MATH 304 - Linear Algebra
Exam 3, May 1

YOU MUST SHOW ALL WORK TO GET CREDIT.
BE VERY CAREFUL WITH YOUR ARITHMETIC.

1. a) (7 points) State a definition of the determinant. State the cofactor expansions (expansions with respect to $i-$th row/ $j-th$ column) of the determinant.

   b) Let $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$

   i) Compute the determinant of A using the definition of the determinant (8 points).

   ii) Use the row/column reduction to compute $\det A$ (8 points).

2. a) (8 points) Find the characteristic polynomial and the eigenvalues of the matrix $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

   b) (8 points) The numbers 2 and 3 are the only eigenvalues of the matrix $B = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{bmatrix}$. Find bases of the corresponding eigenspaces. Is B diagonalizable? Explain your answer.

   c) (8 points) A 3×3 matrix C has eigenvectors $v_1 = (1,1,1)$ with eigenvalue 1, $v_2 = (1,1,0)$ with eigenvalue -1 and $v_3 = (1,0,0)$ with eigenvalue 0. It follows that C is diagonalizable, i.e. there exist a matrix P and a diagonal matrix D such that $C = PDP^{-1}$. Find P, D and then C. What is C^{2003}?

3. a) (8 points) The space $\mathbb{P}_3$ of all polynomials $f(x)$ of degree ≤ 3 has basis $1, x, x^2, x^3$. Find the matrix in this basis of the linear transformation $T : \mathbb{P}_3 \longrightarrow \mathbb{P}_3$ defined by $T(f(x)) = (x+1)f'(x) - f(x)$.

   b) (8 points) Find the change of basis matrix from the basis $(1,0,1), (1,1,0), (0,1,1)$ to the basis $(1,0,0), (1,1,0), (1,1,1)$ of $\mathbb{R}^3$. What are the coordinates of a vector v in the second basis if its coordinates in the first basis are $(1,-1,2)$?

   c) (8 points) A linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is represented in the standard bases by the matrix $\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix}$. What is the matrix representation of T in the bases $(1,1,0), (1,-1,0), (1,1,1)$ of $\mathbb{R}^3$ and $(2,1), (3,2)$ of $\mathbb{R}^2$?

4. a) (3 points) State Cramer's Rule.

   b) (3 points) A matrix A satisfies $AA^t = I$. Prove that $\det A = \pm 1$.

   c) (3 points) Two 13×13 matrices A, B satisfy $AB = -BA$. Prove that at least one of them is not invertible.

The following problem is optional. You can earn 15 extra points if you solve it, but work on it only if you are done with all the other problems

5. A square matrix M is such that $M^2 = M$ (such matrices are called **idempotent**).

   a) Show that if λ is an eigenvalue of M then $\lambda = 0$ or $\lambda = 1$.

   b) Show that the kernel of M is the eigenspace corresponding to 0 and the image of M is the eigenspace corresponding to 1.

   c) Prove that M is diagonalizable.