Solutions to Exam I, Math 222, section 1

Problem 1. a) Let $f(x) = \arctan(\ln x)$. Find a formula for f^{-1}.

Solution: Let $y = \arctan(\ln x)$. Then $\tan y = \tan(\arctan(\ln x)) = \ln x$. It follows that $e^{\tan y} = e^{\ln x} = x$. Thus $f^{-1}(x) = e^{\tan x}$.

b) Let

$$f(x) = \frac{\arctan x}{\frac{\pi}{2} + \arctan x}, \quad x \in (-\infty, \infty).$$

Prove that f has an inverse and find the domain of f^{-1}.

Solution: Let us compute the derivative of f:

$$f'(x) = \frac{(\arctan x)'(\frac{\pi}{2} + \arctan x) - (\arctan x)(\frac{\pi}{2} + \arctan x)'}{(\frac{\pi}{2} + \arctan x)^2} =$$

$$= \frac{(\arctan x)'\frac{\pi}{2} + (\arctan x)' \arctan x - (\arctan x)(\arctan x)'}{(\frac{\pi}{2} + \arctan x)^2} = \frac{(\arctan x)'\frac{\pi}{2}}{(\frac{\pi}{2} + \arctan x)^2} =$$

$$= \frac{\frac{1}{1+x^2}\frac{\pi}{2}}{(\frac{\pi}{2} + \arctan x)^2} = \frac{\pi}{2(1+x^2)(\frac{\pi}{2} + \arctan x)^2}.$$

It is clear now that $f'(x) > 0$ for all x as all the factors in the denominator are positive. Thus f is an increasing function, hence f is one-to-one and it has an inverse. The domain of f^{-1} is the range of f. Since f is increasing and its domain is $(-\infty, \infty)$, the range of f is (a, b), where $a = \lim_{x \to -\infty} f(x)$ and $b = \lim_{x \to \infty} f(x)$. In order to compute the limits recall that $\lim_{x \to -\infty} \arctan x = \frac{-\pi}{2}$ and $\lim_{x \to \infty} \arctan x = \frac{\pi}{2}$. Thus

$$a = \lim_{x \to -\infty} \frac{\arctan x}{\frac{\pi}{2} + \arctan x} = -\infty$$

as the denominator approaches 0 through positive numbers and the numerator approaches $-\pi/2$. Similarly,

$$b = \lim_{x \to \infty} \frac{\arctan x}{\frac{\pi}{2} + \arctan x} = \frac{\frac{\pi}{2}}{\frac{\pi}{2} + \frac{\pi}{2}} = \frac{1}{2}.$$

Thus the domain of f^{-1} is $(-\infty, 1/2)$.

c) The function $f(x) = x + 2e^x$, $x \in (-\infty, \infty)$ is one to one. Compute $(f^{-1})'(2)$.

Solution: Recall that

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

Note that $f'(x) = 1 + 2e^x$. We need to compute $f^{-1}(2)$. Since $f(0) = 2$, we have $f^{-1}(2) = 0$. Thus

$$(f^{-1})'(2) = \frac{1}{f'(0)} = \frac{1}{1+2e^0} = \frac{1}{3}.$$

Problem 2. a) Compute $\cos\left(\arcsin\frac{12}{13}\right)$.

Solution: Consider the right triangle with hypotenuse of length 13 and one arm of length 12. Then the other arm has lenght $\sqrt{13^2 - 12^2} = 5$. Let α be the angle subtended the arm of length 12. Then $\sin\alpha = 12/13$. Thus $\arcsin\frac{12}{13} = \alpha$ and

$$\cos\left(\arcsin\frac{12}{13}\right) = \cos\alpha = \frac{5}{13}.$$

Second solution: Let $\alpha = \arcsin\frac{12}{13}$. Then $\alpha \in (-\pi/2, \pi/2)$ and $\sin\alpha = 12/13$. From the identity $\sin^2\alpha + \cos^2\alpha = 1$ we get $\cos\alpha = \pm\sqrt{1-\sin^2\alpha}$. Since cos is positive on the interval $(-\pi/2, \pi/2)$, we have in fact $\cos\alpha = \sqrt{1-\sin^2\alpha}$. Thus

$$\cos\left(\arcsin\frac{12}{13}\right) = \cos\alpha = \sqrt{1-\left(\frac{12}{13}\right)^2} = \frac{5}{13}.$$

b) Differentaite the following function: $f(x) = x^x \cdot 3^{x^2} \cdot 5^{\sin x}$.

