# Quizzes for Math 222

**QUIZ 1.** I. Let $f(x) = 2x + \ln x$, $x \in (0, \infty)$.

a) Explain why $f$ has an inverse function.

b) Compute $(f^{-1})'(2)$.

c) Find the range of $f$. Explain your answer.

II. What is the inverse funstion of $y = e^{x^3}$?

**Solution:** I. a) Note that $f'(x) = 2 + 1/x > 0$, so $f$ is increasing, hence also one-to-one.

b) First we need to find $f^{-1}(2)$, i.e. we need to find $a$ such that $f(a) = 2a + \ln a = 2$. We do it by experimenting with some small numbers and observing that $a = 1$ works (solving the equation $f(a) = 3$ would be much harder and no nice answer can be given). Recall now that

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{2+1} = \frac{1}{3}.$$

c) Since $f$ is increasing and continuous on the interval $(0, \infty)$, the range of $f$ is $(a, b)$, where $a = \lim_{x \to 0^+} f(x)$ and $b = \lim_{x \to \infty} f(x)$. Recall that $\lim_{x \to 0^+} \ln x = -\infty$, $\lim_{x \to 0^+} 2x = 0$, $\lim_{x \to \infty} \ln x = \infty$, and $\lim_{x \to \infty} 2x = \infty$. It follows that

$$\lim_{x \to 0^+} (2x + \ln x) = -\infty \text{ and } \lim_{x \to \infty} (2x + \ln x) = \infty,$$

so the range of $f$ is $(-\infty, \infty)$.

II. We need to use the equation $y = e^{x^3}$ to express $x$ in terms of $y$. Taking natural logarithms of both sides we get $\ln y = \ln e^{x^3} = x^3$. Thus $x = \sqrt[3]{\ln y}$. The inverse function is $y = \sqrt[3]{\ln x}$.

**QUIZ 2.** a) Differentiate the function $f(x) = \log_{10^x} 2^{x \arctan x}$.

b) Compute $\sin(\arctan(3/4))$. Explain your reasoning.

c) Compute $\int \frac{dx}{\sqrt{x}\sqrt{1-x}}$. Hint: $u = \sqrt{x}$.

**Solution:** a) We use the identity $\log_a b = \ln b / \ln a$. Thus

$$f(x) = \log_{10^x} 2^{x \arctan x} = \frac{\ln 2^{x \arctan x}}{\ln 10^x} = \frac{(x \arctan x) \ln 2}{x \ln 10} =$$
$$(\arctan x) \frac{\ln 2}{\ln 10} = (\log 2) \arctan x.$$

Now we are ready to differentiate: $f'(x) = [(\log 2) \arctan x]' = \frac{\log 2}{(1+x^2)}$.

b) Let $\alpha = \arctan(3/4)$. Then $\tan(\alpha) = 3/4$ and we need to compute $\sin \alpha$. Note that $\alpha \in (-\pi/2, \pi/2)$ and $\tan \alpha > 0$. This tells us that $\sin \alpha > 0$ (and $\alpha \in (0, \pi/2)$). We have $\sin \alpha / \cos \alpha = 3/4$, i.e. $\sin \alpha = \frac{3}{4} \cos \alpha$. By squaring both sides we get

$$\sin^2 \alpha = \frac{9}{16} \cos^2 \alpha = \frac{9}{16} (1 - \sin^2 \alpha).$$

This means that $(1 + \frac{9}{16}) \sin^2 \alpha = \frac{9}{16}$ which yields $\sin^2 \alpha = \frac{9}{25}$. It follows that $\sin \alpha = \frac{3}{5}$ (since we have seen that it is positive).

c) As hinted, we use the substitution $u = \sqrt{x}$. Thus $du = dx/2\sqrt{x}$ and $x = u^2$. We get

$$\int \frac{dx}{\sqrt{x}\sqrt{1-x}} = \int \frac{2du}{\sqrt{1-u^2}} = 2 \arcsin u + C = 2 \arcsin \sqrt{x} + C.$$

**QUIZ 3.** a) Compute the following limits:

$$(1) \quad \lim_{x \to \infty} x(\pi - \arctan x) \qquad (2) \quad \lim_{x \to 0} (1 - \cos x)^x.$$

b) Compute the following integrals:

$$(1) \quad \int 2^{\cosh x} \sinh x dx \qquad (2) \quad \int x^2 e^{2x} dx.$$

**Solution:** a) (1) Note that $\lim_{x \to \infty} x = \infty$ and

$$\lim_{x \to \infty} (\pi - \arctan x) = \pi - \lim_{x \to \infty} \arctan x = \pi - \pi/2 = \pi/2.$$

It follows that this is not an indeterminate form and we have

$$\lim_{x\to\infty} x(\pi - \arctan x) = \infty \cdot \pi/2 = \infty.$$

(2) As $\lim_{x\to 0}(1-\cos x) = 0$, the limit is of the form $0^0$, hence it is of one of the exponential indeterminate forms. So we first compute

$$\lim_{x\to 0} \ln(1-\cos x)^x = \lim_{x\to 0} x\ln(1-\cos x) = \lim_{x\to 0} \frac{\ln(1-\cos x)}{\frac{1}{x}}.$$

The last limit is of the form $\infty/\infty$, hence we can try to use L'Hospital's rule, so we compute the limit of the derivatives:

$$\lim_{x\to 0} \frac{\frac{\sin x}{1-\cos x}}{\frac{-1}{x^2}} = \lim_{x\to 0} \frac{x^2 \sin x}{\cos x - 1}.$$

