MATH 304 - Linear Algebra
Exam 2, March 27

YOU MUST SHOW ALL WORK TO GET CREDIT.
BE VERY CAREFUL WITH YOUR ARITHMETIC.

1. (4 points each) State a definition of

   a) a basis and dimension of a vector space V.

   b) the kernel (null space) and the image (range) of a liner transformation $T : V \longrightarrow W$.

   c) the span of a subset X of a vector space V.

   d) State either the going up or the going down lemma.

2. (20 points) Let $X = \{(1, 0, 1, 0), (1, -1, 0, 2), (1, -3, -2, 6), (1, 1, 1, 1), (2, -5, -2, 7)\}$. Among the vectors in X find a basis of $span(X)$. Express each vector of X as a linear combination of vectors in this basis. What is the dimension of $span(X)$?

3. A linear transformation $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ has matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ -3 & -6 & 4 & -13 & -17 \\ 4 & 8 & -6 & 25 & 29 \\ -1 & -2 & 0 & 10 & 6 \end{bmatrix}$$

   a) (10 points) Find a basis of the image of T.

   b) (10 points) Find a basis of the kernel of T. Verify that the vectors you found are indeed in the kernel.

4. (14 points) Let $X = \{(1, -1, -1, 1), (-1, 1, 1, 0), (0, 1, 0, 1)\}$ and $W = span(X)$. Determine which of the vectors $(1, 0, -1, 3)$, $(1, 1, 1, 1)$ belong to W and those which do express as linear combinations of vectors in X.

5. (10 points) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation such that the composition $T \circ T$ is the zero map (i.e. maps every vector of $\mathbb{R}^n$ to 0).

   a) Prove that image $Im(T)$ is contained in the kernel $\ker(T)$.

   b) Prove that the rank of the matrix of T does not exceed $n/2$.

The following problem is optional. You can earn 15 extra points if you solve it, but work on it only if you are done with all the other problems

6. a) Prove that a linear transformation $T : V \longrightarrow W$ is one-to-one if and only if $\ker(T) = \{0\}$

   b) Suppose that $T : V \longrightarrow V$ is a linear transformation. Suppose v is a vector in V such that $w = T(v) \neq 0$ but $T(w) = 0$. Prove that v and w are linearly independent.