# Solutions to Exam III

**Problem 1.** a) Let $f$ be a function which has derivatives of all orders. The $n-$th Taylor polynomial of $f$ centered at $a$ is

$$T_{a,n}f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The Taylor series of $f$ centered at $a$ is defined as

$$T_a f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The $n-$th remainder $R_n(x)$ is the difference $f(x) - T_{a,n}f(x)$. Taylor's inequality asserts that

$$|R_n(x)| \leq M(x-a)^{n+1}/(n+1)!$$

where $M$ is any number such that $|f^{(n+1)}(t)| \leq M$ for all $t$ between $x$ and $a$.

b) The derivatives of $f(x) = \sin x$ are $f^{(0)}(x) = \sin x$, $f^{(1)}(x) = \cos x$, $f^{(2)}(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x$, ... . Thus $f^{(0)}(\pi/2) = 1$, $f^{(1)}(\pi/2) = 0$, $f^{(2)}(\pi/2) = -1$, $f^{(3)}(\pi/2) = 0$, $f^{(4)}(\pi/2) = 1$, ... . Consequently, the Taylor series of $\sin x$ at $a = \pi/2$ is

$$T_{\pi/2}f(x) = 1-(x-\pi/2)^2/2!+(x-\pi/2)^4/4!-(x-\pi/2)^6/6!+... = \sum_{k=0}^{\infty}(-1)^k(x-\pi/2)^{2k}/(2k)!$$

Since the $(n+1)-$st derivative of $f$ is either $\pm\sin x$ or $\pm\cos x$, we see that $|f^{(n+1)}(t)| \leq 1$ for all $t$ so we may take $M = 1$ in Taylor's inequality, i.e. $|R_n(x)| \leq (x-\pi/2)^{n+1}/(n+1)!$. Since $\lim_{n\to\infty} b^n/n! = 0$ for every $b$, we see that $\lim_{n\to\infty}|R_n(x)| = 0$ for every $x$, i.e the Taylor series converges to $f(x)$ for all $x$.

c) Recall that the Taylor series expansion centered at 0 for $\ln(1-x)$ is $-\sum_{k=0}^{\infty} x^k/k$. It follows that the Taylor series expansion centered at 0 for $\ln(1-x^2)$ is $-\sum_{k=0}^{\infty} x^{2k}/k$ and therefore the Taylor series expansion centered at 0 for $f(x) = x\ln(1-x^2)$ is $-\sum_{k=0}^{\infty} x^{2k+1}/k$. The coefficient at $x^9$ of this power series equals $-1/4$. On the other hand, it equals $f^{(9)}(0)/9!$, so $f^{(9)}(0) = -9!/4$.

**Problem 2.** a) **Ratio test.** Suppose that $a_n \neq 0$ and $\lim_{n\to\infty}|a_{n+1}/a_n| = \lambda$ exists. If $\lambda < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely and if $\lambda > 1$ then this series diverges.

**Root test.** Suppose that $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lambda$ exists. If $\lambda < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely and if $\lambda > 1$ then this series diverges.

b) For the series $\sum_{n=0}^{\infty} \frac{(-3)^{1+3n}}{n^n}$ we use the root test (the ratio test works here too, but it is a little more complicated). We have $a_n = \frac{(-3)^{1+3n}}{n^n} = (-3)(\frac{-27}{n})^n$ so $\sqrt[n]{|a_n|} = \sqrt[n]{3}\frac{37}{n}$. Since $\lim_{n\to\infty} \sqrt[n]{3} = 1$ and $\lim_{n\to\infty} \frac{37}{n} = 0$, we see that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$, so the series converges absolutely.

For the series $\sum_{n=0}^{\infty} \frac{n!}{e^n}$ use the ratio test. We have $|a_{n+1}/a_n| = \frac{(n+1)!}{e^{n+1}} / \frac{n!}{e^n} = n + 1/e$, so $\lim_{n\to\infty} |a_{n+1}/a_n| = +\infty > 1$ and consequently the series diverges.

**Problem 3.** a) **Integral test.** If $a_n = f(n)$, where $f$ is a continuous decreasing and positive function on $[N, +\infty)$ (for some $N$) then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\int_N^{\infty} f(t)dt$ converges.