The last limit is of the form $0/0$, so we try to use L'Hospital's rule again:

$$\lim_{x\to 0} \frac{2x\sin x + x^2\cos x}{-\sin x}.$$

Again, we get a limit of the form $0/0$ so we use L'Hospital's rule one more time:

$$\lim_{x\to 0} \frac{2\sin x + 2x\cos x + 2x\cos x = x^2 \sin x}{-\cos x} = 0/-1 = 0.$$

In the last limit the numeartor approches 0 when $x$ tends to 0 and the denominator approaches $-1$, so the limit equals 0. Going backwards and using L'Hospital's rule, we conclude that

$$\lim_{x\to 0} \ln(1-\cos x)^x = 0.$$

Thus

$$\lim_{x\to 0}(1-\cos x)^x = e^0 = 1.$$

b) (1) We use substitution $u = \cosh x$, so $du = \sinh x dx$ and the integral becomes

$$\int 2^u du = \frac{2^u}{\ln 2} + C = \frac{2^{\cosh x}}{\ln 2} + C.$$

(2) We use integration by parts: $f'(x) = e^{2x}$, $g(x) = x^2$, so $f(x) = e^{2x}/2$, $g'(x) = 2x$ and

$$\int x^2 e^{2x} dx = \frac{x^2 e^{2x}}{2} - \int x e^{2x} dx.$$

We need to sompute the integral on the right hand side, and we use integration by parts again: $f'(x) = e^{2x}$, $g(x) = x$, so $f(x) = e^{2x}/2$, $g'(x) = 1$ and

$$\int xe^{2x}dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}dx}{2} = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C.$$

Returning to our original integratin, we get

$$\int x^2e^{2x}dx = \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4} + C.$$

**QUIZ 4.** a) State the form of the partial fractions decomposition of the rational function

$$f(x) = \frac{x^2+2}{(x^2-3x+2)(x^2+x+1)^2}.$$

b) Evaluate the following integrals:

$$(1) \int \frac{x+2}{(x^2+4x+5)^2}dx \qquad (2) \int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} \qquad (3) \int \frac{\ln(x+1)}{x^2}dx.$$

Hint for (1): think!. Hint for (2): try $x = u^6$.

**Solution:** a) Note that the first quadratic polynomial $x^2 - 3x + 2$ in the denominator has positive discriminant, hence it can be factored into a product of linear polynomials: $x^2 - 3x + 2 = (x-1)(x-2)$. The other quadratic polynomial has negative discriminant. Now we have the denominator factored appropriately and therefore the partial fractions decomposition is of the following form:

$$f(x) = \frac{x^2+2}{(x-1)(x-2)(x^2+x+1)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+x+1)^2}.$$

b) (1) We use the substitution $u = x^2 + 4x + 5$ so $du = (2x+4)dx = 2(x+2)dx$. Thus

$$\int \frac{x+2}{(x^2+4x+5)^2}dx = \int \frac{\frac{1}{2}du}{u^2} = \frac{-1}{2u} + C = \frac{-1}{2(x^2+4x+5)} + C.$$

(2) We follow the hint and use the substitution $x = u^6$, so $dx = 6u^5du$, $\sqrt{x} = u^3$ and $\sqrt[3]{x} = u^2$. Thus

$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = \int \frac{6u^5du}{u^3+u^2} = 6\int \frac{u^3du}{u+1}.$$

We perform the long division $u^3 : (u+1)$ and get $u^3 = (u^2-u+1)(u+1)-1$. Thus

$$\int \frac{u^3du}{u+1} = \int \frac{((u^2-u+1)(u+1)-1)du}{u+1} =$$

$$\int (u^2-u+1)du - \int \frac{du}{u+1} = \frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1| + C.$$

Returning to our original integral, and noting that $u = x^{1/6}$ we get

$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = 6\left(\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1| + C\right) = 2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln(\sqrt[6]{x}+1)+C.$$

(3) We use integration by parts with $f'(x) = 1/x^2$ and $g(x) = \ln(x+1)$. Thus $f(x) = -1/x$, $g'(x) = 1/(x+1)$ and

$$\int \frac{\ln(x+1)}{x^2}dx = \frac{-\ln(x+1)}{x} - \int \frac{-1}{x(x+1)}dx = \frac{-\ln(x+1)}{x} + \int \frac{dx}{x(x+1)}.$$

In order to compute the last integral, we decompose $1/x(x+1)$ into partial fractions:

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{(A+B)x+A}{x(x+1)}.$$

It follows that $1 = (A+B)x + A$, so $A+B = 0$ and $A = 1$, i.e. $A = 1$ and $B = -1$. Thus

$$\int \frac{dx}{x(x+1)} = \int \frac{dx}{x} - \int \frac{dx}{x+1} = \ln|x| - \ln|x+1| + C$$

and consequently

$$\int \frac{\ln(x+1)}{x^2}dx = \frac{-\ln(x+1)}{x} + \ln|x| - \ln|x+1| + C.$$

**QUIZ 5.** a) Is the integral $\int_1^\infty \frac{\arctan x}{\sqrt{x^2-x}}dx$ convergent or divergent? Small hint: $\arctan x$ is increasing. Hint: focus on infinity.

b) Is the integral $\int_1^2 \frac{dx}{\sqrt{x-1}}$ proper? Compute this integral.

c) Compute the following limits:

$$\lim_{n\to\infty} \frac{\sqrt{n^4+n^3+1}}{n^2+1} \qquad \lim_{n\to\infty} \frac{\arctan\sqrt{n}}{\sqrt{4+\sqrt[n]{3}}}.$$

d) A **convergent** sequence $(a_n)$ satisifes the recurssive formula: $a_1 = 2$, $a_{n+1} = \sqrt{2a_n - 1}$ for all $n$. What is $\lim_{n\to\infty} a_n$?