Solution: We use the formula for logarithmic differentiation:

$$f'(x) = f(x)[\ln f(x)]'.$$

Note that

$$\ln f(x) = \ln(x^x \cdot 3^{x^2} \cdot 5^{\sin x}) = \ln x^x + \ln 3^{x^2} + \ln 5^{\sin x} = x\ln x + x^2\ln 3 + \sin x \ln 5.$$

Thus

$$[\ln f(x)]' = (x\ln x)' + (x^2\ln 3)' + (\sin x\ln 5)' = \ln x + x\cdot\frac{1}{x} + 2x\ln 3 + \cos x\ln 5.$$

and consequently

$$f'(x) = x^x \cdot 3^{x^2} \cdot 5^{\sin x}\left(\ln x + 1 + 2x\ln 3 + \cos x\ln 5\right).$$

c) Solve the following equation: $\log_3(x+4) - \log_3(4-x) = 1$.

Solution: Note that

$$\log_3(x+4) - \log_3(4-x) = \log_3\left(\frac{x+4}{4-x}\right) \quad \text{and} \quad 1 = \log_3 3.$$

Our equation takes then the follwing form:

$$\log_3\left(\frac{x+4}{4-x}\right) = \log_3 3.$$

As $\log_3$ is a one-to-one function, we conclude that $\frac{x+4}{4-x} = 3$, i.e. $x+4 = 12-3x$ which is the same as $4x = 8$, i.e. $x = 2$.

Problem 3. Compute the following limits. If you use L'Hospital's rule, show where you use it and explain what type of limit you are using it on.

a) $\lim_{x\to\infty}(e^x+3)^{\frac{1}{x+1}}$ b) $\lim_{x\to 0}\frac{x^2-x}{\cos x}$ c) $\lim_{x\to 0}\frac{\sin x - x}{x^3}$ d) $\lim_{x\to 0}\cot^2 x \cdot \ln\cos x$

Solution: a) The limit is of the form ∞^0, which is an indeterminate form. We use the following fact:

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim_{x\to a} g(x)\ln f(x)}.$$

In our case, we have $f(x) = e^x + 3$ and $g(x) = 1/(x+1)$ and $a = \infty$. Thus we first compute the limit

$$\lim_{x\to\infty}\frac{1}{x+1}\ln(e^x+3) = \lim_{x\to\infty}\frac{\ln(e^x+3)}{x+1}.$$

As the last limit is of the form $\frac{\infty}{\infty}$ we can try to apply L'Hospital's rule and copute the limit of the ratio of the derivatives:

$$\lim_{x\to\infty}\frac{\frac{e^x}{e^x+3}}{1} = \lim_{x\to\infty}\frac{e^x}{e^x+3}.$$

This is again of the form $\frac{\infty}{\infty}$ so we apply L'Hospital's rule again and compute

$$\lim_{x\to\infty}\frac{e^x}{e^x} = 1.$$

By L'Hospitel's rule, we conclude that $\lim_{x\to\infty}\frac{1}{x+1}\ln(e^x+3) = 1$ and therefore

$$\lim_{x\to\infty}(e^x+3)^{\frac{1}{x+1}} = e^1 = e.$$

b) As $\lim_{x\to 0}(x^2-x) = 0$ and $\lim_{x\to 0}\cos x = 1$, we have

$$\lim_{x\to 0}\frac{x^2-x}{\cos x} = \frac{0}{1} = 0.$$

c) The limit $\lim_{x\to 0}\frac{\sin x - x}{x^3}$ is of the form $\frac{0}{0}$. In order to apply L'Hospital's rule we compute the limit $\lim_{x\to 0}\frac{\cos x - 1}{3x^2}$. This is again of the form $\frac{0}{0}$ so we compute $\lim_{x\to 0}\frac{-\sin x}{6x}$. Again, this is of the form $\frac{0}{0}$, so we compute $\lim_{x\to 0}\frac{-\cos x}{6} = \frac{-1}{6}$. By L'Hospital's rule, we have

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{\cos x - 1}{3x^2} = \lim_{x\to 0}\frac{-\sin x}{6x} = \lim_{x\to 0}\frac{-\cos x}{6} = \frac{-1}{6}.$$

d) The limit is of the form $\infty\cdot 0$. We use the identity $fg = \frac{g}{\frac{1}{f}}$. Thus

$$\cot^2 x\cdot\ln\cos x = \frac{\ln\cos x}{\frac{1}{\cot^2 x}} = \frac{\ln\cos x}{\tan^2 x}.$$

So we need to compute $\lim_{x\to 0}\frac{\ln\cos x}{\tan^2 x}$, which is of the form $\frac{0}{0}$. In order to apply L'Hospitel's rule we consider the limit

$$\lim_{x\to 0}\frac{\frac{-\sin x}{\cos x}}{2\tan x\sec^2 x} = \lim_{x\to 0}\frac{-1}{2\sec^2 x} = \frac{-1}{2\cdot 1} = \frac{-1}{2}.$$

By L'Hospital's rule, $\lim_{x\to 0}\cot^2 x\cdot\ln\cos x = \frac{-1}{2}$.

