Solutions to the Midterm Exam, Math 222

Problem 1. Evaluate the following integrals.

a) $\int_1^4 e^{\sqrt{x}}\,dx$

Solution. Use substitution $x = u^2$, $dx = 2udu$ to get

$$\int_1^4 e^{\sqrt{x}}\,dx = \int_1^2 e^u 2u\,du = 2\int_1^2 ue^u\,du.$$

(Equivalently, you can use substitution $u = \sqrt{x}$, $du = dx/2\sqrt{x}$, $dx = 2udu$ to arrive at the same integral). Now integrate by parts with $f(u) = u$, $g'(u) = e^u$, so $f'(u) = 1$, $g(u) = e^u$, to get

$$\int_1^2 ue^u\,du = ue^u|_1^2 - \int_1^2 e^u\,du = 2e^2 - e - (e^2 - e) = e^2.$$

It follows that

$$\int_1^4 e^{\sqrt{x}}\,dx = 2e^2.$$

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b) $\int \frac{3x^2 - x + 1}{x^3 + x}\,dx$

Solution. Note that $x^3 + x = x(x^2+1)$ and x^2+1 is irreducible (it has negative discriminant). The partial fraction decomposition of the rational function we integrate has the following form:

$$\frac{3x^2 - x + 1}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} = \frac{A(x^2+1) + x(Bx + C)}{x(x^2+1)} = \frac{(A+B)x^2 + Cx + A}{x^3 + x}.$$

It follows that $3x^2 - x + 1 = (A+B)x^2 + Cx + A$, so $A + B = 3$, $C = -1$, and $A = 1$. Thus $A = 1$, $B = 2$, $C = -1$. Our integral is

$$\int \frac{3x^2 - x + 1}{x^3 + x}\,dx = \int \frac{dx}{x} + \int \frac{2x - 1}{x^2 + 1}\,dx = \int \frac{dx}{x} + \int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx =$$

$$= \ln|x| + \ln(x^2+1) - \arctan x + C = \ln|x^3 + x| - \arctan x + C.$$

Second solution. The integral could also be evaluated as follows:

$$\int \frac{3x^2 - x + 1}{x^3 + x}\,dx = \int \frac{3x^2 + 1}{x^3 + x}\,dx - \int \frac{x}{x^3 + x}\,dx = \int \frac{3x^2 + 1}{x^3 + x}\,dx - \int \frac{1}{x^2 + 1}\,dx.$$

The first integral on the right can be evaluated using the substitution $u = x^3 + x$, $du = (3x^2 + 1)\ dx$ to be $\ln|x^3 + x| + C$, and the second integral is $\arctan x + C$.

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c) $\displaystyle\int \frac{1}{\sqrt{4x - x^2}}\ dx.$

Solution. We first complete to squares:

$$4x - x^2 = 4 - (4 - 4x + x^2) = 4 - (x - 2)^2.$$

Now we make a substitution $u = x - 2$, $du = dx$ and get

$$\int \frac{1}{\sqrt{4x - x^2}}\ dx = \int \frac{1}{\sqrt{4 - (x-2)^2}}\ dx = \int \frac{1}{\sqrt{4 - u^2}}\ du.$$

Now we make a substitution $u = 2\sin t$, $du = 2\cos t\ dt$, $\sqrt{4 - u^2} = 2\cos t$, $t \in [-\pi/2, \pi/2]$, to get

$$\int \frac{1}{\sqrt{4 - u^2}}\ du = \int \frac{1}{2\cos t} 2\cos t dt = \int dt = t + C.$$

Now $\sin t = u/2$ so $t = \arcsin(u/2)$. As $u = x - 2$, we get

$$\int \frac{1}{\sqrt{4x - x^2}}\ dx = \arcsin\frac{x - 2}{2} + C.$$

Remark. You could also use the formula

$$\int \frac{1}{\sqrt{a^2 - u^2}}\ du = \arcsin\frac{u}{a} + C.$$

Second solution. We make a substitution $u = \sqrt{4 - x}$, $du = \dfrac{-dx}{2\sqrt{4 - x}}$, $x = 4 - u^2$, to get

$$\int \frac{1}{\sqrt{4x - x^2}}\ dx = \int \frac{1}{\sqrt{x}} \frac{dx}{\sqrt{4 - x}} = \int \frac{1}{\sqrt{4 - u^2}}(-2)\ du = -2\int \frac{du}{\sqrt{4 - u^2}}.$$