**Solution:** a) Note that the integral is improper at both ends. Thus we need to study convergence of the integrals

$$\int_2^\infty \frac{\arctan x}{\sqrt{x^2 - x}} dx \quad \text{and} \quad \int_1^2 \frac{\arctan x}{\sqrt{x^2 - x}} dx.$$

We follow the hint and focus on the first integral. Intuitively, for large $x$ the value of $\arctan x$ is about $\pi/2$ and the value of $\sqrt{x^2 - x}$ is about $x$. Thus the quantity $\arctan x / \sqrt{x^2 - x}$ is approximately $\pi/2x$. This suggests that the first integral should behave like $\int_2^\infty \frac{1}{x} dx$ which diverges. To get a precise argument, note that in the interval $[2, \infty)$ we have $\arctan x \geq \arctan 2$ and $\sqrt{x^2 - x} \leq x$. It follows that

$$\frac{\arctan x}{\sqrt{x^2 - x}} \geq \frac{\arctan 2}{x}$$

for all $x > 2$. Since $\displaystyle\int_2^\infty \frac{c}{x} dx$ diverges for any constant $c > 0$, the integral

$$\int_2^\infty \frac{\arctan x}{\sqrt{x^2 - x}} dx$$

also diverges by comparison test. It follows that the original integral

$$\int_1^\infty \frac{\arctan x}{\sqrt{x^2 - x}} dx$$

diverges.

**Remark.** It is natural to ask whether

$$\int_1^2 \frac{\arctan x}{\sqrt{x^2 - x}} dx$$

converges or diverges. Note that $\dfrac{\arctan x}{\sqrt{x^2 - x}}$ and $\dfrac{\arctan 1}{\sqrt{x - 1}}$ are comparable when $x$ is close to 1. As the integral $\displaystyle\int_1^2 \frac{1}{\sqrt{x - 1}} dx$ conveges (see b)), we expect that our integral also converges. More precisely, in $(1, 2]$ we have $\arctan x \leq \arctan 2$ and $\sqrt{x^2 - x} = \sqrt{x}\sqrt{x - 1} \geq \sqrt{x - 1}$. Thus

$$\frac{\arctan x}{\sqrt{x^2 - x}} \leq \frac{\arctan 2}{\sqrt{x - 1}}$$

for $x \in (1,2]$. We will see in b) that $\int_1^2 \frac{1}{\sqrt{x-1}}dx$ converges, so $\int_1^2 \frac{\arctan x}{\sqrt{x^2-x}}dx$ converges by comparison test.

b) The integral is improper since the function escapes to $\infty$ when $x$ approaches 1. Note that

$$\int \frac{1}{\sqrt{x-1}}dx = 2\sqrt{x-1} + C$$

for $x > 1$. Thus

$$\int_1^2 \frac{1}{\sqrt{x-1}}dx = \lim_{t\to 1^+} \int_t^2 \frac{1}{\sqrt{x-1}}dx = \lim_{t\to 1^+} (2\sqrt{2-1} - 2\sqrt{t-1}) = 2$$

converges.

c) We have

$$\frac{\sqrt{n^4+n^3+1}}{n^2+1} = \frac{\sqrt{n^4(1+\frac{1}{n}+\frac{1}{n^4})}}{n^2(1+\frac{1}{n^2})} = \frac{\sqrt{1+\frac{1}{n}+\frac{1}{n^4}}}{1+\frac{1}{n^2}}.$$

Thus

$$\lim_{n\to\infty} \frac{\sqrt{n^4+n^3+1}}{n^2+1} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}+\frac{1}{n^4}}}{1+\frac{1}{n^2}} = \frac{1}{1} = 1.$$

Note now that $\lim_{n\to\infty} \arctan\sqrt{n} = \pi/2$ and $\lim_{n\to\infty} \sqrt[n]{3} = 1$, so

$$\lim_{n\to\infty} \frac{\arctan\sqrt{n}}{\sqrt{4+\sqrt[n]{3}}} = \frac{\frac{\pi}{2}}{\sqrt{4+1}} = \frac{\pi}{2\sqrt{5}}.$$

d) Let $\lim_{n\to\infty} a_n = g$. Passing to the limit in the equalities $a_{n+1} = \sqrt{2a_n - 1}$ we get

$$g = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n-1} = \sqrt{2g-1}.$$

Thus $g^2 = 2g-1$, i.e. $g^2 - 2g + 1 = (g-1)^2 = 0$. Thus $g = 1$.

**QUIZ 6.** a) Determine whether the following series converge or diverge:

$$(1)\ \sum_{n=1}^{\infty} \frac{\arctan n}{2n^2-1} \qquad (2)\ \sum_{n=2}^{\infty} \frac{1}{n\ln n} \qquad (3)\ \sum_{n=1}^{\infty} \left(1-\frac{2}{n}\right)^{2n+1}.$$

b) Compute $\sum_{n=0}^{\infty} \frac{2^n-1}{3^{2n+1}}$.

**Solution:** a) (1) For large $n$ the quantity $\frac{\arctan n}{2n^2-1}$ is like $\frac{\pi}{4n^2}$ so we should compare our series to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges. To make this more precise, we can either use limit comparison test or comparison test.

