MATH 304 - Linear Algebra
Solutions to Exam 1

1. Among the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

A, B are upper triangular, A, E are lower triangular, A is diagonal, A, C are symmetric, B, D are in a row echelon form, D is in the reduced row echelon form.

2. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

   a) $A + 2B = \begin{bmatrix} 3 & 4 & 5 \\ 10 & 11 & 14 \end{bmatrix}$.

   b) $AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix}$.

   c) Note that $E_{2,1}(-1)$ must be of size 3×3 to make sense of the multiplication. Thus $E_{2,1}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $BE_{2,1}(-1) = \begin{bmatrix} -1 & 2 & 3 \\ -1 & 5 & 6 \end{bmatrix}$. Alternatively, recall that multiplication by $E_{2,1}(-1)$ on the right is the same as the operation $E_{1,2}(-1)$ on the columns.

3. The augmented matrix of a system of linear equations is

$$A = \left[\begin{array}{cccccc|c} 2 & 4 & 1 & 4 & 0 & 8 & 4 \\ 3 & 6 & 1 & 5 & 1 & 10 & 8 \\ 2 & 4 & 0 & 2 & 1 & 5 & 5 \end{array}\right]$$

and the reduced row echelon form of A is

$$R = \left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \end{array}\right]$$

We see that the free (independent) variables (which correspond to the non-pivot columns) are x_2, x_4, x_6. To solve the system we express the dependent variables in terms of the free variables: $x_1 = 1 - 2x_2 - x_4 - 3x_6$, $x_3 = 2 - 2x_4 - 2x_6$, $x_5 = 3 + x_6$. The rank of the coefficient matrix equals the number of dependent variables (or pivot columns) so it is 3.

4. a) A linear transformation between vector spaces V and W is a function $T : V \longrightarrow W$ such that $T(u + v) = T(u) + T(v)$ and $T(au) = aT(u)$ for any vectors $u, v \in V$ and any scalar $a \in \mathbb{R}$.

   b) Let T be the reflection of the plane about the line $y = x$. Thus $T(e_1) = e_2$ and $T(e_2) = e_1$. Consequently, the matrix of T is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

   c) A linear transformation $F : \mathbf{R}^4 \longrightarrow \mathbf{R}^2$ satisfies $F(1, 0, 1, 0) = (1, -1)$ and $F(0, 1, 0, 1) = (-1, 1)$. Note that $(1, 2, 1, 2) = (1, 0, 1, 0) + 2(0, 1, 0, 1)$. Since F is a linear transformation, we have

$$F(1, 2, 1, 2) = F((1, 0, 1, 0) + 2(0, 1, 0, 1)) = F(1, 0, 1, 0) + 2F(0, 1, 0, 1) =$$
$$= (1, -1) + 2(-1, 1) = (-1, 1).$$

5. Since A is invertible, we can multiply the equation $XA = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}$ on the right by A^{-1} to get

$$X = (XA)A^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

6. Let $A = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 4 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

a) In order to find the inverse of A we start with the matrix $\left[\begin{array}{ccc:ccc} 2 & 3 & -2 & 1 & 0 & 0 \\ 3 & 4 & 3 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$ and perform a sequence of elementary row operations

$T_{1,3}$, $E_{2,1}(-3)$, $E_{3,1}(-2)$, $E_{3,2}(-1)$, $E_{1,2}(-1)$, $E_{2,3}(-1)$, $S_3(-1/3)$, $E_{1,3}(-5)$

to arrive at $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 5/3 & -8/3 & 17/3 \\ 0 & 1 & 0 & -1 & 2 & -4 \\ 0 & 0 & 1 & -1/3 & 1/3 & -1/3 \end{array}\right]$. It follows that $A^{-1} = \begin{bmatrix} 5/3 & -8/3 & 17/3 \\ -1 & 2 & -4 \\ -1/3 & 1/3 & -1/3 \end{bmatrix}$. We verify our answer by performing the multiplication