As in the first solution, we have

$$\int \frac{1}{\sqrt{4 - u^2}}\ du = \arcsin(u/2) + C$$

so

$$\int \frac{1}{\sqrt{4x - x^2}}\ dx = -2\arcsin\frac{u}{2} + C = -2\arcsin\frac{\sqrt{4 - x}}{2} + C.$$

Remark. The answers provided by each solution seem different. The conclusion is that we have the following identity:

$$\arcsin\frac{x - 2}{2} = -2\arcsin\frac{\sqrt{4 - x}}{2} + C$$

for some constant C and all $x \in [0,4]$. Taking $x = 0$, we get $-\pi/2 = -\pi + C$ so $C = \pi/2$, i.e.

$$\arcsin \frac{x-2}{2} = -2 \arcsin \frac{\sqrt{4-x}}{2} + \frac{\pi}{2}.$$

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d) $\int e^{-2x} \sin x dx$

Solution. We use integration by parts with $f(x) = e^{-2x}$ and $g'(x) = \sin x$, so $f'(x) = -2e^{-2x}$ and $g(x) = -\cos x$:

$$\int e^{-2x} \sin x \, dx = -e^{-2x} \cos x - \int (-2e^{-2x})(-\cos x) \, dx = -e^{-2x} \cos x - 2 \int e^{-2x} \cos x dx.$$

Now we apply integration by parts to $\int e^{-2x} \cos x \, dx$ with $f(x) = e^{-2x}$ and $g'(x) = \cos x$, so $f'(x) = -2e^{-2x}$ and $g(x) = \sin x$:

$$\int e^{-2x} \cos x dx = e^{-2x} \sin x - \int (-2e^{2x}) \sin x \, dx = e^{-2x} \sin x + 2 \int e^{-2x} \sin x \, dx.$$

It follows that

$$\int e^{-2x} \sin x \, dx = -e^{-2x} \cos x - 2e^{-2x} \sin x - 4 \int e^{-2x} \sin x \, dx$$

i.e.

$$5 \int e^{-2x} \sin x \, dx = -e^{-2x} \cos x - 2e^{-2x} \sin x + C.$$

Thus

$$\int e^{-2x} \sin x dx = -\frac{1}{5} e^{-2x} \cos x - \frac{2}{5} e^{-2x} \sin x + C.$$

Second solution. We use integration by parts with $f'(x) = e^{-2x}$ and $g(x) = \sin x$, so $f(x) = -\frac{1}{2} e^{-2x}$ and $g'(x) = \cos x$:

$$\int e^{-2x} \sin x \, dx = -\frac{1}{2} e^{-2x} \sin x - \int (-\frac{1}{2} e^{-2x}) \cos x \, dx = -\frac{1}{2} e^{-2x} \sin x + \frac{1}{2} \int e^{-2x} \cos x \, dx.$$

Now we apply integration by parts to $\int e^{-2x} \cos x \, dx$ with $f'(x) = e^{-2x}$ and $g(x) = \cos x$, so $f(x) = -\frac{1}{2} e^{-2x}$ and $g'(x) = -\sin x$:

$$\int e^{-2x} \cos x \, dx = -\frac{1}{2} e^{-2x} \cos x - \int (-\frac{1}{2} e^{2x})(-\sin x) \, dx = -\frac{1}{2} e^{-2x} \cos x - \frac{1}{2} \int e^{-2x} \sin x \, dx.$$

It follows that

$$\int e^{-2x} \sin x \, dx = -\frac{1}{2} e^{-2x} \sin x - \frac{1}{4} e^{-2x} \cos x - \frac{1}{4} \int e^{-2x} \sin x \, dx$$

i.e.

$$\frac{5}{4}\int e^{-2x}\sin x\ dx = -\frac{1}{2}e^{-2x}\sin x - \frac{1}{4}e^{-2x}\cos x + C.$$

Thus

$$\int e^{-2x}\sin x dx = -\frac{2}{5}e^{-2x}\sin x - \frac{1}{5}e^{-2x}\cos x + C.$$

Question. What would happen if we applied the method from the first solution to the second integration by parts in the second solution?

