Solutions to Exam II

Problem 1. a) In order to determine whether the integral $\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx$ is convergent or divergent we use comparison test. Note that $\sqrt{1+\sqrt{x}} \geq 1$ for all x. It follows that

$$\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} \geq \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}}.$$

Recall now that the integral $\int_1^\infty \frac{1}{x^p}dx$ converges for $p > 1$ and diverges for $p \leq 1$. In our case $p = 1/2 < 1$, so the integral $\int_1^\infty \frac{1}{x^{1/2}}dx$ diverges, and so does the integral $\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx$ by the comparison test.

b) For (i) note first that using integration by parts we can compute an anti-derivative of $\ln x/x^2$. In fact, taking $f'(x) = 1/x^2$ and $g(x) = \ln x$ we have $f(x) = -1/x$, $g'(x) = 1/x$ and

$$\int \frac{\ln x}{x^2}dx = \frac{-\ln x}{x} + \int \frac{1}{x^2}dx = \frac{-\ln x}{x} - \frac{1}{x} + C$$

Now

$$\int_1^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}dx = \lim_{b\to\infty} [(\frac{-\ln b}{b} - \frac{1}{b}) - (\frac{-\ln 1}{1} - \frac{1}{1})] = \lim_{b\to\infty} (\frac{-\ln b}{b} - \frac{1}{b}) + 1 = 1$$

since both $\lim_{b\to\infty} \frac{-\ln b}{b} = 0$ and $\lim_{b\to\infty} \frac{1}{b} = 0$.

For (ii) note that the function $1/x^4$ has discontinuity at $x = 0$. Thus

$$\int_{-2}^3 \frac{dx}{x^4} = \int_{-2}^0 \frac{dx}{x^4} + \int_0^3 \frac{dx}{x^4}.$$

Now $-1/(3x^3)$ is an anti-derivative of $1/x^4$ so

$$\int_0^3 \frac{dx}{x^4} = \lim_{a\to 0+} \int_a^3 \frac{dx}{x^4} = \lim_{a\to 0+} (\frac{-1}{3\cdot 3^3} - \frac{-1}{3\cdot a^3}) = \frac{-1}{81} + \lim_{a\to 0+} \frac{1}{3\cdot a^3}$$

and the last limit clearly does not exist. Thus our integral diverges. Note: similarly one proves that $\int_{-2}^0 \frac{dx}{x^4}$ diverges.

Problem 2. a) The length of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \leq x \leq 2$ is given by the following integral:

$$\int_1^2 \sqrt{1 + (\frac{x^2}{2} - \frac{1}{2x^2})^2}dx.$$

Note now that

$$1 + (\frac{x^2}{2} - \frac{1}{2x^2})^2 = 1 + (\frac{x^2}{2})^2 - 2\frac{x^2}{2}\frac{1}{2x^2} + (\frac{1}{2x^2})^2$$

Since $\frac{x^2}{2}\frac{1}{2x^2} = 1/4$, we can write $1 = 4\frac{x^2}{2}\frac{1}{2x^2}$ so

$$1+(\frac{x^2}{2}-\frac{1}{2x^2})^2 = 4\frac{x^2}{2}\frac{1}{2x^2}+(\frac{x^2}{2})^2-2\frac{x^2}{2}\frac{1}{2x^2}+(\frac{1}{2x^2})^2 = (\frac{x^2}{2})^2+2\frac{x^2}{2}\frac{1}{2x^2}+(\frac{1}{2x^2})^2 = (\frac{x^2}{2}+\frac{1}{2x^2})^2$$

Thus

$$\int_1^2 \sqrt{1+(\frac{x^2}{2}-\frac{1}{2x^2})^2}dx = \int_1^2 \sqrt{(\frac{x^2}{2}+\frac{1}{2x^2})^2}dx = \int_1^2 (\frac{x^2}{2}+\frac{1}{2x^2})dx =$$

$$= (\frac{x^3}{6}-\frac{1}{2x})|_1^2 = \frac{8}{6} - \frac{1/4}{-}\frac{1}{6} + \frac{1}{2} = \frac{17}{12}$$

b) The curve $x^2+y^2 = 1$, $0 \leq x \leq 1$ is a half-circle with radius 1 and center $(0,0)$. By rotating this curve about the $x-$axis we obtain half of a sphere of radius 1. We may write $y = \sqrt{1-x^2}$ (we restrict only to the part of the curve above the $x-$axis) so $y' = -x/\sqrt{1-x^2}$. The surface area is given by

$$2\pi\int_0^1 \sqrt{1-x^2}\sqrt{1+(\frac{-x}{\sqrt{1-x^2}})^2}dx = 2\pi\int_0^1 \sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}dx = 2\pi\int_0^1 dx = 2\pi.$$

As a corollary, the area of a sphere of radius 1 is 4π, and more generally the area of a sphere of radius r is $4\pi r^2$.