**Using limit comparison**: We compute

$$\lim_{n\to\infty} \frac{\frac{\arctan n}{2n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2 \arctan n}{2n^2-1} = \lim_{n\to\infty} \frac{\arctan n}{2-\frac{1}{n^2}} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}.$$

As the limit is positive, the limit comparison test tells us the the eseries $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{\arctan n}{2n^2-1}$ either both converge or both diverge. Since the former series converges, so does the latter.

**Using comparison test**: Note that $\arctan n \leq \pi/2$ and $2n^2-1 = n^2+(n^2-1) \geq n^2$ for all positive integers $n$. It follows that

$$\frac{\arctan n}{2n^2-1} \leq \frac{\frac{\pi}{2}}{n^2}$$

for all positive integers $n$. Since the series $\sum_{n=1}^{\infty} \frac{\pi/2}{n^2}$ converges, the series $\sum_{n=1}^{\infty} \frac{\arctan n}{2n^2-1}$ also converges by the comparison test.

(2) Let $f(x) = \frac{1}{x \ln x}$. Since $x \ln x$ is an increasing function of $x$ for $x > 1$, the function $f(x)$ is decreasing. Alternatively, compute the derivative

$$f'(x) = \frac{-\ln x - 1}{(x \ln x)^2} < 0 \quad \text{for } x > 1,$$

which is negative, so $f(x)$ is decreasing. Clearly $f$ is continuous and positive for $x > 1$. Thus we may apply the integral test to the series $\sum_{n=2}^{\infty} f(n)$. Using substitution $u = \ln x$, $du = dx/x$, we evaluate the integral

$$\int \frac{1}{x \ln x} dx = \int \frac{du}{u} = \ln|u| + C = \ln \ln x + C$$

for $x > 1$. It follows that the integral

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t\to\infty} (\ln \ln t - \ln \ln 2) = \infty$$

diverges. By the integral test, the series $\sum_{n=2}^{\infty} f(n)$ diverges as well.

(3) Recall that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

for every real number $x$. Note that

$$\left(1-\frac{2}{n}\right)^{2n+1} = \left(1-\frac{2}{n}\right)\left[\left(1+\frac{-2}{n}\right)^n\right]^2.$$

Since $\lim_{n\to\infty}\left(1-\frac{2}{n}\right) = 1$ we get

$$\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^{2n+1} = \lim_{n\to\infty}\left(1-\frac{2}{n}\right)\lim_{n\to\infty}\left[\left(1+\frac{-2}{n}\right)^n\right]^2 = 1\cdot[e^{-2}]^2 = e^{-4}.$$

Thus our series diverges by the divergence test.

b) Recall the formula for the sum of a geometric series:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{for } -1 < a < 1.$$

Note that

$$\frac{2^n-1}{3^{2n+1}} = \frac{2^n-1}{3\cdot 9^n} = \frac{1}{3}\left(\frac{2}{9}\right)^n - \frac{1}{3}\left(\frac{1}{9}\right)^n.$$

Now

$$\sum_{n=0}^{\infty}\left(\frac{2}{9}\right)^n = \frac{1}{1-\frac{2}{9}} = \frac{9}{7}$$

and

$$\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n = \frac{1}{1-\frac{1}{9}} = \frac{9}{8}.$$

Thus

$$\sum_{n=0}^{\infty}\frac{2^n-1}{3^{2n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{9}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n = \frac{3}{7}-\frac{3}{8} = \frac{3}{56}.$$

**QUIZ 7.** Determine whether the following series converge conditionally, converge absolutely, or diverge.

(1) $\sum_{n=1}^{\infty}(-1)^n \tan\frac{1}{n}$ (2) $\sum_{n=1}^{\infty}\frac{e^n n!}{(2n)!}$ (3) $\sum_{n=1}^{\infty}\frac{5^n}{3^n+7^n}$ (4) $\sum_{n=1}^{\infty}(-1)^n\frac{(\arctan n)^{2n}}{\left(1+\frac{2}{n}\right)^{n^2}}$

**Solution:** (1) Note that the series is alternating. Thus it is natural to start with the alternating series test with $a_n = \tan(1/n)$. Clearly $\lim_{n\to\infty} a_n = 0$ as $\tan x$ is continuous at 0:

$$\lim_{n\to\infty} \tan\frac{1}{n} = \tan(\lim_{n\to\infty}\frac{1}{n}) = \tan 0 = 0.$$

We now show that $a_n$ is decreasing. In fact, we have $\frac{1}{n+1} < \frac{1}{n}$, and since $\tan x$ is increasing in the interval $(-\pi/2, \pi/2)$, we have $a_{n+1} = \tan(1/(n+1)) < \tan(1/n) = a_n$. By the alternating series test, we conclude that the series (1) converges.

In order to determine whether it converges absolutely, we need to study convergence of the series $\sum_{n=1}^{\infty} \tan\frac{1}{n}$. Recall that $\lim_{x\to 0}\frac{\tan x}{x} = 1$ (this can be seen, for example, by L'Hospital's rule, and it means that for small $x$ the numbers $\tan x$ and $x$ are about the same). It follows that

$$\lim_{n\to\infty}\frac{\tan\frac{1}{n}}{\frac{1}{n}} = 1.$$

By the limit comparison test, the series $\sum_{n=1}^{\infty}\tan\frac{1}{n}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ either both converge or both diverge. As the latter series diverges (harmonic series), we conclude that the former series also diverges. Thus (1) does not converge absolutely, but it converges, hence it converges conditionally.

