Quizzes for Math 222

QUIZ 1. Let $f(x) = 2x + \ln x$, $x \in (0, \infty)$.

a) Explain why f has an inverse function.

b) Compute $(f^{-1})'(2)$.

c) Find the range of f. Explain your answer.

Solution: a) Note that $f'(x) = 2+1/x > 0$, so f is increasing, hence also one-to-one.

b) First we need to find $f^{-1}(2)$, i.e. we need to find a such that $f(a) = 2a+\ln a = 2$. We do it by experimenting with some small numbers and observing that $a = 1$ works (solving the equation $f(a) = 3$ would be much harder and no nice answer can be given). Recall now that

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{2+1} = \frac{1}{3}.$$

c) Since f is increasing and continuous on the interval $(0, \infty)$, the range of f is (a, b), where $a = \lim_{x\to 0^+} f(x)$ and $b = \lim_{x\to\infty} f(x)$. Recall that $\lim_{x\to 0^+} \ln x = -\infty$, $\lim_{x\to 0^+} 2x = 0$, $\lim_{x\to\infty} \ln x = \infty$, and $\lim_{x\to\infty} 2x = \infty$. It follows that

$$\lim_{x\to 0^+} (2x + \ln x) = -\infty \text{ and } \lim_{x\to\infty} (2x + \ln x) = \infty,$$

so the range of f is $(-\infty, \infty)$.

QUIZ 2. a) Solve the equation

$$\ln(x - 2) + \ln(x + 1) = \ln 12 - \ln 3$$

b) Differentiate the function $f(x) = x^{\ln x}$.

Solution: a) Note first that the left hand side is defined only for $x > 2$. Using the basic properties of ln we get

$$\ln[(x-2)(x+1)] = \ln\frac{12}{3} = \ln 4.$$

Since ln is a one-to-one function, we conclude that

$$(x-2)(x+1) = 4 \text{ i.e. } x^2 - x - 6 = 0.$$

The last equation has solutuions $3, -2$ but only $x = 3$ satisifes the original constrain $x > 2$. Thus $x = 3$ is the only solution.

b) We use the formula $h(x)^{g(x)} = e^{g(x)\ln h(x)}$. It follows that $f(x) = e^{\ln^2 x}$. Now the chain rule yields

$$f'(x) = e^{\ln^2 x} 2\ln x \frac{1}{x} = 2x^{\ln x}\ln x/x = 2x^{\ln x - 1}\ln x.$$

QUIZ 3. a) Compute $\lim\limits_{x\to\infty}\left(\dfrac{x-1}{x+1}\right)^{2x}$.

b) Compute $\displaystyle\int_0^1 \frac{e^x dx}{e^{2x}+1}$.

Solution: a) Since $\lim_{x\to\infty}\frac{x-1}{x+1} = 1$ (this type of limits were discussed in Calc I; you can also apply L'Hospital's rule), the limit we need to compute is of the type 1^∞. Thus we first try to compute

$$\lim_{x\to\infty}\ln\left[\left(\frac{x-1}{x+1}\right)^{2x}\right] = \lim_{x\to\infty} 2x(\ln(x-1) - \ln(x+1))$$

The last limit is of the type $\infty \cdot 0$, and we transform it as follows:

$$\lim_{x\to\infty} 2x(\ln(x-1) - \ln(x+1)) = 2\lim_{x\to\infty}\frac{\ln(x-1)-\ln(x+1)}{\frac{1}{x}}.$$

Now we get limit of the type $\frac{0}{0}$, so we can apply L'Hospital's rule:

$$2\lim_{x\to\infty}\frac{\ln(x-1)-\ln(x+1)}{\frac{1}{x}} = 2\lim_{x\to\infty}\frac{\frac{1}{x-1}-\frac{1}{x+1}}{\frac{-1}{x^2}} = 2\lim_{x\to\infty}\frac{-2x^2}{(x-1)(x+1)} = 2\cdot(-2) = -4.$$

(We use here the fact that

$$\lim_{x\to\infty}\frac{-2x^2}{(x-1)(x+1)} = \lim_{x\to\infty}\frac{-2x^2}{x^2-1} = -2$$

which should be clear from what you learned in Calc I, or you could apply L'Hospital's rule again). Thus we computed that

$$\lim_{x\to\infty} \ln\left[\left(\frac{x-1}{x+1}\right)^{2x}\right] = -4$$

and therefore

$$\lim_{x\to\infty} \left(\frac{x-1}{x+1}\right)^{2x} = e^{-4}.$$

b) We use substitution $u = e^x$, $du = e^x dx$ to get

$$\int_0^1 \frac{e^x dx}{e^{2x}+1} = \int_1^e \frac{du}{u^2+1} = \arctan(e) - \arctan(1) = \arctan(e) - \frac{\pi}{2}.$$

QUIZ 4. Compute:

a) $\displaystyle\int e^x \sin 2x dx$

b) $\displaystyle\int \tan^2 x \sec^4 x dx$.

