Solutioms to Exam II, Math 222, section 1

Problem 1. Compute the following infinite sum $\sum_{n=1}^{\infty} \frac{(\ln 2)^n}{2^n n!}$.

Solution: Note that
$$\sum_{n=1}^{\infty} \frac{(\ln 2)^n}{2^n n!} = \sum_{n=1}^{\infty} \frac{\left(\frac{\ln 2}{2}\right)^n}{n!}.$$
Recall now the Taylor series expansion of e^x centered at 0:
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$
which holds for every x. Taking $x = \ln 2/2$, we get
$$1 + \sum_{n=1}^{\infty} \frac{\left(\frac{\ln 2}{2}\right)^n}{n!} = e^{\frac{\ln 2}{2}} = \left(e^{\ln 2}\right)^{\frac{1}{2}} = \sqrt{2}.$$

Thus $\sum_{n=1}^{\infty} \frac{(\ln 2)^n}{2^n n!} = \sqrt{2} - 1$.

Problem 2. Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Explain what test you are applying and verify all the conditions necessary to apply the test.

$$\text{a)} \sum_{n=1}^{\infty} (-1)^n \frac{n^4 - n + 1}{2n^7 - n} \qquad\qquad \text{b)} \sum_{n=1}^{\infty} (-1)^n \tan\left(\frac{1}{\sqrt{n}}\right).$$

Solution: a) The key observation is that for large n the quantity $\frac{n^4 - n + 1}{2n^7 - n}$ is comparable to $\frac{n^4}{2n^7} = \frac{1}{2n^3}$. More precisely, we have
$$\lim_{n\to\infty} \frac{\frac{n^4-n+1}{2n^7-n}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^7 - n^4 + n^3}{2n^7 - n} = \lim_{n\to\infty} \frac{1 - \frac{1}{n^3} + \frac{1}{n^4}}{2 - \frac{1}{n^6}} = \frac{1}{2}.$$
The limit comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{n^4 - n + 1}{2n^7 - n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ either both converge or both diverge. But the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, so also $\sum_{n=1}^{\infty} \frac{n^4 - n + 1}{2n^7 - n}$ converges. Thus the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^4 - n + 1}{2n^7 - n}$ converges absolutely,as
$$\left| (-1)^n \frac{n^4 - n + 1}{2n^7 - n} \right| = \frac{n^4 - n + 1}{2n^7 - n}.$$

Remark. One can also use the comparison test, by observing that $0 < n^4 - n + 1 \leq n^4$ and $2n^7 - n = n^7 + (n^7 - n) \geq n^7$ for evry natural number n, so $\dfrac{n^4 - n + 1}{2n^7 - n} \leq \dfrac{n^4}{n^7} = \dfrac{1}{n^3}$.

b) Note that the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \tan\left(\frac{1}{\sqrt{n}}\right)$ is alternating. Thus it is natural to try the alternating series test with $a_n = \tan(1/\sqrt{n})$. We need to verify that (a_n) satisfies the assumptions of the test, i.e. that (a_n) is decreasing and tends to 0. Clearly $\lim_{n\to\infty} a_n = 0$ as $\tan x$ is continuous at 0:

$$\lim_{n\to\infty} \tan\frac{1}{\sqrt{n}} = \tan(\lim_{n\to\infty}\frac{1}{\sqrt{n}}) = \tan 0 = 0.$$

To show that the sequence (a_n) is decreasing note that $\dfrac{1}{\sqrt{n+1}} < \dfrac{1}{\sqrt{n}}$, and since $\tan x$ is increasing in the interval $(-\pi/2, \pi/2)$, we have $a_{n+1} = \tan(1/\sqrt{n+1}) < \tan(1/\sqrt{n}) = a_n$. By the alternating series test, we conclude that the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \tan\left(\frac{1}{\sqrt{n}}\right)$ converges.