Problem 4. Compute the following integrals:

$$\text{a)}\int_1^e \frac{\sqrt{\ln x}}{x}dx \quad \text{b)}\int e^{2x}\cos x dx \quad \text{c)}\int_1^{\sqrt{3}}\frac{dx}{x^2\sqrt{4-x^2}} \quad \text{d)}\int \tan^8 x\sec^4 x dx.$$

Solution: a) We use substitution $u = \ln x$, $du = dx/x$ to get

$$\int_1^e \frac{\sqrt{\ln x}}{x}dx = \int_0^1 \sqrt{u}du = \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{3}.$$

b) We use integration by parts with $f'(x) = \cos x$ and $g(x) = e^{2x}$, so $f(x) = \sin x$ and $g'(x) = 2e^{2x}$:

$$\int e^{2x}\cos x dx = e^{2x}\sin x - \int 2e^{2x}\sin x dx = e^{2x}\sin x - 2\int e^{2x}\sin x dx.$$

Now we apply integration by parts to $\int e^{2x}\sin x dx$ with $f'(x) = \sin x$ and $g(x) = e^{2x}$, so $f(x) = -\cos x$ and $g'(x) = 2e^{2x}$:

$$\int e^{2x}\sin x dx = -e^{2x}\cos x - \int 2e^{2x}(-\cos x)dx = -e^{2x}\cos x + 2\int e^{2x}\cos x dx.$$

It follows that

$$\int e^{2x}\cos x dx = e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x dx$$

i.e.

$$5\int e^{2x}\cos x dx = e^{2x}\sin x + 2e^{2x}\cos x + C.$$

Thus

$$\int e^{2x}\sin x dx = \frac{1}{5}e^{2x}\sin x + \frac{2}{5}e^{2x}\cos x + C.$$

c) We use substitution $x = 2\sin t$, $dx = 2\cos t dt$, $\sqrt{4-x^2} = 2\cos t$. Thus

$$\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{4-x^2}} = \int_{\pi/6}^{\pi/3} \frac{2\cos t dt}{(2\sin t)^2 2\cos t} = \int_{\pi/6}^{\pi/3} \frac{dt}{4\sin^2 t} = \frac{1}{4}\int_{\pi/6}^{\pi/3} \csc^2 t dt =$$

$$= \frac{1}{4}(-\cot t)\Big|_{\pi/6}^{\pi/3} = \frac{1}{4}\left(-\frac{\sqrt{3}}{3} + \sqrt{3}\right) = \frac{\sqrt{3}}{6}.$$

d) We use the substitution $u = \tan x$, $du = \sec^2 x dx$ and the identity $\sec^2 x = \tan^2 x + 1 = u^2 + 1$ to get

$$\int \tan^8 x \sec^4 x dx = \int \tan^8 x \sec^2 x \sec^2 x dx = \int u^8(u^2+1)du =$$

$$= \int u^{10}du + \int u^8 du = \frac{u^{11}}{11} + \frac{u^9}{9} + C = \frac{\tan^{11} x}{11} + \frac{\tan^9 x}{9} + C.$$

Problem 5. Using integration by parts derive the following formula for $n \geq 2$:

$$\int \frac{f(x)}{(x-a)^n}dx = \frac{f(x)}{(1-n)(x-a)^{n-1}} + \frac{1}{n-1}\int \frac{f'(x)}{(x-a)^{n-1}}dx.$$

Solution: We use integration by parts with $f(x)$ and $g'(x) = (x-a)^{-n}$, so $g(x) = (x-a)^{1-n}/(1-n)$. Thus

$$\int \frac{f(x)}{(x-a)^n}dx = f(x)\frac{(x-a)^{1-n}}{1-n} - \int f'(x)\frac{(x-a)^{1-n}}{1-n}dx = \frac{f(x)}{(1-n)(x-a)^{n-1}} + \frac{1}{n-1}\int \frac{f'(x)}{(x-a)^{n-1}}dx.$$

Problem 6. Suppose that $f(0) = 3$, $f(3) = 6$, $f'(3) = 4$ and $f''(x)$ is continuous. Calculate $\int_0^3 xf''(x)dx$.

Solution: We use integration by parts:

$$\int_0^3 xf''(x)dx = xf'(x)\Big|_0^3 - \int_0^3 f'(x)dx = 3f'(3) - 0f'(0) - f(x)\Big|_0^3 =$$

$$= 3f'(3) - (f(3) - f(0)) = 3\cdot 4 - (6-3) = 9.$$