Solution: a) We use integration by parts with $f'(x) = e^x$ and $g(x) = \sin 2x$, so $f(x) = e^x$ and $g'(x) = 2\cos 2x$:

$$\int e^x \sin 2x dx = e^x \sin 2x - \int 2e^x \cos 2x dx = e^x \sin 2x - 2\int e^x \cos 2x dx.$$

Now we apply integration by parts to $\displaystyle\int e^x \cos 2x dx$ with $f'(x) = e^x$ and $g(x) = \cos 2x$, so $f(x) = e^x$ and $g'(x) = -2\sin 2x$:

$$\int e^x \cos 2x dx = e^x \cos 2x - \int (-2)e^x \sin 2x dx = e^x \cos 2x + 2\int e^x \sin 2x dx.$$

It follows that

$$\int e^x \sin 2x dx = e^x \sin 2x - 2e^x \cos 2x - 4\int e^x \sin 2x dx$$

i.e.

$$5\int e^x \sin 2x dx = e^x \sin 2x - 2e^x \cos 2x + C.$$

Thus

$$\int e^x \sin 2x dx = \frac{1}{5}e^x \sin 2x - \frac{2}{5}e^x \cos 2x + C.$$

b) We use the identity $\sec^2 x = 1 + \tan^2 x$ as follows:

$$\int \tan^2 x \sec^4 x dx = \int \tan^2 x \sec^2 x \sec^2 x dx = \int \tan^2 x (1 + \tan^2 x) \sec^2 x dx.$$

Now we subsitute $u = \tan x$, $du = \sec^2 x dx$ to get

$$\int \tan^2 x \sec^4 x dx = \int u^2(1+u^2) du = \int (u^2 + u^4) du = \int u^2 du + \int u^4 du =$$

$$= \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C.$$

QUIZ 5. Compute the following integrals:

$$a) \int_0^1 \frac{dx}{(1+x^2)^2} \qquad b) \int_0^{1/2} \sqrt{1-4x^2} dx$$

Solution: a) We use the substitution $x = \tan u$, $dx = \sec^2 u du$:

$$\int_0^1 \frac{dx}{(1+x^2)^2} = \int_0^{\pi/4} \frac{\sec^2 u du}{(1+tan^2 u)^2} = \int_0^{\pi/4} \frac{\sec^2 u du}{\sec^4 u} = \int_0^{\pi/4} \cos^2 u du.$$

Now we use the identity $\cos^2 u = (\cos 2u + 1)/2$ to get

$$\int_0^{\pi/4} \cos^2 u du = \int_0^{\pi/4} \frac{\cos 2u + 1}{2} du = (\frac{\sin 2u}{4} + \frac{u}{2})|_0^{\pi/4} = \frac{1}{4} + \frac{\pi}{8}.$$

b) We use the substitution $x = \frac{\sin u}{2}$, $dx = \frac{\cos u du}{2}$:

$$\int_0^{1/2} \sqrt{1-4x^2} dx = \int_0^{\pi/2} \sqrt{1 - \sin^2 u} \frac{\cos u du}{2} = \frac{1}{2} \int_0^{\pi/2} \cos^2 u du.$$

Now we use the identity $\cos^2 u = (\cos 2u + 1)/2$ to get

$$\int_0^{\pi/2} \cos^2 u du = \int_0^{\pi/2} \frac{\cos 2u + 1}{2} du = (\frac{\sin 2u}{4} + \frac{u}{2})|_0^{\pi/2} = \frac{\pi}{4}.$$