Problem 2. Circle the correct form for the partial fraction decomposition of the rational function $\dfrac{x^5 - x^3 + 1}{(x^2 - x - 2)^2(x^2 + x + 2)}$:

a) $\dfrac{Ax+B}{x^2-x-2} + \dfrac{Cx+D}{(x^2-x-2)^2} + \dfrac{Ex+F}{x^2+x+2}$

b) $\dfrac{Ax+B}{(x^2-x-2)^2} + \dfrac{Cx+D}{x^2+x+2}$

c) $\dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2} + \dfrac{Ex+F}{x^2+x+2}$

d) $\dfrac{A}{x-2} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+x+2}$

e) $\dfrac{A}{x^2-x-2} + \dfrac{B}{(x^2-x-2)^2} + \dfrac{C}{x^2+x+2}$

f) $\dfrac{A}{(x-2)^2} + \dfrac{B}{(x+1)^2} + \dfrac{Cx+D}{x^2+x+2}$.

Solution. Note that the discriminant of $x^2 + x + 2$ is $1 - 4 \cdot 2 = -7 < 0$, which is negative so this polynomial is irreducible. On the other hand, $x^2 - x - 2$ has positive discriminant $1 + 4 \cdot 2 = 9$ and the polynomial factors into linear factors $x^2 - x - 2 = (x-2)(x+1)$. Thus the denominator factors into irreducibles as $(x-2)^2(x+1)^2(x^2+x+2)$, so the correct form for the partial fraction decomposition is c):

$$\frac{x^5 - x^3 + 1}{(x^2 - x - 2)^2(x^2 + x + 2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+x+2}.$$

Problem 3. a) Does $\displaystyle\int_0^\infty \frac{2x}{1+x^4}\ dx$ converge or diverge? Why? Evaluate it if it converges.

Solution. The convergence of the integral can be easily established using the comparison test, as $\dfrac{2x}{1+x^4} < \dfrac{2x}{x^4} = \dfrac{2}{x^3}$. This however is unnecessary, as we can actually compute the integral. First we use the substitution $u = x^2$, $du = 2x\ dx$ to get

$$\int \frac{2x}{1+x^4}\ dx = \int \frac{du}{1+u^2} = \arctan u + C = \arctan x^2 + C.$$

Thus

$$\int_0^\infty \frac{2x}{1+x^4}\,dx = \lim_{t\to\infty}\int_0^t \frac{2x}{1+x^4}\,dx = \lim_{t\to\infty}(\arctan t^2 - \arctan 0) = \frac{\pi}{2}.$$

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b) Verify that the integral test can be applied to the series $\sum_{n=1}^{\infty} \frac{n}{1+n^4}$ and use it to determine if this series converges or diverges.

Solution. Let $f(x) = \frac{x}{1+x^4}$. Clearly f is continuous and positive on $[1,\infty)$. Now

$$f'(x) = \frac{(1+x^4) - x(4x^3)}{(1+x^4)^2} = \frac{1-3x^4)}{(1+x^4)^2} < 0 \text{ for } x > 1$$

so $f(x)$ is decreasing on $[1,\infty)$. Thus we can apply the integral test, which says that the integral $\int_1^\infty f(x)dx$ converges if and only if the sum $\sum_{n=1}^{\infty} f(n)$ converges. As the convergence of the integral has been established in part a), the infinite series $\sum_{n=1}^{\infty} \frac{n}{1+n^4}$ converges.

Remark. Strictly speaking, the integral in a) is $\int_0^\infty \frac{2x}{1+x^4}\,dx$ and the integral in b) is $\int_1^\infty \frac{x}{1+x^4}\,dx$. Note however that there is a simple relation between them:

$$\int_0^\infty \frac{2x}{1+x^4}\,dx = \int_0^1 \frac{2x}{1+x^4}\,dx + 2\int_1^\infty \frac{x}{1+x^4}\,dx.$$

In particular, one converges if and only if the other does. You could also handle the integral in b) directly.

Problem 4. Does $\int_1^4 \frac{1}{x-2}\,dx$ converge or diverge? Why? Evaluate it if it converges.