(2) We apply the ratio test. Thus we compute the following limit:

$$\lim_{n\to\infty}\frac{\frac{e^{n+1}(n+1)!}{(2(n+1))!}}{\frac{e^n n!}{(2n)!}} = \lim_{n\to\infty}\frac{e^{n+1}}{e^n}\frac{(n+1)!}{n!}\frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} e(n+1)\frac{1}{(2n+1)(2n+2)} = \lim_{n\to\infty}\frac{e}{4n+2} = 0.$$

As the limit exist and is less than 1, the series converges absolutely.

(3) Note that $0 < \frac{5^n}{3^n+7^n} < \frac{5^n}{7^n}$ for every $n$. As the series $\sum_{n=1}^{\infty}(5/7)^n$ converges (it is a geometric series), the series (3) converges as well by the comparison test. As this is a series with positive terms, it converges absolutely.

**Remark.** This problem can be also solved using limit comparison test, or the root test (note that $\lim_{n\to\infty}\sqrt[n]{3^n+7^n} = 7$), or the ratio test.

(4) We try to apply the root test, so we compute the following limit:

$$\lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \frac{(\arctan n)^{2n}}{\left(1+\frac{2}{n}\right)^{n^2}}\right|} = \lim_{n\to\infty} \frac{(\arctan n)^2}{\left(1+\frac{2}{n}\right)^n} = \frac{\left(\frac{\pi}{2}\right)^2}{e^2} = \left(\frac{\pi}{2e}\right)^2 < 1.$$

As the limit exists and is less than 1, the series converges absolutely.

**QUIZ 8.** a) Determine the radius of convergence and the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{n(x+1)^n}{4^n}.$$

b) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(n!)^2 x^{2n+1}}{(2n)!}.$$

c) Express $\dfrac{x}{x+2}$ as a power series centered at 0.

**Solution:** a) Note that the center of the power series is at $-1$. To find the radius of convergence we apply the root test to the series, so we compute the following limit:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n(x+1)^n}{4^n}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{n}|x+1|}{4} = \frac{|x+1|}{4}.$$

By the root test, the series converges if $1 > |x+1|/4$, i.e. $|x+1| < 4$, and the series diverges if $1 < |x+1|/4$, i.e. $|x+1| > 4$. Thus the radius of convergence is 4 and the ends of the interval of convergence are at $-1+4=3$ and $-1-4=-5$.

We now need to test convergence at the ends of the interval of convergence. For $x=3$ we get the series

$$\sum_{n=1}^{\infty} n$$

which diverges by the divergence test. Similarly, for $x=-5$, we get the series

$$\sum_{n=1}^{\infty} n(-1)^n$$

which diverges by the divergence test. Thus the interval of convergence is the open interval $(-5,3)$.

**Remark.** For the first part of the problem, the ratio test works as well.

b) We apply the ratio test to the series (assuming $x \neq 0$), so we compute

$$\lim_{n\to\infty}\left|\frac{\frac{((n+1)!)^2x^{2(n+1)+1}}{(2(n+1))!}}{\frac{(n!)^2x^{2n+1}}{(2n)!}}\right| = \lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\right)^2\frac{(2n)!}{(2n+2)!}|x|^2 = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)}|x|^2 =$$

$$= \lim_{n\to\infty}\frac{n+1}{4n+2}|x|^2 = \frac{|x|^2}{4}.$$

By the ratio test, the series converges when $1 > |x|^2/4$, i.e. $|x| < 2$, and it diverges when $1 < |x|^2/4$, i.e. $|x| > 2$. Thus the radius of convergence is 2.

**Remark.** Testing convergence at the ends of the interval $(-2, 2)$ is a harder problem. It turns out that the series diverges at both ends. To see that note that

$$4^n = (1+1)^{2n} = \sum_{k=0}^{2n}\binom{2n}{k} > \binom{2n}{n} = \frac{(2n)!}{(n!)^2}.$$

It follows that

$$\left|\frac{(n!)^2(\pm 2)^{2n+1}}{(2n)!}\right| = 2\frac{4^n(n!)^2}{(2n)!} > 2$$

for every $n$, so the series

$$\sum_{n=0}^{\infty}\frac{(n!)^2(\pm 2)^{2n+1}}{(2n)!}$$

diverges by the divergence test.

c) Note that

$$\frac{x}{x+2} = \frac{1}{2}x\frac{1}{1+\frac{x}{2}}.$$

Recall now that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots = \sum_{n=0}^{\infty}(-1)^n x^n$$

and the interval of convergence is $(-1, 1)$. It follows that

$$\frac{1}{1+\frac{x}{2}} = 1 - \frac{x}{2} + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \ldots = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^n$$

and it converges iff $x/2 \in (-1, 1)$, i.e. $x \in (-2, 2)$. Thus

$$\frac{x}{x+2} = \frac{1}{2}\cdot x\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^n}x^n$$

and the interval of convergence is $(-2, 2)$.

**QUIZ 9.** a) State Newton's binomial formula.

b) Express the function $f(x) = \sqrt{1-2x^2}$ as a power series centered at 0 and state the radius of convergence. Use the power series to compute $f^{(4)}(0)$.

c) Compute the third Taylor polynomial $T_3(x)$ of the function $\sqrt{x}$ centered at $a = 4$ (i.e. find $T_3(x) = T_3(\sqrt{x}, 4)(x)$). Use Taylor's inequality to show that $|\sqrt{5} - T_3(5)| \leq 5/2^{14}$.