QUIZ 6. a) Write the form of the partial fraction decomposition of the rational function

$$f(x) = \frac{x^2 + 2}{(x^2 - 3x + 3)(x^2 + x + 1)^2}.$$

b) Compute the integral

$$\int \frac{x+2}{(x^2 + 4x + 5)^2} dx.$$

Solution: a) Since the numerator has degree smaller that the denominator, there is no need for long division here. We need to decompose the denominator into product of linear factors and quadratic factors which do not have real roots. Note that x^2+x+1 has no real roots (its discrimant is -1, which is negative). On the other hand, $x^2-3x+2=(x-1)(x-2)$. Thus we have

$$f(x)=\frac{x^2+2}{(x-1)(x-2)(x^2+x+1)^2}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C_1x+D_1}{x^2+x+1}+\frac{C_2x+D_2}{(x^2+x+1)^2}.$$

b) We first complete to square the denominator:

$$x^2+4x+5=x^2+2\cdot x\cdot 2+2^2-2^2+5=(x+2)^2+1.$$

We substitute $u=x+2$ to get

$$\int\frac{x+2}{(x^2+4x+5)^2}dx=\int\frac{u}{(u^2+1)^2}dx.$$

Now substitute $w=u^2+1$, $dw=2udu$ to get

$$\int\frac{x+2}{(x^2+4x+5)^2}dx=\int\frac{\frac{1}{2}}{w^2}dw=\frac{-1}{2w}+C=\frac{-1}{2(u^2+1)}+C=\frac{-1}{2((x+2)^2+1)}+C.$$

Note: one could substitute $u=x^2+4x+5$ and do the computations in one step.

QUIZ 7. a) Determine whether the following integral converges or diverges

$$\int_1^\infty\frac{1}{\sqrt{x^2-x}}dx.$$

b) Compute

$$\lim_{n\to\infty}\frac{\sqrt{n^4+n^3+1}}{n^2+1}.$$

Solution: a) Note that the integral is improper at both ends. Thus we need to study convergence of the integrals

$$\int_2^\infty\frac{1}{\sqrt{x^2-x}}dx \quad\text{and}\quad \int_1^2\frac{1}{\sqrt{x^2-x}}dx.$$

Intuitively, for large x the quantity $1/\sqrt{x^2-x}$ is comparable to $1/x$. This suggests that the first integral should behave like $\int_2^\infty \frac{1}{x}dx$ which diverges. To get a precise argument, note that

$$\frac{1}{\sqrt{x^2-x}}>\frac{1}{x}$$

for all $x > 2$. Since $\int_2^\infty \frac{1}{x} dx$ diverges, the integral

$$\int_2^\infty \frac{1}{\sqrt{x^2 - x}} dx$$

also diverges by comparison test. It follows that the original integral

$$\int_1^\infty \frac{1}{\sqrt{x^2 - x}} dx$$

diverges.

Remark. It is natural to ask whether

$$\int_1^2 \frac{1}{\sqrt{x^2 - x}} dx$$

converges or diverges. Note that $\dfrac{1}{\sqrt{x^2 - x}}$ and $\dfrac{1}{\sqrt{x - 1}}$ are comparable when x tends to 1. Since

$$\int \frac{1}{\sqrt{x - 1}} dx = 2\sqrt{x - 1} + C$$

we see that

$$\lim_{t \to 1^+} \int_t^2 \frac{1}{\sqrt{x - 1}} dx = \lim_{t \to 1^+} (2\sqrt{2 - 1} - 2\sqrt{t - 1}) = 2$$

converges. Note now that

$$\frac{1}{\sqrt{x^2 - x}} = \frac{1}{\sqrt{x(x - 1)}} \leq \frac{1}{\sqrt{x - 1}}$$

for $x > 1$. Thus the integral

$$\int_1^2 \frac{1}{\sqrt{x^2 - x}} dx$$

converges by comparison test.

b) We have

$$\frac{\sqrt{n^4 + n^3 + 1}}{n^2 + 1} = \frac{\sqrt{n^4(1 + \frac{1}{n} + \frac{1}{n^4})}}{n^2(1 + \frac{1}{n^2})} = \frac{\sqrt{1 + \frac{1}{n} + \frac{1}{n^4}}}{1 + \frac{1}{n^2}}.$$

Thus

$$\lim_{n \to \infty} \frac{\sqrt{n^4 + n^3 + 1}}{n^2 + 1} = \lim_{n \to \infty} \frac{\sqrt{1 + \frac{1}{n} + \frac{1}{n^4}}}{1 + \frac{1}{n^2}} = \frac{1}{1} = 1.$$

QUIZ 8. a) Is the series $\sum_{n=1}^{\infty} \arctan n$ convergent? Explain your answer.

b) Compute $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^n}$.

