Solutions to Exam III

Problem 1. a) Consider the parametric curve $x = t^2 + t + 1$, $y = 4t^3 + 3t^2 + 2$. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ as functions of t. Find the equation of the line tangent to the curve at the point $(1, 1)$.

b) Sketch the curve given in polar coordinates by the equation $r = \theta^2$, $\theta \in [0, 2\pi]$. Find a parametric equation of this curve in Cartesian coordinates (use θ as the parameter). Compute the length of this curve.

c) The curve $x = t^2 + 2$, $y = \dfrac{1}{3}t^3 - t + 2$, $t \in [0, 1]$ is revolved about the y-axis. Compute the area of the resulting surface.

Solution: a) Recall that $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$. Since $\frac{dy}{dt} = 12t^2 + 6t = 6t(2t+1)$ and $\frac{dx}{dt} = 2t + 1$, we get $\dfrac{dy}{dx} = \dfrac{6t(2t+1)}{2t+1} = 6t$.

Similarly,

$$\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx} = \frac{\frac{d(\frac{dy}{dx})}{dt}}{\frac{dx}{dt}} = \frac{\frac{d(6t)}{dt}}{2t+1} = \frac{6}{2t+1}.$$

The point $(1, 1)$ corresponds to the parameter $t = -1$ (you solve the system $t^2+t+1 = 1, 4t^3 + 3t^2 + 2 = 1$). Thus the slope of the tangent is the value of $\frac{dy}{dx}$ at $t = -1$, which is -6. Thus the tangent line has equation $y - 1 = -6(x - 1)$, i.e. $y = -6x + 7$.

b) The point with polar coordinates (r, θ) has Cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$. Since on our curve $r = \theta^2$, our curve in Cartesian coordinates is given by parametric equation $x = \theta^2\cos\theta$, $y = \theta^2\sin\theta$. The formula for the length of a parametric curve $x = \phi(\theta)$, $y = \psi(\theta)$, $\theta \in [a, b]$ is given by

$$\int_a^b \sqrt{\phi'(\theta)^2 + \psi'(\theta)^2}\,d\theta = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta.$$

Since $\dfrac{dx}{d\theta} = 2\theta\cos\theta - \theta^2\sin\theta$, $\dfrac{dy}{d\theta} = 2\theta\sin\theta + \theta^2\cos\theta$, we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (2\theta\cos\theta - \theta^2\sin\theta)^2 + (2\theta\sin\theta + \theta^2\cos\theta)^2 = 4\theta^2 + \theta^4$$

and therefore the length of the curve is

$$\int_0^{2\pi} \sqrt{4\theta^2 + \theta^4}\,d\theta = \int_0^{2\pi} \theta\sqrt{4 + \theta^2}\,d\theta.$$

Using substitution $u = \theta^2 + 4$, $du = 2\theta d\theta$, we get that the length is equal to

$$\int_4^{4+4\pi^2} \frac{1}{2}\sqrt{u}\,du = \frac{1}{3}((4 + 4\pi^2)^{3/2} - 4^{3/2}) = \frac{8}{3}((1 + \pi^2)^{3/2} - 1).$$

c) Recall that the area of the surface obtained by revolving the curve $x = \phi(t)$, $y = \psi(t)$, $t \in [a, b]$ about the y-axis is

$$2\pi \int_a^b \phi(t)\sqrt{\phi'(t)^2 + \psi'(t)^2}dt$$

(assuming that $\phi(t) > 0$ for $t \in [a, b]$). Since $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = t^2 - 1$, the surface area is

$$2\pi \int_0^1 (t^2+2)\sqrt{(2t)^2 + (t^2-1)^2}dt = 2\pi \int_0^1 (t^2+2)\sqrt{(t^2+1)^2}dt = 2\pi \int_0^1 (t^2+2)(t^2+1)dt =$$

$$= 2\pi \int_0^1 (t^4 + 3t^2 + 2)dt = \frac{32}{5}\pi.$$

Problem 2. Compute the following infinite sums:

a) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{-2}{3}\right)^{n-1}$ b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}}\right)$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{(1-\frac{1}{e})^n}{n}$