**Solution:** a) Newton's binomial formula states that for any exponent $\alpha$ we have

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n,$$

where $\binom{\alpha}{n}$ are the binomial coefficients defined as follows:

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\ldots(\alpha-(n-1))}{n!} \quad \text{for every positive integer } n.$$

The formula holds for all $x$ in the interval of convergence of the power series on the right. This interval of convergence is as follows:

- $(-1, 1)$ if $\alpha \leq -1$;
- $(-1, 1]$ if $-1 < \alpha < 0$;
- $[-1, 1]$ if $\alpha > 0$ and $\alpha$ is not an integer.
- $(-\infty, \infty)$ if $\alpha$ is a non-negative integer (i.e. $\alpha = 0, 1, 2, \ldots$).

In any case, the radius of convergence is 1 except that it is $\infty$ when $\alpha$ is a non-negative integer.

b) By Newton's binomial formula, we have

$$f(x) = \sqrt{1-2x^2} = (1 + (-2x^2))^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-2x^2)^n = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-2)^n x^{2n}$$

which is the required power series expansion. This series converges if $|-2x^2| < 1$, i.e. $|x| < 1/\sqrt{2}$ and diverges when $|-2x^2| > 1$, i.e. $|x| > 1/\sqrt{2}$. It follows that the radius of convergence is $1/\sqrt{2}$.

In particular, the series on the right is the Taylor series of $f$ centered at 0 (Maclaurin series). This means that $f^{(k)}(0)/k!$ is the coefficient at $x^k$ for every $k$. When $k = 4$, we look at the coefficient at $x^4$ (so $n = 2$) and get

$$\frac{f^{(4)}(0)}{4!} = \binom{\frac{1}{2}}{2}(-2)^2 = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\cdot 4 = \frac{-1}{2}$$

so $f^{(4)}(0) = -12$.

c) Let $f(x) = \sqrt{x}$. Then

$$T_3(x) = T_3(f,4)(x) = f(4) + f^{'}(4)(x-4) + f^{''}(4)\frac{(x-4)^2}{2!} + f^{'''}(4)\frac{(x-4)^3}{3!}.$$

We have

- $f^{'}(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ so $f^{'}(4) = \frac{1}{4}$.
- $f^{''}(x) = \frac{1}{2}\frac{-1}{2}x^{-3/2} = \frac{-1}{4\sqrt{x^3}}$ so $f^{''}(4) = \frac{-1}{32}$.
- $f^{'''}(x) = \frac{1}{2}\frac{-1}{2}\frac{-3}{2}x^{-5/2} = \frac{3}{8\sqrt{x^5}}$ so $f^{'''}(4) = \frac{3}{256}$.
- $f^{(4)}(x) = \frac{1}{2}\frac{-1}{2}\frac{-3}{2}\frac{-5}{2}x^{-7/2} = \frac{-15}{16\sqrt{x^7}}$.

Thus $T_3(x) = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512}$.

Taylor's inequality (which we will review below) implies that

$$|\sqrt{5} - T_3(5)| \leq M\frac{|5-4|^4}{4!} = \frac{M}{24}$$

where $M$ is an upper bound for $|f^{(4)}(t)|$ on the inteval $[4,5]$, i.e. $M$ satisfies $|f^{(4)}(t)| \leq M$ for all $t \in [4,5]$. We need to find $M$. Recall that $|f^{(4)}(t)| = \frac{15}{16\sqrt{t^7}}$, which is clearly a decreasing function of $t$. Thus $|f^{(4)}(t)| \leq |f^{(4)}(4)|$ for all $t \in [4,5]$, so we can take $M = |f^{(4)}(4)| = \frac{15}{2^{11}}$. Using this value of $M$ we get

$$|\sqrt{5} - T_3(5)| \leq \frac{M}{24} = \frac{5}{2^{14}},$$

as required.

A simple computation yields $T_3(5) = 2\frac{121}{512} = 2.236328125$ and $5/2^{14} = 0.00030517578125$. Hence we proved that $|\sqrt{5} - 2.236328125| < 0.00031$. In fact, $\sqrt{5} = 2.2360679774997896964091736687313$

**Remark.** Let us review Taylor's inequality. Let $f$ be a function which has derivatives of all orders in a neighborhood of $a$. The $k-$th Taylor polynomial of $f$ centered at $a$ is

$$T_k(f,a)(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The Taylor series of $f$ centered at $a$ is defined as

$$T(f,a)(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The $k-$th remainder $R_k(x)$ is the difference $f(x) - T_k(f,a)(x)$. To prove that the Taylor series at $x$ converges to $f(x)$ is equivalent to proving that $\lim_{k\to\infty} R_k(x) = 0$. This can be done if we can understand $R_k(x)$, which can often be achieved by using the following theorem:

**Taylor's Formula.** Under the above assumptions we have

$$f(x) - \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n = R_k(x) = \int_a^x f^{(k+1)}(t)\frac{(x-t)^k}{k!}.$$

(a proof consists of performing integration by parts several times).

Using Taylor's formula, we can get very useful estimate for $R_k(x)$, called Taylor's inequality:

**Taylor's Inequality.**

$$|R_k(x)| \le M\frac{(x-a)^{k+1}}{(k+1)!}$$

where $M$ is any number such that $|f^{(n+1)}(t)| \le M$ for all $t$ between $x$ and $a$. In other words, $M$ is an upper bound for $|f^{(n+1)}(t)|$ on the interval between $a$ and $x$. A key step in using Taylor's inequality is to find $M$.