In order to determine whether it converges absolutely, we need to study convergence of the series $\displaystyle\sum_{n=1}^{\infty}\tan\frac{1}{\sqrt{n}}$. Recall that $\displaystyle\lim_{x\to 0}\frac{\tan x}{x} = 1$ (this can be seen, for example, by L'Hospital's rule, and it means that for small x the numbers $\tan x$ and x are about the same). It follows that

$$\lim_{n\to\infty}\frac{\tan\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = 1.$$

By the limit comparison test, the series $\displaystyle\sum_{n=1}^{\infty}\tan\frac{1}{\sqrt{n}}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ either both converge or both diverge. As the latter series diverges (p-series, with $p = 1/2 < 1$), we conclude that the former series also diverges. Thus the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \tan\left(\frac{1}{\sqrt{n}}\right)$ does not converge absolutely, but it converges, hence it converges conditionally.

Problem 3. Determine the interval of convergence of the following power series:

$$\text{a)} \sum_{n=1}^{\infty}(\arctan n)^{-n}x^n \quad , \quad \text{b)} \sum_{n=1}^{\infty}\frac{(x-1)^{2n-1}}{n(n+1)4^n} \quad .$$

Solution: a) We apply the root test, so we compute

$$\lim_{n\to\infty}\sqrt[n]{|(\arctan n)^{-n}x^n|} = \lim_{n\to\infty}(\arctan n)^{-1}|x| = \lim_{n\to\infty}\frac{|x|}{\arctan n} = \frac{|x|}{\frac{\pi}{2}}.$$

It follows that the series converges if $\dfrac{|x|}{\frac{\pi}{2}} < 1$, i.e. $|x| < \frac{\pi}{2}$ and it diverges if $\dfrac{|x|}{\frac{\pi}{2}} > 1$, i.e. $|x| > \dfrac{\pi}{2}$. The radius of convergence is threfore equal to $\dfrac{\pi}{2}$.

To determine the interval of convergence, we need to test the series with $x = \pi/2$ and $x = -\pi/2$. When $x = \pi/2$, we get

$$\sum_{n=1}^{\infty} (\arctan n)^{-n} \left(\frac{\pi}{2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{\frac{\pi}{2}}{\arctan n}\right)^n.$$

Note now that $\frac{\frac{\pi}{2}}{\arctan n} > 1$ for all n (as $arctanx < \pi/2$ for all x). It follows that $\left(\frac{\frac{\pi}{2}}{\arctan n}\right)^n > 1$ for all n, so the series diverges at $x = \pi/2$ by the divergence test.

Same argument works for $x = -\pi/2$, i.e. the series

$$\sum_{n=1}^{\infty} (\arctan n)^{-n} \left(-\frac{\pi}{2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{\frac{-\pi}{2}}{\arctan n}\right)^n$$

diverges by the divergence test, as $\left|\frac{\frac{-\pi}{2}}{\arctan n}\right|^n > 1$ for all n.

Thus the interval of convergence is $(-\pi/2, \pi/2)$.

b) We use the ratio test with $a_n = \left|\frac{(x-1)^{2n-1}}{n(n+1)4^n}\right|$. Thus we compute

$$\lim_{n\to\infty} \frac{\left|\frac{(x-1)^{2(n+1)-1}}{(n+1)(n+2)4^{n+1}}\right|}{\left|\frac{(x-1)^{2n-1}}{n(n+1)4^n}\right|} = \lim_{n\to\infty} \frac{n|x-1|^2}{4(n+2)} = \frac{|x-1|^2}{4}.$$

It follows that the series converges if $\frac{|x-1|^2}{4} < 1$, i.e. $|x-1| < 2$ and it diverges if $\frac{|x-1|^2}{4} > 1$, i.e. $|x-1| > 2$. The radius of convergence is threfore equal to 2. To determine the interval of convergence, we need to test the series with $x = 3$ and $x = -1$. When $x = 3$, we get

$$\sum_{n=1}^{\infty} \frac{(3-1)^{2n-1}}{n(n+1)4^n} = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n(n+1)2^{2n}} = \sum_{n=1}^{\infty} \frac{1}{2n(n+1)}.$$

This series convergence, for example, by comaprison test with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Similarly, When $x = -1$, we get

$$\sum_{n=1}^{\infty} \frac{(-1-1)^{2n-1}}{n(n+1)4^n} = \sum_{n=1}^{\infty} \frac{(-2)^{2n-1}}{n(n+1)2^{2n}} = \sum_{n=1}^{\infty} \frac{-1}{2n(n+1)} = -\sum_{n=1}^{\infty} \frac{1}{2n(n+1)},$$

so this series converges as well.