$$\begin{bmatrix} 2 & 3 & -2 \\ 3 & 4 & 3 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5/3 & -8/3 & 17/3 \\ -1 & 2 & -4 \\ -1/3 & 1/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

b) From a) we see that

$$E_{1,3}(-5)S_3(-1/3)E_{2,3}(-1)E_{1,2}(-1)E_{3,2}(-1)E_{3,1}(-2)E_{2,1}(-3)T_{1,3}A = I.$$

Thus

$$A = (E_{1,3}(-5)S_3(-1/3)E_{2,3}(-1)E_{1,2}(-1)E_{3,2}(-1)E_{3,1}(-2)E_{2,1}(-3)T_{1,3})^{-1} =$$
$$= T_{1,3}E_{2,1}(3)E_{3,1}(2)E_{3,2}(1)E_{1,2}(1)E_{2,3}(1)S_3(-3)E_{1,3}(5).$$

c) Recall that $(A^T)^{-1} = (A^{-1})^T$. From a) we see that $(A^T)^{-1} = \begin{bmatrix} 5/3 & -1 & -1/3 \\ -8/3 & 2 & 1/3 \\ 17/3 & -4 & -1/3 \end{bmatrix}$.

7. Let A be a skew-symmetric matrix, so $A^T = -A$. Thus

$$(A^2)^T = (A \cdot A)^T = A^T A^T = (-A)(-A) = A^2,$$

i.e. A^2 is symmetric. Note that in general, for any n, we have

$$(A^n)^T = (A \cdot \ldots \cdot A)^T = A^T \cdot \ldots \cdot A^t = (A^T)^n = (-A)^n = (-1)^n A^n.$$

Thus A^n is symmetric for n even and skew-symmetric for n odd. Note also that A^{-1} is skew-symmetric, since $(A^{-1})^T = (A^T)^{-1} = (-A)^{-1} = -A^{-1}$. The diagonal entries of A^T and A coincide: $(A^T)_{i,i} = A_{i,i}$ for all i. From $A^T = -A$ we conclude that $A_{i,i} = -A_{i,i}$, so $A_{i,i} = 0$. Thus the main diagonal of a skew-symmetric matrix consists of 0's.

8. Let F be a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$ such that $F(v) = \mathbf{e_1}$ and $F(u) = \mathbf{e_2}$ for some $u, v \in \mathbf{R}^3$.

   a) In order to show that F is onto we need to argue that for any element $(a, b) \in \mathbf{R}^2$ there is an element $w \in \mathbf{R}^3$ such that $F(w) = (a, b)$. Note that $(a, b) = a\mathbf{e_1} + b\mathbf{e_2} = aF(v) + bF(u) = F(av) + F(bu) = F(av + bu)$. Thus $w = av + bu$ works.

   b) Recall that F, being a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$, is given by a 2×3 matrix A. Since the number of rows of A is larger than the number of columns, A can not be one-to-one. Thus F is not one-to-one, i.e there are two different vectors w_1, w_2 such that $F(w_1) = F(w_2)$. Taking $w = w_1 - w_2$ we see that $w \neq (0, 0, 0)$ and $F(w) = F(w_1) - F(w_2) = (0, 0)$.

   c) Let A be the matrix of F. Since A is onto by a), the system of equations

   $$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

   has a solution. Note that the augmented matrix of this system has at most 2 pivot columns. Thus the system has free variables, i.e. if it has solutions then it has infinitely many of them. Note that this argument can be applied to give another proof of b). We can also use b) to show c). In fact, our system of equations is equivalent to $F(x) = (1, 2)$. Note that if x is a solution (which we know exists by a)), then for w as in b) and any real number t, $F(x + tw) = F(x) + tF(w) = (1, 2) + t(0, 0) = (1, 2)$. Thus for every real number t the vector $x + tw$ is a solution, in particular we have infinitely many solutions.