Solution: a) Recall the formula for the sum of a geometric series: $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, for $|x| < 1$. Note that

$$\sum_{n=2}^{\infty} \left(\frac{-2}{3}\right)^{n-1} = \frac{-2}{3} + \left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \ldots = \frac{-2}{3}[1 + \frac{-2}{3} + \left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \ldots] =$$

$$= \frac{-2}{3}\sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n = \frac{-2}{3}\frac{1}{1-(\frac{-2}{3})} = \frac{-2}{5}.$$

b) This is an example of the so called telescoping. The n-th partial sum of the series is:

$$s_N = \sum_{n=1}^{N} \left(\frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}}\right) =$$

$$= \left(\frac{1}{e} - \frac{1}{2e^2}\right) + \left(\frac{1}{2e^2} - \frac{1}{3e^3}\right) + \left(\frac{1}{3e^3} - \frac{1}{4e^4}\right) + \ldots + \left(\frac{1}{Ne^N} - \frac{1}{(N+1)e^{N+1}}\right) = \frac{1}{e} - \frac{1}{(N+1)e^{N+1}}.$$

It follows that

$$\sum_{n=1}^{\infty} \left(\frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}}\right) = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \left(\frac{1}{e} - \frac{1}{(N+1)e^{N+1}}\right) = \frac{1}{e}.$$

c) Recall the power series expansion of $\ln(1-x)$:

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

which is valid for $x \in [-1, 1)$. Taking $x = 1 - \frac{1}{e}$, we get

$$\ln(1 - (1 - \frac{1}{e})) = -\sum_{n=1}^{\infty} \frac{(1-\frac{1}{e})^n}{n}$$

i.e.

$$\sum_{n=1}^{\infty} \frac{(1-\frac{1}{e})^n}{n} = -\ln\left(\frac{1}{e}\right) = 1.$$

Problem 3. Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Explain what test you are applying and verify all the conditions necessary to apply the test.

a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{1}{n}\right)$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4-n^2+1}}$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{\ln n+1}}$ d) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2n}-1\right)^{2n^2}$

Solution: a) Recall the divergence test: if $\lim_{n\to\infty} a_n$ is not equal to 0 then the series $\sum a_n$ diverges. In our case $a_n = (-1)^n \cos\left(\frac{1}{n}\right)$ does not tend to 0, since $\displaystyle\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$. Thus the series diverges.

b) We compare the series to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$, which is a p-series with $p = 4/3 > 1$, hence it is convergent. We use limit comparison test, which we can since the terms of both series are positive:

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n^4}}}{\frac{1}{\sqrt[3]{n^4-n^2+1}}} = \lim_{n\to\infty} \frac{\sqrt[3]{n^4-n^2+1}}{\sqrt[3]{n^4}} = \lim_{n\to\infty} \sqrt[3]{1-\frac{1}{n^2}+\frac{1}{n^4}} = 1.$$

Since the limit is a positive real number, the limit comparison test tells us that either both series diverge or they both converge. Since we know that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$ converges, the other series converges as well. Since the series has positive terms, it converges absolutely.

c) The sign of the terms of our series alternates. Thus we would like to apply the alternating series test with $a_n = \dfrac{1}{n\sqrt{\ln n+1}}$. Clearly the function $x\sqrt{\ln x+1}$ is increasing for $x \geq 1$. Thus the function $\dfrac{1}{x\sqrt{\ln x+1}}$ is decreasing and therefore the sequence a_n is decreasing. Moreover, $\lim_{n\to\infty} a_n = 0$, so we can apply the alternating series test to conclude that the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges. To test for absolute convergence we need to determine whether the series

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n\sqrt{\ln n+1}}\right| = \sum_{n=1}^{\infty}\frac{1}{n\sqrt{\ln n+1}}$$

converges. We have $a_n = f(n)$, where $f(x) = \dfrac{1}{x\sqrt{\ln x+1}}$ is positive and decreasing for $x \geq 1$. Thus we may apply the integral test to conclude that our series converges absolutely iff the integral $\displaystyle\int_1^{\infty} \frac{dx}{x\sqrt{\ln x+1}}$ converges. Using the substitution $u = 1+\ln x$ we see that

$$\int_1^t \frac{dx}{x\sqrt{\ln x+1}} = \int_1^{1+\ln t} \frac{du}{\sqrt{u}} = 2\sqrt{1+\ln t}-2.$$