The interval of convergence of our series is therefore $[-1, 3]$.

Problem 4. a) Express the function $f(x) = \dfrac{1}{\sqrt{1-2x^2}}$ as a power series centered at 0 and state the radius of convergence. Use the power series to compute $f^{(6)}(0)$ (the answer must be given as a fraction in lowest terms).

Solution: By Newton's binomial formula, we have

$$f(x) = \frac{1}{\sqrt{1-2x^2}} = (1 + (-2x^2))^{-1/2} = \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-2x^2)^n = \sum_{n=0}^{\infty} \binom{\frac{-1}{2}}{n} (-2)^n x^{2n}$$

which is the required power series expansion. This series converges if $|-2x^2| < 1$, i.e. $|x| < 1/\sqrt{2}$ and diverges when $|-2x^2| > 1$, i.e. $|x| > 1/\sqrt{2}$. It follows that the radius of convergence is $1/\sqrt{2}$.

In particular, the series on the right is the Taylor series of f centered at 0 (Maclaurin series). This means that $f^{(k)}(0)/k!$ is the coefficient at x^k for every k. When $k = 6$, we look at the coefficient at x^6 (so $n = 3$) and get

$$\frac{f^{(6)}(0)}{6!} = \binom{\frac{-1}{2}}{3}(-2)^3 = \frac{\frac{-1}{2}(\frac{-1}{2}-1)(\frac{-1}{2}-2)}{6} \cdot (-8) = \frac{5}{2}$$

so $f^{(6)}(0) = 6! \cdot 5/2 = 1800$.

b) Find the Taylor series centered at 0 of the function $f(x) = \dfrac{\arctan x - x}{x^3}$.

Solution: Recall that the Taylor series centered at 0 of $\arctan x$ is $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$. Moreover we have the equality

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

for all $x \in [-1, 1]$. It follows that

$$\arctan x - x = -\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

and

$$\frac{\arctan x - x}{x^3} = -\frac{1}{3} + \frac{x^2}{5} - \frac{x^4}{7} + \dots = \sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^{2n}}{2n+3}.$$

The series on the right is therefore the Taylor series centered at 0 for f.

Problem 5. The curve $x = t^2 + 2$, $y = \dfrac{1}{3}t^3 - t + 2$, $t \in [0, 1]$ is revolved about the y-axis.

a) Compute the area of the resulting surface.

Solution: We have $x' = 2t$ and $y' = t^2 - 1$. As $x(t) > 0$ for $t \in [0,1]$, the area of the surface obtained by revolving the curve about the y-axis is

$$2\pi \int_0^1 x(t)\sqrt{x'(t)^2 + y'(t)^2}dt = 2\pi \int_0^1 (t^2+2)\sqrt{(2t)^2 + (t^2-1)^2}dt =$$

$$= 2\pi \int_0^1 (t^2+2)\sqrt{4t^2 + t^4 - 2t^2 + 1}dt = 2\pi \int_0^1 (t^2+2)\sqrt{t^4 + 2t^2 + 1}dt =$$

$$= 2\pi \int_0^1 (t^2+2)\sqrt{(t^2+1)^2}dt = 2\pi \int_0^1 (t^2+2)(t^2+1)dt =$$

$$= 2\pi \int_0^1 (t^4 + 3t^2 + 2)dt = 2\pi \left(\frac{1}{5} + 1 + 2\right) = \frac{32\pi}{5}.$$

b) Compute the length of this curve.

