MATH 304 - Linear Algebra
Inner product spaces

1. a) Find an orthonormal basis of the subspace $W$ of $\mathbb{R}^4$ spanned by the vectors $(1,1,1,1),(3,1,3,1),(1,1,3,-1)$ by applying the Gram-Schmidt orthogonalization process. Find an orthogonal basis of the orthogonal complement $W^\perp$ to this subspace.

   b) Let $V$ be an inner product space with orthogonal basis $v_1,...,v_n$. Prove that for any $v \in V$ we have

$$v = \frac{< v, v_1 >}{< v_1, v_1 >} v_1 + \frac{< v, v_2 >}{< v_2, v_2 >} v_2 + ... + \frac{< v, v_n >}{< v_n, v_n >} v_n.$$

   c) Let $W$ be a subspace of an inner product space $V$. Suppose that $w_1,...,w_k$ is an orthogonal basis for $W$. Let $v \in V$. Prove that the vector

$$\pi(v) = \frac{< v, w_1 >}{< w_1, w_1 >} w_1 + \frac{< v, w_2 >}{< w_2, w_2 >} w_2 + ... + \frac{< v, w_k >}{< w_k, w_k >} w_k$$

   is the unique vector in $W$ such that $v - \pi(v)$ is orthogonal to $W$ (i.e. it is orthogonal to each vector $w_i$). The function $\pi : v \mapsto \pi(v)$ from $V$ to $W$ is called the **orthogonal projection onto** $W$. It is a linear transformation. What is the kernel of $\pi$? Show that $\pi^2 = \pi$. Among all the vectors in $W$ the vector $\pi(v)$ is closest to $v$, i.e. the length of $v - \pi(v)$ is minimal.

   d) Find the orthogonal projection of $v = (1,0,0,0)$ onto the subspace $W$ of $\mathbb{R}^4$ spanned by $(1,-1,-1,1),(1,1,-1,-1)$.

2. Consider the vector space $P_2$ of all polynomials of degree $\leq 2$ with the inner product $< f, g >= \int_0^1 f(x)g(x)dx$. Apply the Gram-Schmidt process to the basis $1, x, x^2$ to get an orthogonal basis. Express the polynomial $f = 1 + x + x^2$ in this orthogonal basis. What is the length of $f$?