MATH 304 - Linear Algebra
Exam 1, February 27

YOU MUST SHOW ALL WORK TO GET CREDIT.

1. (8 points) Consider the following matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Which of these matrices are upper triangular, lower triangular, diagonal, symmetric? Which are in a row echelon form, reduced row echelon form?

2. (9 points) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

   a) What is $A + 2B$?

   b) What is AA^T?

   c) What is $BE_{2,1}(-1)$?

3. The augmented matrix of a system of linear equations is

$$A = \left[\begin{array}{cccccc:c} 2 & 4 & 1 & 4 & 0 & 8 & 4 \\ 3 & 6 & 1 & 5 & 1 & 10 & 8 \\ 2 & 4 & 0 & 2 & 1 & 5 & 5 \end{array}\right]$$

and the reduced row echelon form of A is

$$R = \left[\begin{array}{cccccc:c} 1 & 2 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \end{array}\right]$$

   a) (10 points) Solve this system of linear equations.

   b) (3 points) What are the independent variables?

   c) (2 points) What is the rank of the coefficient matrix?

4. (5 points each)

   a) Define a linear transformation.

   b) The reflection of the plane about the line $y = x$ is a linear transformation. Find the matrix of this transformation.

   c) A linear transformation $F : \mathbf{R}^4 \longrightarrow \mathbf{R}^2$ satisfies $F(1,0,1,0) = (1,-1)$ and $F(0,1,0,1) = (-1,1)$. What is $F(1,2,1,2)$?

5. (8 points) Find a matrix X such that $XA = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}$ knowing that $A^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 4 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

a) (6 points) Find the inverse of A. Verify your answer.

b) (5 points) Express A as a product of elementary matrices.

c) (4 points) What is the inverse of A^T?

7. (10 points) A matrix A is called skew-symmetric if $A^T = -A$. Prove that if A is skew-symmetric then A^2 is symmetric. What can you say about higher powers of A? What can you say about the main diagonal of a skew-symmetric matrix?

The following problem is optional. You can earn 15 extra points if you solve it, but work on it only if you are done with all the other problems

8. Let F be a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$ such that $F(v) = \mathbf{e_1}$ and $F(u) = \mathbf{e_2}$ for some $u, v \in \mathbf{R}^3$.

a) Show that F is onto.

b) Prove that there is a non-zero vector $w \in \mathbf{R}^3$ such that $F(w) = (0, 0)$.

c) Let A be the matrix of F. Show that the system of equations

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

has infinitely many solutions.