**Alternating series test.** If $a_n$ is a decreasing sequence such that $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

b) Let $f(x) = 1/x \ln x$. Clearly $f$ is a positive decreasing continuous function on $[2, \infty)$ (since both $x$ and $\ln x$ are increasing). Since $\int dx/x \ln x = \int du/u = \ln u + C = \ln(\ln x) + C$ (here $u = \ln x$), we see that $\int_2^{\infty} dx/x \ln x = \lim_{t\to\infty}(\ln(\ln t) - \ln(\ln 2)) = \infty$ so the series $\sum_{n=2}^{\infty} 1/n \ln n$ diverges by the integral test. On the other hand, since $1/n \ln n$ decreases to 0, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by the alternating series test. Thus the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges conditionally.

For the series $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$ we use the integral test. Let $f(x) = x/(x^2+1)$. We see that $f'(x) = (1-x^2)/(x^2+1)^2 < 0$ for $x > 1$. It follows that $f$ is a decreasing function on $[1, \infty)$ and clearly it is continuous and positive. The integral $\int x dx/(x^2+1) = \int du/2u = (\ln u)/2 = (\ln(x^2+1))/2$. Thus $\int_1^{\infty} x dx/(x^2+1) = \lim_{t\to\infty}(\ln(t^2+1))/2 - (\ln 2)/2) = \infty$, so the series $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$ diverges.

Alternatively, note that $n/(n^2+1) > n/(n^2+n) = 1/(n+1)$. Since the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges, the series $\sum_{n=0}^{\infty} \frac{n}{n^2+1}$ diverges by the comparison test (you may also use limit comparison test here).

**Problem 4.**

$$\text{a) } \lim_{n\to\infty}(1-\frac{2}{n})^{3n+1} = \lim_{n\to\infty}[(1+\frac{-2}{n})^n]^3(1-\frac{2}{n}) = [e^{-2}]^3 \cdot 1 = e^{-6}$$

Alternatively, you could use L'Hospital's rule.

b) We use the squeeze theorem. We have $3^n < 2^n + 3^n < 2 \cdot 3^n$ so $3 < \sqrt[n]{2^n + 3^n} < 3\sqrt[n]{2}$. Since $\lim_{n\to\infty} \sqrt[n]{2} = 1$, we see that both the most left and most right sequences tend to 3, so $\lim_{n\to\infty} \sqrt[n]{2^n + 3^n} = 3$.

$$\text{c) } \lim_{n\to\infty} \frac{n2^n + 1}{3^n + 2} = \lim_{n\to\infty} \frac{n(2/3)^n + 1/3^n}{1 + 2/3^n}$$

Since $\lim_{n\to\infty} 1/3^n = 0$ and $\lim_{n\to\infty} n(2/3)^n = 0$ we see that $\lim_{n\to\infty} \frac{n2^n+1}{3^n+2} = 0$. We use here the fact that for any $a$ such that $|a| < 1$ the sequence $na^n$ tends to 0. It can be proved by using L'Hospital's rule or by observing that the series $\sum_{n=1}^{\infty} na^n$ converges by the ratio test, so $\lim_{n\to\infty} na^n = 0$ by the divergence test.

**Problem 5.** A curve $C$ is given by parametric equations $x = \sin t$, $y = \cos^3 t$, $t \in [0, 2\pi]$.

a) We have $dy/dx = (dy/dt)/(dx/dt) = 3\cos^2 t(-\sin t)/\cos t = -3\sin t\cos t$ and $d^2y/dx^2 = (d(dy/dx)/dt)/(dx/dt) = -3(\cos^2 t - \sin^2 t)/\cos t = -3\cos 2t/\cos t$.

b) The tangent to the curve $C$ is horizontal iff $dy/dx = 0$, i.e. iff $-3\sin t\cos t = 0$ . Thus $t = 0, \pi/2, \pi, 3\pi/2 and 2\pi$ are the only parameters in $[0, 2\pi]$ for which the tangent to the curve $C$ is horizontal.

c) Note that the part of $C$ corresponding to $0 \le t \le \pi/2$ is a graph of a function and the area under it is $1/4$ of the area of the region bounded by the curve $C$. Thus the area in question equals $4\int_0^{\pi/2} \cos^3 x \cos x dx = 4\int_0^{\pi/2} \cos^4 x dx$. We have $\cos^4 x = (\cos^2 x)^2 = [(1+\cos 2x)/2]^2 = (1 + 2\cos 2x + \cos^2 2x)/4 = (1 + 2\cos 2x + (1+\cos 4x)/2)/4$. Thus

$$4\int_0^{\pi/2} \cos^4 x dx = \int_0^{\pi/2} (1 + 2\cos 2x + (1+\cos 4x)/2) dx = 3\pi/4$$

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