c) There are 2 formulas which may be used to solve this problem. First formula states that the area is equal to

$$2\pi\int_0^1 x\sqrt{1+(-2x)^2}dx.$$

In order to state the second formula we first need to find the inverse function of $f(x) = 1-x^2$. If $y = 1-x^2$ then $x = \pm\sqrt{1-y}$, and since we are interested in positive x, we see that the inverse function is $g(y) = \sqrt{1-y}$. We have $g'(y) = -1/(2\sqrt{1-y})$, $f(0) = 1$ and $f(1) = 0$ so the second formula for the surface area is

$$2\pi\int_0^1 \sqrt{1-y}\sqrt{1+(\frac{-1}{2\sqrt{1-y}})^2}dy = 2\pi\int_0^1 \sqrt{1-y+\frac{1}{4}}dy = 2\pi\int_0^1 \sqrt{-y+\frac{5}{4}}dy.$$

The integrals in both formulas are rather easy to compute. In the first formula substitute $w = 1+4x^2$, in the second formula substitute $w = 5/4-y$. A simple computation tels us that the surface area in question equals $\pi(5\sqrt{5}-1)/6$.

Problem 3. a) We have $f(x,y) = x+2y$. Euler's method with step size d starts at the initial value $y_0 = y(x_0)$ and constructs a sequence of numbers recursively by the formula $y_{n+1} = y_n + f(x_0+nd, y_n)d$. In our case $x_0 = 0$, $y_0 = 1$, $d = 1$ and we are interested in y_3. We have $y_1 = 1+2 = 3$, $y_2 = 3+7 = 10$ and $y_3 = 10+22 = 32$, so the approximation to $y(3)$ is 32.

Remark. One can verify easily that $y(x) = -x/2 - 1/4 + 5e^{2x}/4$ is the solution to our differential equation. We see that $y(3) = (5e^6-7)/4$ which is much larger than 32.

b) Note that $2 + 2x^2 + y + x^2y = (y+2)(x^2+1)$. So our differential equation is $y' = (y+2)(x^2+1)$, which is a separable equation. There is a constant solution $y = -2$, and if $y \neq -2$ then $y'/(y+2) = x^2 + 1$.Thus

$$\int \frac{dy}{y+2} = \int (x^2+1)dx$$

so

$$\ln|y+2| = \frac{x^3}{3} + x + C$$

i.e.

$$|y+2| = e^{\frac{x^3}{3}+x+C} = Ce^{\frac{x^3}{3}+x}$$

Equivalently, we may write

$$y + 2 = \pm Ce^{\frac{x^3}{3}+x}$$

but the $\pm$ can be absorbed by the constant so finally

$$y(x) = Ce^{\frac{x^3}{3}+x} - 2.$$

c) Let $b(t)$ be the number of bacteria after t hours. The problem tells us that $b(t)$ satisfies a differential equation $b'(t) = kb(t)$ for some constant k. We know that the solutions to this equation are given by the formula $b(t) = Ce^{kt}$ for some constant C. We need to find C and k. We know that $b(0) = 1000$ which tells us that $C = 1000$. Thus $b(t) = 1000e^{kt}$. We also know that $b(2) = 9000$, so $9000 = 1000e^{2k}$, i.e. $e^{2k} = 9$. Taking logarithms of both sides yields $2k = \ln 9 = 2\ln 3$, so $k = \ln 3$. Thus $b(t) = 1000e^{k \ln 3} = 1000 \cdot 3^t$. This answers i). To answer ii) we just compute $b(3) = 1000 \cdot 3^3 = 27000$. Finally, in iii) we look for t such that $2000 = b(t)$, i.e. $2000 = 1000 \cdot 3^t$, so $3^t = 3$ and $t = \ln 2/\ln 3$.

Problem 4. a) Note that

$$(1+\frac{2}{n})^{3n-1} = (1+\frac{2}{n})^{3n}/(1+\frac{2}{n}) = [(1+\frac{2}{n})^n]^3/(1+\frac{2}{n})$$

Recall that

$$\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$$

so

$$\lim_{n\to\infty}(1+\frac{2}{n})^{3n-1} = \lim_{n\to\infty} = [(1+\frac{2}{n})^n]^3/(1+\frac{2}{n}) = [e^2]^3/1 = e^6$$

Alternatively, one can write

$$(1+\frac{2}{n})^{3n-1} = e^{n\ln(1+\frac{2}{n})}$$

and then use L'Hospitele's rule to compute

$$\lim_{x\to\infty}\frac{\ln(1+\frac{2}{x})}{1/x}.$$

b) Note that $2^n < 2^n n^2 + n < 2n^2 2^n$ for every n. Taking $n-$th roots gives

$$2 < \sqrt[n]{2^n n^2 + n} < \sqrt[n]{2n^2 2^n} = \sqrt[n]{2}(\sqrt[n]{n})^2 \cdot 2 \qquad (*).$$

Recall that $\lim_{n\to\infty} \sqrt[n]{2} = 1$ and $\lim_{n\to\infty} \sqrt[n]{n} = 1$, so both the most left and most right sides of $(*)$ tend to 2. By squeeze theorem, $\lim_{n\to\infty} \sqrt[n]{2^n n^2 + n} = 2$.

c)

$$\lim_{n\to\infty} \frac{2^n + 3}{3^n + 2} = \lim_{n\to\infty} \frac{2^n/3^n + 3/3^n}{1 + 2/3^n} = 0/1 = 0$$

Note that $2^n/3^n = (2/3)^n$ which tends to 0, since $|2/3| < 1$. Also $3/3^n$ and $1/3^n$ tend to 0, so the numerator tends to 0 and the denominator tends to 1 as claimed.