Thus the integral

$$\int_1^{\infty} \frac{dx}{x\sqrt{\ln x+1}} = \lim_{t\to\infty}\int_1^t \frac{dx}{x\sqrt{\ln x+1}} = \lim_{t\to\infty}(2\sqrt{1+\ln t}-2) = +\infty$$

diverges and therefore our series is not convergent absolutely. In other words, our series is conditionally convergent.

d) We apply the root test. We have

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{1}{2n}-1\right)^{2n^2}\right|} = \lim_{n\to\infty}\left(1-\frac{1}{2n}\right)^{2n} = \lim_{n\to\infty}\left(1+\frac{-1}{2n}\right)^{2n} = e^{-1} < 1$$

(recall that $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$). Since the limit is smaller than 1, the series converges absolutely.

Problem 4. a) Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+2}(x-1)^n$. Carefully justify your answer.

b) Give the power series expansion centered at 0 for the following functions and state radius of convergence:

$$\text{i) } \sqrt{1-2x^2} \qquad \text{ii) } \frac{d}{dx}\left(\frac{1}{1-x}\right) \qquad \text{iii) } \int \cos(\sqrt{x})dx$$

c) Use i) of part b) to compute the 6th derivative of $\sqrt{1-2x^2}$ at 0.

Solution: a) We use the ratio test method to compute the radius of convergence. We have

$$\lim_{n\to\infty} \frac{\frac{2(n+1)+1}{3(n+1)^2+2}}{\frac{2n+1}{3n^2+2}} = \lim_{n\to\infty} \frac{2n+3}{2n+1}\frac{3n^2+2}{3n^2+6n+5} = 1.$$

It follows that the radius of convergence $R = 1/1 = 1$. The power series in centered at 1, so the interval of convergence has left end at $1-R = 1-1 = 0$ and right and at $1+R = 2$.

To test convergence at the left end we look at the series when $x = 0$, i.e. the series $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+2}(-1)^n$. The sign of the terms of this series alternates. Consider $a_n = \frac{2n+1}{3n^2+2}$. We claim that this sequence is decreasing. This means that $a_n > a_{n+1}$, i.e.

$$\frac{2n+1}{3n^2+2} > \frac{2n+3}{3n^2+6n+5}$$

which is equivalent to $(2n+1)(3n^2+6n+5) > (2n+3)(3n^2+2)$, i.e. $6n^3+15n^2+16n+5 > 6n^3+9n^2+4n+6$, i.e. $9n^2+12n > 1$, which is clearly true . Alternatively, compute the derivative of $f(x) = \frac{2x+1}{3x^2+2}$, which is $f'(x) = \frac{4-6x}{(3x^2+2)^2} < 0$ for $x \geq 1$. This means that the function is decreasing for $x \geq 1$ and therefore the sequence $a_n = f(n)$ is decreasing. Clearly, $\lim_{n\to\infty} a_n = 0$. By the alternating series test, the series $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+2}(-1)^n$ converges.

To test convergence at the right end we look at the series when $x = 2$, i.e. the series $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+2}$, which has positive terms. We compare it to the harmonic series. We use limit comparison test:

$$\lim_{n\to\infty} \frac{\frac{2n+1}{3n^2+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^2+n}{3n^2+2} = \frac{2}{3}.$$

Since the limit exists and it is positive, the limit comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+2}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ either both converge or both diverge. Since the harmonic series diverges, the series $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+2}$ diverges as well.