Solution: a) Since $\lim_{n\to\infty} \arctan n = \pi/2 \neq 0$, the series $\sum_{n=1}^{\infty} \arctan n$ diverges by the divergence test.

b) The series is reduced to a geometric series as folows:

$$\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^n} = \sum_{n=1}^{\infty} \frac{2 \cdot 4^n}{5^n} = 2\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n = 2 \cdot \frac{4}{5} \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n = \frac{8}{5} \frac{1}{1 - \frac{4}{5}} = 8.$$

QUIZ 9. Use appropriate test to determine convergence/divergence of

a)

$$\sum_{n=1}^{\infty} \frac{1}{n(1 + (\ln n)^2)}$$

b)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2 + n - 1}.$$

Solution: a) Let $a_n = \frac{1}{n(1 + (\ln n)^2)}$. Thus $a_n = f(n)$, where $f(x) = \frac{1}{x(1 + (\ln x)^2)}$. Note that the function f is positive and continuous. Moreover, since both x and $\ln x$ are increasing functions of x, the function $x(1 + (\ln x)^2)$ is increasing and consequently $f(x)$ is decreasing. Alterantively, note that

$$f'(x) = \frac{-1 - (\ln x)^2 - 2\ln x}{x^2(1 + (\ln x)^2)^2} < 0.$$

Therefore we may apply the integral test and see that our series converges iff the integral $\int_1^{\infty} \frac{1}{x(1 + (\ln x)^2)} dx$ converges. Note that the substituiton $u = \ln x$, $du = dx/x$ yields

$$\int \frac{1}{x(1 + (\ln x)^2)} dx = \int \frac{1}{1 + u^2} du = \arctan u + C = \arctan \ln x + C.$$

Consequently,

$$\int_1^\infty \frac{1}{x(1+(\ln x)^2)}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x(1+(\ln x)^2)}dx = \lim_{t\to\infty} \arctan \ln t = \pi/2.$$

We see that the integral $\displaystyle\int_1^\infty \frac{1}{x(1+(\ln x)^2)}dx$ converges and therefore so does the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n(1+(\ln n)^2)}$

b) It is intuitively clear that $a_n = \dfrac{\sqrt{n+1}}{n^2+n-1}$ behaves very similarly to $b_n = \dfrac{\sqrt{n}}{n^2} = \dfrac{1}{n^{3/2}}$. We try then the limit comparison test for a_n and b_n:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{\sqrt{n+1}}{\sqrt{n}}\frac{n^2}{n^2+n-1}\right) = \lim_{n\to\infty}\left(\sqrt{\frac{n+1}{n}}\frac{n^2}{n^2+n-1}\right) = 1.$$

Since $\sum b_n = \sum \dfrac{1}{n^{3/2}}$ converges, the series $\sum a_n$ converges by the limit comparison test.

QUIZ 10. Consider the power series $\displaystyle\sum_{n=1}^\infty \frac{n(x+1)^n}{4^n}$.

a) Find the radius of convergence of this power series.

b) Find the interval of convergence of this power series.
Justify your answer with appropriate tests.

Solution: a) We use the following result from class:

Theorem 1. *Let* $\displaystyle\sum_{n=1}^\infty c_n(x-a)^n$ *be a power series with radius of convergence* R.

1. *If the limit* $\displaystyle\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|} = L$ *exists or it is* ∞ *then* $R = 1/L$;
2. *If the limit* $\displaystyle\lim_{n\to\infty}\sqrt[n]{|c_n|} = L$ *exists or it is* ∞ *then* $R = 1/L$;

(Note that it means that if $L = 0$ then $R = \infty$ and if $L = \infty$ then $R = 0$)

In our case we have $c_n = n/4^n$, so

$$\lim_{n\to\infty}\sqrt[n]{|c_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{n}{4^n}} = \lim_{n\to\infty}\frac{\sqrt[n]{n}}{4} = \frac{1}{4}.$$

Alternatively,
$$\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|}=\lim_{n\to\infty}\frac{\frac{n+1}{4^{n+1}}}{\frac{n}{4^n}}=\lim_{n\to\infty}\frac{n+1}{4n}=\frac{1}{4}.$$
Thus the radius of convergence $R=4$.