Solution. The integral in question is improper since the function $\frac{1}{x-2}$ is unbounded around $x=2$. Thus we must break up the integral and consider separately the improper integrals $\int_1^2 \frac{1}{x-2}\,dx$ and $\int_2^4 \frac{1}{x-2}\,dx$. If either of these integrals diverges, then so does the original integral. If both of them converge, then

$$\int_1^4 \frac{1}{x-2} = \int_1^2 \frac{1}{x-2}\,dx + \int_2^4 \frac{1}{x-2}\,dx.$$

Now

$$\int_1^2 \frac{1}{x-2} = \lim_{t\to 2^-}\int_1^t \frac{1}{x-2} = \lim_{t\to 2^-}(\ln|t-2| - \ln|1-2|) = -\infty,$$

so the integral $\int_1^2 \frac{1}{x-2}\,dx$ diverges, and hence our original integral diverges as well (similarly one shows that $\int_2^4 \frac{1}{x-2}\,dx$ diverges).

Problem 5. Use the Comparison Theorem to determine whether the following improper integral converges or diverges. DO NOT COMPUTE THE EXACT VALUE OF THE INTEGRAL, but show all work needed for the Comparison Theorem.

$$\int_2^\infty \frac{x}{\sqrt{x^6+4}}\,dx$$

Solution. Clearly $x^6+4 > x^6$, so $\sqrt{x^6+4} > \sqrt{x^6} = x^3$ and $\frac{x}{\sqrt{x^6+4}} < \frac{x}{x^3} = \frac{1}{x^2}$ for all $x > 2$. Now we know that the integral $\int_2^\infty \frac{1}{x^2} dx$ converges. Thus the integral $\int_2^\infty \frac{x}{\sqrt{x^6+4}}\,dx$ also converges by the comparison test.

Problem 6. Compute limits of the following sequences:

a) $a_n = \dfrac{\sqrt{2n^6-n^3+3n}}{3n^3+2n^2-3\cos n}$

Solution. We have

$$a_n = \frac{\sqrt{2n^6-n^3+3n}}{3n^3+2n^2-3\cos n} = \frac{\sqrt{n^6(2-\frac{n^3}{n^6}+\frac{3n}{n^6})}}{n^3(3+\frac{2n^2}{n^3}-\frac{3\cos n}{n^3})} = \frac{\sqrt{2-\frac{1}{n^3}+\frac{3}{n^5}}}{3+\frac{2}{n}-\frac{3\cos n}{n^3}}.$$

Recall now that $\lim_{n\to\infty} \frac{c}{n^p} = 0$ for any $p > 0$ and any constant c. Also, $0 \le |\frac{3\cos n}{n^3}| \le \frac{3}{n^3}$, so $\lim_{n\to\infty} \frac{3\cos n}{n^3} = 0$ by the squeeze theorem. Thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{2-\frac{n^3}{n^6}+\frac{3n}{n^6}}}{3+\frac{2}{n}-\frac{3\cos n}{n^3}} = \frac{\sqrt{2-0+0}}{3+0-0} = \frac{\sqrt{2}}{3}.$$

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b) $a_n = \ln(2n+\sqrt{n}) - \ln(n)$

Solution. We have

$$a_n = \ln(2n+\sqrt{n}) - \ln(n) = \ln\frac{2n+\sqrt{n}}{n} = \ln\left(2+\frac{1}{\sqrt{n}}\right).$$

As $\lim_{n\to\infty}\left(2+\frac{1}{\sqrt{n}}\right) = 2$ and $\ln x$ is continuous at 2, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(2+\frac{1}{\sqrt{n}}\right) = \ln\left(\lim_{n\to\infty}\left(2+\frac{1}{\sqrt{n}}\right)\right) = \ln 2.$$

c) $a_n = \left(1+\frac{2}{n}\right)^{3n+2}$.

Solution. We have

$$a_n = \left(1+\frac{2}{n}\right)^{3n+2} = \left(1+\frac{2}{n}\right)^{2}\left(1+\frac{2}{n}\right)^{3n} = \left(1+\frac{2}{n}\right)^{2}\left[\left(1+\frac{2}{n}\right)^{n}\right]^3.$$

Recall now that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for all x. So $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2$. Also, $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^2 = 1$. Thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{2}\lim_{n\to\infty}\left[\left(1+\frac{2}{n}\right)^{n}\right]^3 = 1\cdot[e^2]^3 = e^6.$$

Remark. One can also compute $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x+2}$ using L'Hospital's rule.

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d) $a_n = \sqrt[n]{3^n+4^n}$ (Hint: Use the Squeeze Theorem.)