Alternatively, compute $\lim_{x\to\infty} \frac{2^x+3}{3^x+2}$ using L'Hospitale's rule.

Problem 5. a) We have

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{\pi^n} = \sum_{n=0}^{\infty} \frac{3 \cdot 3^n}{\pi^n} = 3\sum_{n=0}^{\infty} (\frac{3}{\pi})^n$$

The last sum is the geometric series with $r = 3/\pi$, so it converges (since $|r| < 1$) to $1/(1-r)$, i.e.

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{\pi^n} = \frac{3}{1 - 3/\pi} = \frac{3\pi}{\pi - 3}.$$

Note that $\lim_{n\to\infty} \dfrac{n^2 - 1}{n^2 + 3n - 1} = 1 \neq 0$. Thus the series $\sum_{n=0}^{\infty} \frac{n^2-1}{n^2+3n-1}$ diverges by the divergence test.

We have $\dfrac{1}{n^2 - 1} = 1/2(\dfrac{1}{n-1} - \dfrac{1}{n+1})$. Thus the $k-$th partial sum

$$s_k = \sum_{n=2}^{k} \frac{1}{n^2 - 1} = 1/2\sum_{n=2}^{k} (\frac{1}{n-1} - \frac{1}{n+1}) =$$

$$= 1/2[(1-\frac{1}{3})+(\frac{1}{2}-\frac{1}{4})+(\frac{1}{3}-\frac{1}{5})+...+(\frac{1}{k-2}-\frac{1}{k})+(\frac{1}{k-1)}-\frac{1}{k+1})] = 1/2(1+\frac{1}{2}-\frac{1}{k}-\frac{1}{k+1})$$

We see that

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} 1/2(1 + \frac{1}{2} - \frac{1}{k} - \frac{1}{k+1}) = 3/4.$$

b) Note that $\sum_{n=0}^{\infty} 2^{n+1}x^n = 2\sum_{n=0}^{\infty}(2x)^n$. The sum is a geometric series with $r = 2x$, so it converges iff $|2x| < 1$, i.e. iff $|x| < 1/2$.

Problem 6. First we assume that $\lim_{n\to\infty} a_n = g$ exists. Passing to the limit in the recursive formula $a_{n+1} = \sqrt[3]{3a_n + 2}$ we get that $g = \sqrt[3]{3g + 2}$. Thus $g^3 - 3g - 2 = 0$, i.e. $(g - 2)(g + 1)^2 = 0$, so $g = 2$ or $g = -1$. But looking at the sequence we see that it is always positive, so $g = 2$. **This is not a precise argument, just a prediction,**

a precise justification will follow below. We predict now that a_n is increasing and bounded above by 2 (this in particular will justify our last claim about a_n being positive).

So we want to prove that $a_{n+1} \geq a_n$ for all n. If not, then there is smallest m such that $a_{m+1} < a_m$. Note that $m > 1$, since $a_1 < a_2$. Using the recursive formula, we may write

$$\sqrt[3]{3a_m + 2} < \sqrt[3]{3a_{m-1} + 2}$$

so taking third powers of both sides gives

$$3a_m + 2 < 3a_{m-1} + 2$$

i.e.

$$3a_m < 3a_{m-1}$$

and

$$a_m < a_{m-1}.$$

This however contradicts our assumption about minimality of m. A contradiction shows that indeed $a_{n+1} \geq a_n$ for all n.

Now we want to prove that $a_n < 2$ for all n. If not, then there is smallest m such that $a_m \geq 2$. Note that $m > 1$, since $a_1 < 2$. Since $a_m = \sqrt[3]{3a_{m-1} + 2}$, we have

$$\sqrt[3]{3a_{m-1} + 2} \geq 2$$

so

$$3a_{m-1} + 2 \geq 2^3 = 8$$

i.e.

$$3a_{m-1} \geq 6$$

which means that

$$a_{m-1} \geq 2.$$

Again, this contradicts our assumption about minimality of m. A contradiction shows that indeed $a_n < 2$ for all n.

We showed that a_n is increasing and bounded above. This implies that $a_n > 0$ for all n and that a_n converges (by the monotone convergence theorem). Now our consideration at the beginning shows that $\lim_{n\to\infty} a_n = 2$