b) The power series is centered at $a=-1$. Thus the interval of conergence has end points $a-R=-1-4=-5$ and $a+R=-1+4=3$. We need to test the convergence of the power series for $x=-5$ and $x=3$. For $x=-5$ we get
$$\sum_{n=1}^{\infty}\frac{n(-5+1)^n}{4^n}=\sum_{n=1}^{\infty}\frac{n(-4)^n}{4^n}=\sum_{n=1}^{\infty}(-1)^n n.$$
Since $a_n=(-1)^n n$ does not converge to 0, the series $\sum_{n=1}^{\infty}(-1)^n n$ diverges by the divergence test, so our power series diverges for $x=-5$.

For $x=3$ we get
$$\sum_{n=1}^{\infty}\frac{n(3+1)^n}{4^n}=\sum_{n=1}^{\infty}\frac{n4^n}{4^n}=\sum_{n=1}^{\infty}n.$$
Since $a_n=n$ does not converge to 0, the series $\sum_{n=1}^{\infty}n$ diverges by the divergence test, so our power series diverges for $x=3$.

It follows that the interval of convergence is $(-5,3)$.

QUIZ 11. a) Find a power series expansion at $a=0$ of the function $f(x)=\dfrac{x}{x^2+16}$.

b) Find the taylor series at $a=1$ of the function $f(x)=x^4-4x^3+9x^2-9x+5$.

Solution: a) Note that
$$f(x)=\frac{x}{x^2+16}=\frac{x}{16}\cdot\frac{1}{1+\frac{x^2}{16}}.$$
Recall now that
$$\frac{1}{1+x}=1-x+x^2-x^3+...=\sum_{n=0}^{\infty}(-1)^n x^n$$
which converges for $x\in(-1,1)$. Substituting $x^2/16$ for x we get
$$\frac{1}{1+\frac{x^2}{16}}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^2}{16}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n}x^{2n}.$$
Thus
$$f(x)=\frac{x}{x^2+16}=\frac{x}{16}\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n}x^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{16^{n+1}}x^{2n+1}.$$

Note that this converges for x such that $-1 < x^2/16 < 1$, i.e. $-4 < x < 4$.

Remark. Using this expansion we can get derivatives of all orders of f at 0. In fact, recall that if $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ then

$$c_n = \frac{f^{(n)}(a)}{n!},$$

so

$$f^{(n)}(a) = n!c_n.$$

Thus, for example, in our case we have $f^{(11)}(0) = 11!$ times the coefficient at $x^{11} = 11!\frac{(-1)^5}{16^6}$.

b) Recall that the Taylor series $T(f,a)(x)$ of a function f at a is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $f^{(n)}(a)$ is the n-th derivative of f at a. Thus we need to compute the derivatives of f at $a = 1$.

$$f^{(0)}(x) = x^4 - 4x^3 + 9x^2 - 9x + 5, \quad f^{(0)}(1) = 2;$$

$$f^{(1)}(x) = 4x^3 - 12x^2 + 18x - 9, \quad f^{(1)}(1) = 1;$$

$$f^{(2)}(x) = 12x^2 - 24x + 18, \quad f^{(2)}(1) = 6;$$

$$f^{(3)}(x) = 24x - 24, \quad f^{(3)}(1) = 0;$$

$$f^{(4)}(x) = 24, \quad f^{(4)}(1) = 24;$$

$$f^{(5)}(x) = 0, \text{ so } f^{(n)}(1) = 0 \text{ for all } n \geq 5.$$

Thus

$$T(f,1)(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = 2 + 1(x-1) + \frac{6}{2!}(x-1)^2 + \frac{0}{3!}(x-1)^3 + \frac{24}{4!}(x-4)^4 =$$

$$= 2 + (x-1) + 3(x-1)^2 + (x-1)^4.$$

QUIZ 12. a) Compute the arc-length function of the curve $y = \arcsin x + \sqrt{1-x^2}$ with starting point $(0,1)$.

b) Find the Taylor series of the function $f(x) = \dfrac{1}{\sqrt{1-x}}$ centered at 0 (i.e. the Maclaurin series). What is the radius of convergence of this series?