Solution. Clearly $4^n < 3^n+4^n < 2\cdot 4^n$ for every natural number n. Thus $\sqrt[n]{4^n} < \sqrt[n]{3^n+4^n} < \sqrt[n]{2\cdot 4^n}$, i.e. $4 < a_n < 4\sqrt[n]{2}$. Now $\lim_{n\to\infty} 4 = 4 = \lim_{n\to\infty} 4\sqrt[n]{2}$, so $\lim_{n\to\infty}\sqrt[n]{3^n+4^n} = 4$ by the squeeze theorem.

Problem 7. A **convergent** sequence of **positive** numbers satisfies the recursive relation $a_n a_{n+1} = a_n + 2$. Find $\lim_{n\to\infty} a_n$.

Solution. Let $\lim_{n\to\infty} a_n = g$ (the problem tells us that it exists). Then $\lim_{n\to\infty} a_{n+1} = g$ and $\lim_{n\to\infty} a_n a_{n+1} = g^2$. Thus

$$g^2 = \lim_{n\to\infty} a_n a_{n+1} = \lim_{n\to\infty}(a_n+2) = g+2.$$

The quadratic equation $x^2 = x+2$ has 2 solutions: $x=2$ and $x=-1$, so g must be one of them. However, the sequence consists of positive numbers, so its limit can not be negative. It follows that $g=2$ is the only possibility.

Problem 8. Compute the following limits

a) $\lim_{x\to\infty} x(3^{1/x}-1)$

Solution. The limit is of the form $\infty \cdot 0$. We use the identity $fg = \frac{g}{\frac{1}{f}}$ to make it into the $\frac{0}{0}$ form and then apply L'Hospital's rule:

$$\lim_{x\to\infty} x(3^{1/x} - 1) = \lim_{x\to\infty} \frac{3^{1/x} - 1}{\frac{1}{x}} = \lim_{x\to\infty} \frac{3^{1/x} \ln 3 \frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} 3^{1/x} \ln 3 = 3^0 \ln 3 = \ln 3.$$

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b) $\lim_{x\to 0}(1 + 3x)^{1/\sin x}$

Solution. The limit is of the form 1^∞, which is an indeterminate form. We use the following fact:

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim_{x\to a} g(x)\ln f(x)}.$$

In our case, we have $f(x) = 1 + 3x$ and $g(x) = 1/\sin x$ and $a = 0$. Thus we first compute the limit

$$\lim_{x\to 0} \frac{\ln(1 + 3x)}{\sin x}.$$

As the last limit is of the form $\frac{0}{0}$ we can try to apply L'Hospital's rule and compute the limit of the ratio of the derivatives:

$$\lim_{x\to 0} \frac{\frac{1}{1+3x} \cdot 3}{\cos x} = \frac{\frac{1}{1+3\cdot 0} \cdot 3}{1} = 3.$$

Thus

$$\lim_{x\to 0}(1 + 3x)^{1/\sin x} = e^3.$$

Problem 9. Determine whether each series converges or diverges. If the series converges, find the sum. Explain the reason for your answer. Make sure to mention what test your using and explain why the test is appropriate.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{\pi^n}$

Solution. We have

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{\pi^n} = \sum_{n=0}^{\infty} 3 \cdot \frac{3^n}{\pi^n} = 3\sum_{n=0}^{\infty} \left(\frac{3}{\pi}\right)^n.$$

The last series is a geometric series with the common ratio $r = 3/\pi$, which is between -1 and 1. Thus the geometric series converges to $1/(1 - r)$, so

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{\pi^n} = 3\sum_{n=0}^{\infty} \left(\frac{3}{\pi}\right)^n = 3 \cdot \frac{1}{1 - \frac{3}{\pi}} = \frac{3\pi}{\pi - 3}.$$

b) $\sum_{n=1}^{\infty} \sqrt[n]{2}$

Solution. Since $\lim_{n\to\infty} \sqrt[n]{2} = 1 \neq 0$, the series diverges by the divergence test.

