$ is of the indeterminate type $\infty^0$. We have

$$\lim_{x\to\infty}(e^x+x)^{1/(x+1)} = \lim_{x\to\infty} e^{\ln(e^x+x)/(x+1)}$$

Thus we need to compute $\lim_{x\to\infty}\ln(e^x+x)/(x+1)$, which is of the type $\infty/\infty$. We use L'Hospital's rule and compute first the limit of the ratio of the derivatives of the numerator and denominator:

$$\lim_{x\to\infty}(e^x+1)/(e^x+x)$$

This is again of the form $\infty/\infty$ (and it is rather easy to see that the limit is 1) so we apply L'Hospital's rule again and get

$$\lim_{x\to\infty} e^x/(e^x+1)$$

It is easy to see directly that the limit is 1, but since it is of the form $\infty/\infty$, we may also apply L'Hospital's rule one more time to get

$$\lim_{x\to\infty} e^x/e^x = 1$$

Thus the original limit $\lim_{x\to\infty}(e^x+x)^{1/(x+1)} = e^1 = e$.

b) Since $\lim_{x\to 0}(x^2-x) = 0$ and $\lim_{x\to 0}\cos x = 1$, we see that $\lim_{x\to 0}\frac{x^2-x}{\cos x} = 0/1 = 0$.

c) The limit $\lim_{x\to 0}\frac{\sin x - x}{x^3}$ is of the form $0/0$, so we apply L'Hospital's rule and compute first $\lim_{x\to 0}\frac{\cos x-1}{3x^2}$. This is again of the form $0/0$, so we apply L'Hospital's rule again and compute $\lim_{x\to 0}\frac{-\sin x}{6x}$. This is again of the form $0/0$, so we apply L'Hospital's rule one more time and compute $\lim_{x\to 0}\frac{-\cos x}{6} = -1/6$. Thus

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = -1/6$$

**Problem 4.** a) Integrate by parts with $f'(x) = x^2$ and $g(x) = \ln(x)$. Thus $f(x) = x^3/3$ and $g'(x) = 1/x$, so

$$\int x^2\ln x dx = \frac{(\ln x)x^3}{3} - \int \frac{x^3}{3}\frac{1}{x}dx = \frac{(\ln x)x^3}{3} - \frac{1}{3}\int x^2 dx = \frac{(\ln x)x^3}{3} - \frac{x^3}{9} + C$$

b) Substitute $u = x^3+1$, $du = 3x^2dx$ to get

$$\int_0^1 x^2\sqrt{x^3+1}dx = \int_1^2 \frac{1}{3}\sqrt{u}du = \frac{2u^{3/2}}{9}|_1^2 = \frac{4\sqrt{2}-2}{9}$$

c) <u>1st method</u> We use the identity $\sin a \cos b = (\sin(a+b) + \sin(a-b))/2$. Thus $\sin 2x \cos 3x = (\sin 5x + \sin(-x))/2$ and

$$\int \sin 2x \cos 3x dx = \int \frac{\sin 5x + \sin(-x)}{2} dx = \frac{-\cos 5x}{10} + \frac{\cos x}{2} + C$$

<u>2nd method</u> We integrate by parts with $f'(x) = \sin 2x$ and $g(x) = \cos 3x$. Thus $f(x) = -\cos 2x/2$, $g'(x) = -3\sin 3x$ and

$$\int \sin 2x \cos 3x dx = \frac{-\cos 2x \cos 3x}{2} - \int \frac{3\cos 2x \sin 3x}{2} dx = \frac{-\cos 2x \cos 3x}{2} - \frac{3}{2}\int \cos 2x \sin 3x dx$$

We integrate by parts again with $f'(x) = \cos 2x$, $g(x) = \sin 3x$. Thus $f(x) = \sin 2x/2$, $g'(x) = 3\cos 3x$ and

$$\int \cos 2x \sin 3x dx = \frac{\sin 2x \sin 3x}{2} - \int \frac{3\sin 2x \cos 3x}{2} dx = \frac{\sin 2x \sin 3x}{2} - \frac{3}{2}\int \sin 2x \cos 3x dx$$

It follows that

$$\int \sin 2x \cos 3x dx = \frac{-\cos 2x \cos 3x}{2} - \frac{3\sin 2x \sin 3x}{4} + \frac{9}{4}\int \sin 2x \cos 3x dx$$

so

$$(1 - \frac{9}{4}) \int \sin 2x \cos 3x dx = \frac{-\cos 2x \cos 3x}{2} - \frac{3\sin 2x \sin 3x}{4} + C$$

i.e.

$$\int \sin 2x \cos 3x dx = \frac{2\cos 2x \cos 3x}{5} + \frac{3\sin 2x \sin 3x}{5} + C$$

**Remark.** Note that the answers given by the first and the second methods seem to be different. But they must be the same, which means that we proved an identity of the form

$$\frac{-\cos 5x}{10} + \frac{\cos x}{2} + C = \frac{2\cos 2x \cos 3x}{5} + \frac{3\sin 2x \sin 3x}{5}$$

for some constant $C$. Evaluating at $x = 0$ shows that in fact $C = 0$. Can you prove this identity directly?

d) We use trigonometric substitution $x = \tan t$, $dx = \sec^2 t dt$, so

$$\int \frac{dx}{\sqrt{(1+x^2)^5}} = \int \frac{\sec^2 t dt}{\sqrt{\sec^{10} t}} = \int \frac{dt}{\sec^3 t} = \int \cos^3 t dt$$

Now we use the fact that $\cos^3 t = \cos^2 t \cos t = (1 - \sin^2 t)\cos t$ and we substitute $w = \sin t$, $dw = \cos t dt$ so

$$\int \cos^3 t dt = \int (1 - w^2) dw = w - \frac{w^3}{3} + C = \sin t - \frac{\sin^3 t}{3} + C$$