Solution: a) Recall that the length of the curve $y = f(x)$ between points $(a, f(a))$ and $(b, f(b))$, $a \leq b$ is given by

$$\int_a^b \sqrt{1+[f'(t)]^2}dt.$$

The arc-length function computes the length of the curve $y = f(x)$ between the starting point and the point $(x, f(x))$ as a function of x. In our case $f(x) = \arcsin x + \sqrt{1-x^2}$ so

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{1-x}{\sqrt{1-x^2}},$$

and that the arc length function is

$$l(x) = \int_0^x \sqrt{1+\left[\frac{1-t}{\sqrt{1-t^2}}\right]^2}dt = \int_0^x \sqrt{1+\frac{1-t}{1+t}}dt = \int_0^x \sqrt{\frac{2}{1+t}}dt = \sqrt{2}\int_0^x \frac{1}{\sqrt{1+t}}dt.$$

Note that

$$\int \frac{1}{\sqrt{1+t}}dt = 2\sqrt{1+t}+C.$$

Thus

$$l(x) = \sqrt{2}\int_0^x \frac{1}{\sqrt{1+t}}dt = 2\sqrt{2}(\sqrt{1+x}-1).$$

Remark. Note that the above formula holds for $x \geq 0$. If $x < 0$, then the above formula yields negative numbers, and the actual length is the negative of the formula above.

b) Recall the binomial series:

$$(1+x)^s = \sum_{n=0}^{\infty} \binom{s}{n} x^n,$$

where $\binom{s}{n} = \dfrac{s(s-1)(s-2)...(s-(n-1))}{n!}$ (recall that, in particular, $\binom{s}{0} = 1$ and $\binom{s}{1} = s$ for any s). The binomial series has radius of converge 1 if s is not a non-negative integer and when $s = 0, 1, 2, ...$ is a non-negative integer then the series converges for all x (it becomes a polynomial of degree s in this case). If $s \leq -1$

then the series has interval of convergence $(-1,1)$, if $-1 < s < 0$ then the interval of convergence is $(-1,1]$ and for $s \geq 0$ the series converges for both $x = -1$ and $x = 1$.

In our problem we have

$$\frac{1}{\sqrt{1-x}} = (1+(-x))^{-1/2} = \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-x)^n = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{-1}{2}}{n} x^n.$$

The binomial coefficients can be transformed into a nicer form as follows

$$\binom{\frac{-1}{2}}{n} = \frac{\frac{-1}{2}(\frac{-1}{2}-1)...(\frac{-1}{2}-(n-1))}{n!} = (-1)^n \frac{\frac{1}{2}\frac{3}{2}\frac{5}{2}...\frac{2(n-1)+1}{2}}{n!} = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot ... \cdot (2n-1)}{2^n n!}.$$

Thus

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{-1}{2}}{n} x^n = \sum_{n=0}^{\infty} (-1)^n (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot ... \cdot (2n-1)}{2^n n!} x^n =$$

$$= \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot ... \cdot (2n-1)}{2^n n!} x^n = 1 + \frac{1}{2}x + \frac{1 \cdot 3}{2^2 \cdot 2!}x^2 + \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4 \cdot 4!}x^4 + ...$$

From the properties of the binomial series, the series has radius of convergence $R = 1$ (and interval of convergence $[-1,1)$).

QUIZ 13. a) Find the area of the surface obtained by rotating the curve $y = \frac{x^2}{4} - \frac{\ln x}{2}$, $1 \leq x \leq 2$ about the y-axis.

b) Find the equation of the line tangent to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ at the point $(4,3)$. What is the length of this curve between points corresponding to $t = 0$ and $t = 1$?

Solution: Let us recall some basic facts about parametric curves. Let $x = \phi(t)$, $y = \psi(t)$, $t \in [a,b]$ be a parametric curve.

1. the derivative of y as a function of x at a point corresponding to a parameter t is equal to $\frac{\psi'(t)}{\phi'(t)}$ (provided $\phi'(t) \neq 0$). In particular, the tangent line to the curve at a point corresponding to a parameter t has slope $\frac{\psi'(t)}{\phi'(t)}$, provided $\phi'(t) \neq 0$. If $\phi'(t) = 0$ and $\psi'(t) \neq 0$ then the tangent is vertical.

2. the length of the curve between points corresponding to $t = a$ and $t = b$ is

$$\int_a^b \sqrt{\phi'(t)^2 + \psi'(t)^2} dt$$

(assuming that there are no overlaps; in general the integral expresses the distance traveled along the curve).