Solution: The length of our curve between points corresponding to $t = 0$ and $t = 1$ is

$$\int_0^1 \sqrt{x'(t)^2 + y'(t)^2}dt.$$

In the silution to part a) we have see that $\sqrt{x'(t)^2 + y'(t)^2} = t^2 + 1$. Thus the length is equal to

$$\int_0^1 (t^2+1)dt = \frac{1}{3} + 1 = \frac{4}{3}.$$

Problem 6. Consider the simple closed curve $x = \sin t + \cos t$, $y = \cos t$, $t \in [0, 2\pi]$.

a) Find the equation of the line tangent to this curve at the point corresponding to $t = \pi/2$.

Solution: We have $x'(t) = \cos t - \sin t$ and $y'(t) = -\sin t$. The equation of the tangent line at the point $(x(t), y(t))$ is given by

$$y - y(t) = \frac{y'(t)}{x'(t)}(x - x(t))$$

if $x'(t) \neq 0$, and by $x = x(t)$ if $x'(t) = 0$ and $y'(t) \neq 0$. When $t = \pi/2$ we have $x'(\pi/2) = -1$, $y'(\pi/2) = -1$, $x(\pi/2) = 1$, $y(\pi/2 = 0$. Thus the tangent at the point $(1,0)$ is $y = x - 2$.

b) Compute the area enclosed by this curve. Hint: $\int_0^{2\pi} \cos^2 x dx = \pi$.

Solution: As we are told that our curve is a simple closed curve, and when t varies from 0 to 2π the curve is traveled around clockwise, the area enclosed by the curve is

$$\int_0^{2\pi} y(t)x'(t)\ dt = \int_0^{2\pi} \cos t(\cos t - \sin t)\ dt = \int_0^{2\pi} \cos^2 t\ dt - \int_0^{2\pi} \sin t \cos t\ dt = \pi$$

as $\sin t \cos t$ is the derivative of $(\sin^2 t)/2$, so the last integral is 0.

Problem 7. Let $f(x) = \sqrt[3]{x}$.

a) Find the second Taylor polynomial of f centered at 8 (i.e. $T_2(\sqrt[3]{x}, 8)(x)$).

Solution: We have

$$T_2(x) = T_2(f, 8)(x) = f(8) + f^{'}(8)(x-8) + f^{''}(8)\frac{(x-8)^2}{2!}.$$

Now

- $f(8) = 2$.
- $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(8) = \frac{1}{3} \cdot 8^{-2/3} = \frac{1}{12}$.
- $f^{''}(x) = \frac{1}{3}\frac{-2}{3}x^{-5/3} = \frac{-2}{9}x^{-5/3}$ and $f^{''}(8) = \frac{-2}{9} \cdot 8^{-5/3} = \frac{-1}{144}$.
- $f^{'''}(x) = \frac{1}{3}\frac{-2}{3}\frac{-5}{3}x^{-8/3} = \frac{10}{27\sqrt[3]{x^8}}$.

  Thus $T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$.

b) Use Taylor's inequality to show that

$$|\sqrt[3]{9} - T_2(\sqrt[3]{x}, 8)(9)| \le \frac{5}{3^4 \cdot 2^8}.$$

Solution: Taylor's inequality tells us that

$$|\sqrt[3]{9} - T_2(9)| \le M\frac{|9-8|^3}{3!} = \frac{M}{6}$$

where M is an upper bound for $|f^{'''}(t)|$ on the inteval $[8, 9]$, i.e. M satisfies $|f^{'''}(t)| \le M$ for all $t \in [8, 9]$. We need to find M. Recall that $|f^{'''}(t)| = \frac{10}{27\sqrt[3]{t^8}}$, which is clearly a decreasing function of t. Thus $|f^{'''}(t)| \le |f^{'''}(8)|$ for all $t \in [8, 9]$, so we can take $M = |f^{'''}(8)| = \frac{10}{27 \cdot 2^8} = \frac{5}{3^3 \cdot 2^7}$. Using this value of M we get

$$|\sqrt[3]{9} - T_2(9)| \le \frac{M}{6} = \frac{5}{3^4 \cdot 2^8}.$$

Remark. A simple computation yields $T_2(9) = 2.079861111\ldots$ and $\frac{5}{3^4 \cdot 2^8} = 0.000241\ldots$. Also $\sqrt[3]{9} = 2.08008382\ldots$.