It follows that the interval of convergence is $[0, 2)$.

b) For i) we need the binomial series:

$$(1+x)^u = \sum_{n=0}^{\infty} \binom{u}{n} x^n.$$

When $u = 1/2$ this series has radius of convergence 1. Substitute $-2x^2$ for x to get

$$\sqrt{1-2x^2} = (1-2x^2)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-2x^2)^n = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-2)^n x^{2n}.$$

This series converges for $2x^2 < 1$, i.e. for $|x| < \sqrt{2}/2$, so the radius of convergence is $\sqrt{2}/2$.

For ii) recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and the radius of convergence is 1. Differentiation yields

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n.$$

The radius of convergence remains the same under differentiation, so the series has radius of convergence 1.

For iii) we need to recall the power series expansion of $\cos x$:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

which converges for all x. Substitute $\sqrt{x}$ for x to get

$$\cos \sqrt{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$$

which converges for all x (note that the right hand side makes sense also for negative x). Integrating we get

$$\int \cos(\sqrt{x}) dx = \int \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!} \right) dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)(2n)!}.$$

Since the radius of convergence remains the same under integration, the series convergence for all x.

c) Recall that if a function is given by a power series, then the power series coincides with the Taylor series. Thus, for $f(x) = \sqrt{1-2x^2}$ we get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-2)^n x^{2n}.$$

It follows that $f^{(n)}(0) = 0$ for odd n and $\frac{f^{(2n)}(0)}{(2n)!} = \binom{\frac{1}{2}}{n}(-2)^n$. Taking $n = 3$, we get $\frac{f^{(6)}(0)}{(6)!} = \binom{\frac{1}{2}}{3}(-2)^3$. Recall that

$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} = \frac{1}{16}.$$

Thus

$$f^{(6)}(0) = 6! \cdot \frac{1}{16} \cdot (-2)^3 = \frac{-6!}{2} = -360.$$

Problem 5. a) Find the Taylor series for $f(x) = \sin x$ centered at $\pi/2$. Use Taylor's inequality to prove that the Taylor series converges to $f(x)$ for all x.

b) Find the degree 3 Taylor polynomial centered at $\pi/4$ for $f(x) = \ln \cos x$.

Solution: a) The Taylor series of a function f centered at $\pi/2$ is the series

$$T(f, \pi/2)(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)}{n!} \left(x - \frac{\pi}{2}\right)^n.$$

When $f(x) = \sin x$, the derivatives of f are $\sin x$, $\cos x$, $-\sin x$, $-\cos x$, $\sin x, \ldots$. Since $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$, we see that $f^{(n)}(\pi/2) = 0$ for n odd and $f^{(2n)}(\pi/2) = (-1)^n$. It follows that

$$T(\sin, \pi/2)(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n}.$$

Since $|f^{(n)}(x)| \le 1$ for all n and all x, Taylor's inequality tells us that for any x we have

$$|\sin x - T_N(\sin, \pi/2)(x)| \le \frac{\left|x - \frac{\pi}{2}\right|^{N+1}}{(N+1)!}.$$

When N tends to infinity, the right hand side tends to 0 and therefore the series

$$T(\sin, \pi/2)(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n}$$

converges to $\sin x$ for every value of x.

b) Recall that the degree 3 Taylor polynomial for a function f, centered at $\pi/4$, is

$$T_3(f, \pi/4)(x) = \sum_{n=0}^{3} \frac{f^{(n)}(\pi/4)}{n!} \left(x - \frac{\pi}{4}\right)^n =$$

$$= f(\pi/4) + f'(\pi/4)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''(\pi/4)}{6}\left(x - \frac{\pi}{4}\right)^3.$$

When $f(x) = \ln \cos x$, then $f(\pi/4) = \ln(\sqrt{2}/2) = (-\ln 2)/2$. Furthermore,

$$f'(x) = -\tan x \text{ and } f'(\pi/4) = -1,$$

$$f''(x) = -\sec^2(x) \text{ and } f''(\pi/4) = -2,$$

$$f'''(x) = -2\sec^2(x)\tan x \text{ and } f''(\pi/4) = -4.$$

Thus

$$T_3(\ln\cos, \pi/4)(x) = \frac{-\ln 2}{2} - \left(x - \frac{\pi}{4}\right) - \left(x - \frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{4}\right)^3.$$