**QUIZ 10.** a) Express $\int \cos(\sqrt{x})\ dx$ as a power series centered at 0. What is the radius of convergence?

b) Consider a curve given by the parametric equation $x = t^2+2t+2$, $y = 4t^3+3t^2+2$. Find all points at which the tangent to this curve is vertical or horizontal. Find the tangent lines at the points corresponding to $t = -1$ and $t = 1$.

**Solution:** a) Recall the Taylor series centered at 0 (the Maclaurin series) for $\cos x$:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

which holds for all $x$ (so the radius of convergence is $\infty$). It follows that

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!},$$

which again holds for all $x$. Integrating term by term we get

$$\int \cos(\sqrt{x})dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)(2n)!},$$

($C$ an arbitrary constant) which is valid for all $x$ (so the radius of convergence is $\infty$).

b) Let us recall the following facts about parametric curves. Let $x = x(t)$, $y = y(t)$, $t \in [a, b]$ be a parametric curve. The derivative of $y$ as a function of $x$ at a point corresponding to a parameter $t$ is equal to $\dfrac{y'(t)}{x'(t)}$ (provided $x'(t) \neq 0$). In particular, the tangent line to the curve at a point corresponding to a parameter $t$ has slope $\dfrac{y'(t)}{x'(t)}$, provided $x'(t) \neq 0$. If $x'(t) = 0$ and $y'(t) \neq 0$ then the tangent is vertical. The equation of the tangent line at the point $(x(t), y(t))$ is given by

$$y - y(t) = \frac{y'(t)}{x'(t)}(x - x(t))$$

if $x'(t) \neq 0$, and by $x = x(t)$ if $x'(t) = 0$ and $y'(t) \neq 0$.

Now we solve part b). We have $x'(t) = 2t + 2$ and $y'(t) = 12t^2 + 6t$. The tangent is horizontal if $y'(t) = 0$ and $x'(t) \neq 0$. Solving $y'(t) = 12t^2 - 6t = 0$ yields $y = 0$ or $y = -1/2$, and in both cases we have $x'(t) \neq 0$. Thus the tangent is horizontal at the point $(2, 2)$ (corrsponding to $t = 0$) and $(5/4, 9/4)$ (corresponding to $t = -1/2$).

The tangent is vetical if $x'(t) = 0$ and $y'(t) \neq 0$. Solving $x'(t) = 2t + 2 = 0$ yields $t = -1$, and $y'(-1) \neq 0$. Thus tangent is vertical at the point $(1, 1)$ corresponding to $t = -1$.

As we have observed above, the tangent at the point $(1, 1)$ corresponding to $t = -1$ is vertical, hence has equation $x = 1$. The point corresponding to $t = 1$ is

$(5,9)$ and the slope of the tangent at this point is $y'(1)/x'(1) = 18/4 = 9/2$. Thus the tangent line at the point $(5,9)$ has equation $y - 9 = \frac{9}{2}(x-5)$, i.e. $y = \frac{9}{2}x - \frac{27}{2}$.

**QUIZ 11.** a) The graph of the function $y = \frac{x^2}{4} - \frac{\ln x}{2}$, $1 \le x \le 2$ is revolved about the $y$-axis. Compute the area of the resulting surface.

b) Compute the length of the parametric curve $x = 3t^2 + 1$, $y = 2t^3 + 1$, $t \in [0,1]$.

c) compute the area enclosed by the loop of the curve $x = t^2$, $y = t^3 - 3t$, when the parameter varies in $[-\sqrt{3}, \sqrt{3}]$.

**Solution:** Let us recall some basic facts about parametric curves. Let $x = x(t)$, $y = y(t)$, $t \in [a,b]$ be a parametric curve.

1. the length of the curve between points corresponding to $t = a$ and $t = b$ is
$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt$$
(assuming that there are no overlaps; in general the integral expresses the distance traveled along the curve). The quantity $\sqrt{x'(t)^2 + y'(t)^2}$ should be considered as the speed. The speed is the length of the vector $[x'(t), y'(t)]$, the velocity.

2. the area of the surface obtained by revolving the curve about the $x$-axis is
$$2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2}dt$$
(assuming that $y(t) > 0$ for $t \in [a,b]$; otherwise use $|y(t)|$).

3. the area of the surface obtained by revolving the curve about the $y$-axis is
$$2\pi \int_a^b x(t)\sqrt{x'(t)^2 + y'(t)^2}dt$$
(assuming that $x(t) > 0$ for $t \in [a,b]$, otherwise use $|x(t)|$).

4. the graph of a function $y = f(x), x \in [a,b]$ can be considered as a special case of a parametric curve with $x(t) = t$, $y(t) = f(t)$, $t \in [a,b]$.

5. if $x(t)$ is increasing and $y(t) \geq 0$ on $[a, b]$ then the curve is a graph of a function and the area under the graph is equal to

$$\int_a^b y(t)x'(t)\ dt$$

6. if the parametrization describes a simple (no self intersections) closed curve which is traveled around clockwise once when $t$ varies from $a$ to $b$ then the area enclosed by the curve is equal to

$$\int_a^b y(t)x'(t)\ dt$$

(if the parametrization goes counteclokwise, we get minus the area).