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c) $\sum_{n=2}^{\infty} \frac{n^2-5n}{n^3+n^2-1}$

Solution. For large n the quantity $\frac{n^2-5n}{n^3+n^2-1}$ is about $\frac{n^2}{n^3} = \frac{1}{n}$ and both are positive. Thus we compare the series to the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$, which diverges. We use the limit comparison test, hence we compute

$$\lim_{n\to\infty} \frac{\frac{n^2-5n}{n^3+n^2-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-5n^2}{n^3+n^2-1} = \lim_{n\to\infty} \frac{1-\frac{5}{n}}{1+\frac{1}{n}-\frac{1}{n^3}} = 1.$$

The limit comparison test tells us that the series $\sum_{n=2}^{\infty} \frac{n^2-5n}{n^3+n^2-1}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ either both converge or both diverge. Since the harmonic series diverges, our series $\sum_{n=2}^{\infty} \frac{n^2-5n}{n^3+n^2-1}$ diverges as well.

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d) $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

Solution. We use partial fractions to get the identity $\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$. Consider now the n-th partial sum s_n of our series:

$$s_n = \sum_{k=1}^{n} \frac{2}{k(k+2)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+2}\right) =$$

$$= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+2}\right) =$$

$$= \left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}\right) - \left(\frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n+2}\right) = \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}.$$

Problem 10.(extra credit) Use the identity $\tan\frac{x}{2} = \cot\frac{x}{2} - 2\cot x$ to compute

$$\sum_{n=1}^{\infty} \frac{1}{2^n}\tan\frac{a}{2^n}.$$

Solution. Let us substitute $x = a/2^{n-1}$ into the given identity: $\tan\frac{a}{2^n} = \cot\frac{a}{2^n} - 2\cot\frac{a}{2^{n-1}}$. Dividing both sides by 2^n, we get

$$\frac{1}{2^n}\tan\frac{a}{2^n} = \frac{1}{2^n}\cot\frac{a}{2^n} - \frac{1}{2^{n-1}}\cot\frac{a}{2^{n-1}}.$$

Set $a_n = \frac{1}{2^n}\tan\frac{a}{2^n}$ and $b_n = \frac{1}{2^n}\cot\frac{a}{2^n}$ so the the last equality becomes $a_n = b_n - b_{n-1}$. It follows that the n-th partial sum of our series is

$$s_n = a_1 + a_2 + \ldots + a_n = (b_1 - b_0) + (b_2 - b_1) + \ldots + (b_n - b_{n-1}) = b_n - b_0$$

(the process is often called "telescoping"). Thus

$$\sum_{n=1}^{\infty} \frac{1}{2^n}\tan\frac{a}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(b_n - b_0) = \lim_{n\to\infty} b_n - b_0 = \lim_{n\to\infty}\frac{1}{2^n}\cot\frac{a}{2^n} - \cot a.$$

It remains to compute $\lim_{n\to\infty}\frac{1}{2^n}\cot\frac{a}{2^n}$. Not that

$$\frac{1}{2^n}\cot\frac{a}{2^n} = \frac{1}{2^n}\frac{\cos\frac{a}{2^n}}{\sin\frac{a}{2^n}} = \frac{1}{a}\frac{\cos\frac{a}{2^n}}{\frac{\sin\frac{a}{2^n}}{\frac{a}{2^n}}}.$$

When n tends to infinity, the quantity $a/2^n$ tends to 0, so $\lim_{n\to\infty}\cos\frac{a}{2^n} = 1$ and $\lim_{n\to\infty}\frac{\sin\frac{a}{2^n}}{\frac{a}{2^n}} = 1$ (recall that $\lim_{x\to\ 0}\frac{\sin x}{x} = 1$). It follows that

$$\lim_{n\to\infty}\frac{1}{2^n}\cot\frac{a}{2^n} = \lim_{n\to\infty}\frac{1}{a}\frac{\cos\frac{a}{2^n}}{\frac{\sin\frac{a}{2^n}}{\frac{a}{2^n}}} = \frac{1}{a}.$$

Thus

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\tan\frac{a}{2^n} = \frac{1}{a} - \cot a.$$

Remark. Let us justify the identity used in the problem: $\tan\frac{x}{2} = \cot\frac{x}{2} - 2\cot x$. This is the same as

$$\cot\frac{x}{2} - \tan\frac{x}{2} = 2\cot x.$$

Recall now that $\cos^2 t - \sin^2 t = \cos 2t$ and $2\sin t\cos t = \sin 2t$. Thus

$$2\cot x = 2\frac{\cos x}{\sin x} = 2\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} - \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \cot\frac{x}{2} - \tan\frac{x}{2}.$$

which is exactly what we need to justify.