Now recall that $x = \tan t$. Thus $x \cos t = \sin t$ and $x^2 \cos^2 t = \sin^2 t$. Using the fact that $\cos^2 t = 1 - \sin^2 t$ we see that $x^2 - x^2 \sin^2 t = \sin^2 t$, which gives $\sin^2 t = x^2/(1+x^2)$ and $\sin t = x/\sqrt{1+x^2}$ (be careful about signs here, since we are taking square roots). Alternatively, use the right triangle method to compute $\sin t$ in terms of $x$. Thus the final answer is

$$\int \frac{dx}{\sqrt{(1+x^2)^5}} = \frac{x}{\sqrt{1+x^2}} - \frac{x^3}{3(x^2+1)\sqrt{1+x^2}} + C$$

e) We first complete to squares the quadratic polynomial in the denominator:

$$x^2 - 6x + 13 = x^2 - 2 \cdot 3 \cdot x + 3^2 + 4 = (x-3)^2 + 2^2$$

Then we make a substitution $2w = x - 3$, $dx = 2dw$ to get

$$\int \frac{dx}{x^2 - 6x + 13} = \int \frac{2dw}{4w^2+4} = \frac{1}{2} \int \frac{dw}{w^2+1} = \frac{\arctan w}{2} + C = \frac{\arctan(x-3)/2}{2} + C$$

f) Since the degree of the numerator is not smaller that the degree of the denominator, we divide first $x^5 - 4$ by $x^3 - x^2 - x - 2$. The result of the division is $x^2 + x + 2$ and the remainder is $5x^2 + 4x$. Thus

$$\int \frac{x^5 - 4}{x^3 - x^2 - x - 2} = \int (x^2+x+2)dx + \int \frac{5x^2+4x}{x^3-x^2-x-2}dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x + \int \frac{5x^2+4x}{x^3-x^2-x-2}dx$$

Now we observe that 2 is a root of $x^3 - x^2 - x - 2$, so $x^3 - x^2 - x - 2 = (x-2)(x^2+x+1)$. The polynomial $x^2 + x + 1$ has no real root so it can not be decomposed further. It is time to find the partial fractions decomposition:

$$\frac{5x^2+4x}{x^3-x^2-x-2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1} = \frac{(A+B)x^2 + (A-2B+C)x + A - 2C}{x^3-x^2-x-2}$$

Thus $A + B = 5$, $A - 2B + C = 4$ and $A - 2C = 0$, i.e. $A = 2C$, $B = 5 - A = 5 - 2C$ and $2C - 2(5-2C) + C = 4$. This gives $C = 2$, $B = 1$ and $A = 4$ so

$$\frac{5x^2+4x}{x^3-x^2-x-2} = \frac{4}{x-2} + \frac{x+2}{x^2+x+1}$$

and

$$\int \frac{5x^2+4x}{x^3-x^2-x-2}dx = \int \frac{4}{x-2}dx + \int \frac{x+2}{x^2+x+1}dx = 4\ln(x-2) + \int \frac{x+2}{x^2+x+1}dx$$

In order to compute $\int \frac{x+2}{x^2+x+1}dx$ we complete the denominator to squares: $x^2+x+1 = (x+1/2)^2 + 3/4$. We make a substitution $x + 1/2 = \sqrt{3}w/2$, $dx = dw\sqrt{3}/2$ so $x = \sqrt{3}w/2 - 1/2$ and

$$\int \frac{x+2}{x^2+x+1}dx = \int \frac{\sqrt{3}w/2 + 3/2}{3/4w^2 + 3/4} \frac{\sqrt{3}}{2} dw = \frac{2\sqrt{3}}{3} \left( \frac{\sqrt{3}}{2} \int \frac{wdw}{w^2+1} + \frac{3}{2} \int \frac{dw}{w^2+1} \right) =$$

$$= \frac{2\sqrt{3}}{3}\left(\frac{\sqrt{3}}{4}\ln(w^2+1) + \frac{3}{2}\arctan w\right) + C = \frac{\ln(w^2+1)}{2} + \sqrt{3}\arctan w + C =$$

$$= \frac{\ln(4(x^2+x+1)/3)}{2} + \sqrt{3}\arctan\frac{2x+1}{\sqrt{3}} + C$$

The final answer is then

$$\int \frac{x^5-4}{x^3-x^2-x-2} = \frac{x^3}{3} + \frac{x^2}{2} + 2x + 4\ln(x-2) + \frac{\ln(4(x^2+x+1)/3)}{2} + \sqrt{3}\arctan\frac{2x+1}{\sqrt{3}} + C$$

**Problem 5.** The Midpoint's Rule sum with $n = 8$ for $\int_0^4 e^{2x}dx$ is $\frac{4}{8}(e^{1/2} + e^{3/2} + e^{5/2} + ... + e^{15/2})$. Recall that the error in the Midpoint's Rule approximation is not larger than $B(b-a)^3/24n^2$, where $B$ is a bound for $|f''(x)|$ on $[a, b]$. In our case, $a = 0$, $b = 4$, $f = e^{2x}$. Thus $f''(x) = 4e^{2x}$. This is an increasing, positive function, so $|f''(x)| \leq 4e^8$ for all $x \in [0, 4]$. In other words, we may take $B = 4e^8$. Thus any $n$ which satisfies

$$4e^8\frac{4^3}{24n^2} \leq 10^{-4}$$

would satisfy our requirement. Solving for $n$ gives $n \geq 800e^4/\sqrt{6}$. Taking any $n$ such that $n \geq 800e^4/\sqrt{6}$ in the Midpoint Rule gives approximation to $\int_0^4 e^{2x}dx$ with error smaller than $10^{-4}$.