Now we can solve our problem.

a) The graph has parametrization $x = t$, $y = \frac{t^2}{4} - \frac{\ln t}{2}$, $t \in [1, 2]$. We have $x'(t) = 1$ and $y'(t) = \frac{t}{2} - \frac{1}{2t}$. The surface area is then given by

$$2\pi \int_1^2 x(t)\sqrt{x'(t)^2 + y'(t)^2}\ dt = 2\pi \int_1^2 t\sqrt{1 + \left(\frac{t}{2} - \frac{1}{2t}\right)^2}\ dt = 2\pi \int_1^2 t\sqrt{1 + \frac{t^2}{4} - \frac{1}{2} + \frac{1}{4t^2}}\ dt =$$

$$= 2\pi \int_1^2 t\sqrt{\left(\frac{t}{2} + \frac{1}{2t}\right)^2}\ dt = 2\pi \int_1^2 t\left(\frac{t}{2} + \frac{1}{2t}\right) = 2\pi \left(\frac{t^3}{6} + \frac{t}{2}\right)\Big|_1^2 = \frac{10\pi}{3}$$

b) Note that $x'(t) = 6t$, $y'(t) = 6t^2$. The lenght of the curve is computed using 1:

$$\text{length } = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\ dt = \int_0^1 \sqrt{(6t)^2(t^2 + 1)}\ dt = \int_0^1 6t\sqrt{t^2 + 1}\ dt.$$

Using the substitution $u = t^2 + 1, du = 2tdt$, we get

$$\text{length } = \int_1^2 3\sqrt{u}du = 2(2^{3/2} - 1) = 4\sqrt{2} - 2.$$

c) When $t$ varies between $-\sqrt{3}$ and $\sqrt{3}$, the loop is traveled once counterclockwise. The area enclosed by the loop is then given by

$$\text{area} = -\int_{-\sqrt{3}}^{\sqrt{3}} y(t)x'(t)\ dt = -\int_{-\sqrt{3}}^{\sqrt{3}} (t^3 - 3t)2t\ dt = -2\int_{-\sqrt{3}}^{\sqrt{3}} (t^4 - 3t^2)\ dt =$$

$$= -2\left(\frac{t^5}{5} - t^3\right)\Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{24\sqrt{3}}{5}.$$

**QUIZ 12.** a) Consider the polar curve $r = 1 + 2\sin\theta$, $\theta \in [0, 2\pi]$. When $\theta$ varies through $[0, 2\pi]$, the curve passes through the origin twice. What are the values of $\theta$ for which the curve passes through the origin?

b) Find the area enclosed by the loop made between the first and second pass through the origin (inner loop on the picture).

c) Find the equation of the two tangent lines to the curve at the origin.

d) Find the length of the polar curve $r = \sin\theta + \cos\theta$, $\theta \in [0, \pi]$.

**Solution:** a) The origin is the point for which $r = 0$. Thus we need to find all $\theta$ such that $1 + 2\sin\theta = 0$. This means that $\sin\theta = -1/2$, so $\theta = 7\pi/6$ or $\theta = 11\pi/6$.

b) The area of a polar region $0 \le r \le f(\theta)$, $\theta \in [a, b]$ is given by the formula $\frac{1}{2}\int_a^b f(\theta)^2\, d\theta$. In our case, $a = 7\pi/6$, $b = 11\pi/6$, $f(\theta) = 1 + 2\sin\theta$, so the area of the inner loop is

$$\frac{1}{2}\int_{7\pi/6}^{11\pi/6} (1 + 2\sin\theta)^2\, d\theta = \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (1 + 4\sin\theta + 4\sin^2\theta)\, d\theta =$$

$$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (4\sin\theta + 3 - 2\cos(2\theta))\, d\theta = \frac{1}{2}(-4\cos\theta + 3\theta - \sin(2\theta))\Big|_{7\pi/6}^{11\pi/6} = \pi + \frac{\sqrt{3}}{2}.$$

We used above the identity $2\sin^2\theta = 1 - \cos(2\theta)$.

c) The parametric equation of the polar curve $r = f(\theta)$ is

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta.$$

Thus our curve has parametric equation

$$x = (1 + 2\sin\theta)\cos\theta, \quad y = (1 + 2\sin\theta)\sin\theta.$$

It follows that

$$x' = -\sin\theta - 2\sin^2\theta + 2\cos^2\theta, \quad y' = \cos\theta + 4\sin\theta\cos\theta.$$

The slope of the tangent line at a point corresponding to $\theta$ is equal to $y'(\theta)/x'(\theta)$.

When $\theta = 7\pi/6$ we get the slope euqal to $\sqrt{3}/3$ so the tangent line has equation $y = \dfrac{\sqrt{3}}{3}x$. When $\theta = 11\pi/6$ we get the slope euqal to $-\sqrt{3}/3$ so the tangent line has equation $y = \dfrac{-\sqrt{3}}{3}x$

d) The length of the polar curve $r = f(\theta)$, $\theta \in [a, b]$ is given by $\displaystyle\int_a^b \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$ (assuming the curve has no "backtracking"; in general we get the length "traveled" along the curve). In our case, $a = 0$, $b = \pi$, $f(\theta) = \sin\theta + \cos\theta$, $f'(\theta) = \cos\theta - \sin\theta$, so the length of our curve is

$$\int_0^\pi \sqrt{(\cos\theta - \sin\theta)^2 + (\sin\theta + \cos\theta)^2}\, d\theta = \int_0^\pi \sqrt{2}\, d\theta = \sqrt{2}\pi.$$

**Remark.** It is not hard to see that our curve is a circle with center at $(1/2, 1/2)$ and radius $\sqrt{2}/2$.