3. the area of the surface obtained by revolving the curve about the x-axis is

$$2\pi \int_a^b \psi(t) \sqrt{\phi'(t)^2 + \psi'(t)^2} dt$$

(assuming that $\psi(t) > 0$ for $t \in [a, b]$).

4. the area of the surface obtained by revolving the curve about the y-axis is

$$2\pi \int_a^b \phi(t) \sqrt{\phi'(t)^2 + \psi'(t)^2} dt$$

(assuming that $\phi(t) > 0$ for $t \in [a, b]$).

5. the graph of a function $y = f(x), x \in [a, b]$ can be considered as a special case of a parametric curve with $x = t$, $y = f(t)$, i.e. $\phi(t) = t$, $\psi(t) = f(t)$, so the above formulas can be applied in this case and yield the formulas we derived earlier for graphs of functions.

Now we can solve our problem.

a) We use the formula in 4. above. Note that in this case $\phi(t) = t$ and $\psi(t) = \frac{t^2}{4} - \frac{\ln t}{2}$. Thus $\phi'(t) = 1$ and $\psi'(t) = \frac{t}{2} - \frac{1}{2t}$. Note that

$$1 + \psi'(t)^2 = 1 + (\frac{t}{2} - \frac{1}{2t})^2 = (\frac{t}{2} + \frac{1}{2t})^2.$$

Thus the surface area is equal to

$$2\pi \int_1^2 t\sqrt{(\frac{t}{2} + \frac{1}{2t})^2} dt = 2\pi \int_1^2 t(\frac{t}{2} + \frac{1}{2t}) dt = \pi \int_1^2 (t^2 + 1) dt = \frac{10\pi}{3}.$$

b) Note that $\phi(t) = 3t^2 + 1$, $\psi(t) = 2t^3 + 1$ so $\phi'(t) = 6t$, $\psi'(t) = 6t^2$. The point $(4, 3)$ corresponds to the parameter $t = 1$. Thus the slope of the tangent at the point

$(4,3)$ is $\dfrac{\psi'(1)}{\phi'(1)} = 1$. Thus the tangent line at $(4,3)$ has equation $y - 3 = 1 \cdot (x-4)$, i.e. $y = x - 1$.

The lenght of the curve is computed using 2:

$$\text{length } = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}dt = \int_0^1 \sqrt{(6t)^2(t^2+1)}dt = \int_0^1 6t\sqrt{t^2+1}dt.$$

Using the substitution $u = t^2 + 1, du = 2tdt$, we get

$$\text{length } = \int_1^2 3\sqrt{u}du = 2(2^{3/2} - 1) = 4\sqrt{2} - 2.$$

QUIZ 14 a) Find the area of the region enclosed by the curve $r = 4 + 3\sin\theta$.

b) Find a function $y(x)$ sych that $y' = 2(y^2+1)x$ and $y(0) = 1$.

Solution: a) The area A of a polar region $0 \le r \le f(\theta)$, $\theta \in [a,b]$ is given by the formula

$$\frac{1}{2}\int_a^b f^2 d\theta.$$

In our case $a = 0$, $b = 2\pi$, $f(\theta) = 4 + 3\sin\theta$, so the area is equal to

$$\frac{1}{2}\int_0^{2\pi} (4+3\sin\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi} (16 + 24\sin\theta + 9\sin^2\theta)d\theta =$$

$$= \int_0^{2\pi} 8d\theta + \int_0^{2\pi} 12\sin\theta d\theta + \frac{9}{2}\int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta = 16\pi + 0 + \frac{9}{4}\int_0^{2\pi} d\theta - \frac{9}{4}\int_0^{2\pi} \cos 2\theta d\theta =$$

$$= 16\pi + \frac{9}{2}\pi - 0 = \frac{41\pi}{2}.$$

b) This is a separable differential equation. The separation of variables yields $dy/(y^2+1) = 2xdx$, so

$$\int \frac{dy}{y^2+1} = \int 2xdx$$

i.e.

$$\arctan y = x^2 + C.$$

It follows that $y(x) = \tan(x^2 + C)$. The condition $y(0) = 1$ implies that $\tan C = 1$ and we may take $C = \pi/4$. Thus $y(x) = \tan(x^2 + \pi/4)$.