# Solutions to Exam I

**Problem 1.** a) Let $f(x) = x + 2e^x$.

i) Since the derivative $f'(x) = 1 + 2e^x$ is clearly positive for all $x$, the function $f$ is increasing hence one-to-one. Consequently, it has an inverse function $g$. Since $f$ is continuous and $\lim_{x\to+\infty}(x + 2e^x) = +\infty$, $\lim_{x\to-\infty}(x + 2e^x) = -\infty$, the range of $f$ consists of all real numbers (by the Intermediate Value Theorem). Thus the domain of $g$ is $\mathbb{R}$.

ii) Clearly $f(0) = 0 + 2e^0 = 2$. Thus $g(2) = g(f(0)) = 0$ (recall that $g(f(x)) = x$ for all $x$). From the formula $g'(x) = 1/f'(g(x))$ we get

$$g'(2) = \frac{1}{f'(0)} = \frac{1}{1 + 2e^0} = \frac{1}{3}.$$

iii) Since $f$ and $g$ are inverse functions, $f(g(x)) = x$ for all $x$ in the domain of $g$. Thus $f(g(7)) = 7$ and $f(g(f(g(7)))) = f(g(7)) = 7$.

b) In order to find the inverse function of $f(x) = e^{\sqrt{x}}$, $x \in (0, \infty)$ we want to express $x$ in terms of $y$ from the equation $y = e^{\sqrt{x}}$. By taking logarithms of both sides we get $\ln y = \sqrt{x}$ so $x = (\ln y)^2$. Thus the inverse function of $f$ is $g(x) = (\ln x)^2$.

**Problem 2.** a) Note that $x$ must satisfy $x < 4$ and $x > -2$. By the main properties of logarithms, we have

$$\log_3(x + 2) + \log_3(4 - x) = \log_3[(x + 2)(4 - x)]$$

and $2 = \log_3 3^2$. Thus our equation can be written as

$$\log_3[(x + 2)(4 - x)] = \log_3 3^2.$$

Since the logarithmic function is one-to-one, we conclude that $(x + 2)(4 - x) = 3^2$, i.e. $x^2 - 2x + 1 = 0$, which is the same as $(x - 1)^2 = 0$. Thus $x = 1$ is the only solution.

b) To find the derivative of $f(x) = x^{\arctan x}$ we use the formula $f(x)^{g(x)} = e^{g(x)\ln f(x)}$. Thus $f(x) = x^{\arctan x} = e^{\arctan x \ln x}$. Now we can use the chain rule:

$$f'(x) = e^{\arctan x \ln x}(\arctan x \ln x)' = x^{\arctan x}\left(\frac{\ln x}{1 + x^2} + \frac{\arctan x}{x}\right).$$

In order to differentiate $g(x) = \log_{e^x} 2$ we use the lormula $\log_a b = \ln b / \ln a$. Thus $f(x) = \ln 2/ \ln e^x = \ln 2/x$. Thus $f'(x) = -\ln 2/x^2$.

**Problem 3.** a) The limit $\lim_{x\to\infty}(e^x + 3)^{1/(x+1)}$ is of the indeterminate type $\infty^0$. Thus we first compute

$$\lim_{x\to\infty} \ln(e^x + 3)^{1/(x+1)} = \lim_{x\to\infty} \frac{\ln(e^x + 3)}{x+1}$$

The last limit is of the type $\infty/\infty$. We use L'Hospital's rule so we compute the limit of the ratio of the derivatives, which is:

$$\lim_{x\to\infty} \frac{e^x}{e^x + 3}.$$

This is again of the form $\infty/\infty$ (it is fairly easy to see that the limit is 1 though), so we apply L'Hospital's rule again and get

$$\lim_{x\to\infty} \frac{e^x}{e^x} = 1.$$

Thus the original limit $\lim_{x\to\infty}(e^x + x)^{1/(x+1)} = e^1 = e$.

b) Since $\lim_{x\to 0}(x^2 - x) = 0$ and $\lim_{x\to 0} \cos x = 1$, we see that $\lim_{x\to 0} \frac{x^2-x}{\cos x} = 0/1 = 0$.

c) The limit $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3}$ is of the form $0/0$, so we apply L'Hospital's rule and compute first $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{3x^2}$. This is again of the form $0/0$, so we apply L'Hospital's rule again and compute $\lim\limits_{x\to 0} \dfrac{-\sin x}{6x}$. This is again of the form $0/0$, so we apply L'Hospital's rule one more time and compute $\lim\limits_{x\to 0} \dfrac{-\cos x}{6} = -1/6$. Thus

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = -1/6.$$

d) The limit $\lim_{x\to\infty} x(\frac{\pi}{2} - \arctan x)$ is of the form $\infty \cdot 0$. Using the formula $f(x)g(x) = g(x)/\frac{1}{f(x)}$ we transform it to the form $\infty/\infty$ and then apply L'Hospital's rule:

$$\lim_{x\to\infty} x(\frac{\pi}{2} - \arctan x) = \lim_{x\to\infty} \frac{\frac{\pi}{2} - \arctan x}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{1+x^2}.$$

The last limit is of the form $\infty/\infty$, so one more application of L'Hospital's rule yields

$$\lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{2x}{2x} = 1.$$

Thus $\lim_{x\to\infty} x(\frac{\pi}{2} - \arctan x) = 1$

**Problem 4.** a) Substitute $u = \ln x$, $du = dx/x$ to get

$$\int_1^e \frac{\sqrt{\ln x}}{x} dx = \int_0^1 \sqrt{u} du = \frac{2}{3} u^{3/2}|_0^1 = \frac{2}{3}.$$

b) Substitute $u = x^2$, $du = 2x dx$ to get

$$\int_0^{1/\sqrt{2}} \frac{2x dx}{\sqrt{1-x^4}} = \int_0^{1/2} \frac{du}{\sqrt{1-u^2}} = \arcsin x|_0^{1/2} = \arcsin \frac{1}{2} - \arcsin 0 = \pi/6.$$

c) Substitute $u = 2e^{2x}$, $du = 4e^{2x} dx$ to get

$$\int \frac{4e^{2x} dx}{1 + 4e^{4x}} = \int \frac{du}{1+u^2} = \arctan u + C = \arctan(2e^{2x}) + C.$$

**Problem 5.** Let $b(t)$ be the number of bacteria after t hours. The problem tells us that $b(t)$ satisfies a differential equation $b'(t) = kb(t)$ for some constant $k$. We know that the solutions to this equation are given by the formula $b(t) = Ce^{kt}$ for some constant $C$. We need to find $C$ and $k$. We know that $b(0) = 1500$ which tells us that $C = 1500$. Thus $b(t) = 1500e^{kt}$. We also know that $b(3) = 12000$, so $12000 = 1500e^{3k}$, i.e. $e^{3k} = 8$. Taking logarithms of both sides yields $3k = \ln 8 = 3 \ln 2$, so $k = \ln 2$. Thus $b(t) = 1500e^{t \ln 2} = 1500 \cdot 2^t$. This answers 1).

To answer 2) we look for $t$ such that $3000 = b(t)$, i.e. $3000 = 1500 \cdot 2^t$, so $2^t = 2$ and $t = 1$.