The following problems should help you learn the material and prepare for the exams. Some of the problems here are more difficult than an avarage test problem.

Problem 1. State a definition of:

a) a linearly independent subset S of a vector space V;

b) a basis and dimension of a vector space;

c) the span of a subset S of a vector space V;

d) the kernel and range of a linear transformation $T : V \longrightarrow W$;

Problem 2. Find the dimension and a basis of the subspace of $\mathbb{R}^5$ spanned by the vectors $(1, 1, 0, -1, -1)$, $(1, 0, -1, 0, 0)$, $(1, 2, 1, -2, -2)$, $(2, 1, 1, 1, 2)$, $(4, 3, -1, -3, -3)$. Find a basis for this subspace which starts with the vector $(2, 1, -1, -1, -1)$ and whose all other vectors are taken from the above list of spanning vectors.

Problem 3. a) Find a basis of the kernel and range of the linear transformation $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ whose matrix representation in the standard basis is

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix}$$

b) Find a basis for the kernel of the linear transformation $T : R^5 \longrightarrow R^4$ given by $T(a, b, c, d, e) = (a + b - 2c - d + e, a + b - d - e, 3a - 2c - e, 2a - b + d - 2e)$. What is the dimension of the image of T?

c) Find a basis for the image of the linear transformation $T : \mathbb{R}^4 \mapsto \mathbb{R}^5$ whose matrix representation in the standard bases is

$$\begin{pmatrix} 1 & 4 & 1 & 2 \\ 2 & 5 & -1 & 1 \\ 3 & 10 & 1 & 4 \\ 2 & 5 & -1 & 1 \\ 1 & 4 & 1 & 2 \end{pmatrix}$$

Problem 4. Answer true or false. In each case provide an explanation.

a) If vector 0 is included among some set of vectors then this set is linearly dependent.

b) If $v_1, ..., v_k$ are linearly independent and v_{k+1} is not a linear combination of these vectors then $v_1, ..., v_{k+1}$ are linearly independent.

c) If w is a liner combination of $v_1, ..., v_k$ and each v_i is a linear combination of $u_1, ..., u_t$ then w is a linear combination of $u_1, ..., u_t$.

d) If $v_1, ..., v_k$ are linearly independent then none of them is a linear combination of the others. How about the converse?

e) If $v_1, ..., v_k$ are linearly dependent then each of these vectors is a linear combination of the others.

f) If w is not a linear combination of $v_1, ..., v_k$ then $w, v_1, ..., v_k$ are linearly independent

g) If any $k-1$ vectors from the set $v_1, ..., v_k$ are linearly independent, then $v_1, ..., v_k$ are linearly independent

h) If $V = \text{span}(\{v_1, ..., v_k\})$ and if every v_i is a linear combination of no more than r vectors from $v_1, ..., v_k$ excluding v_i then $\dim V < (r+1)$.

i) Suppose that $S_1 \subseteq S_2$ are subsets of a vector space V such that S_1 spans V and S_2 is linearly independent. Then $S_1 = S_2$.

j) If $T : V \longrightarrow W$ is a linear transformation and $\dim V > \dim W$ then $T(v) = 0$ for some $v \neq 0$.

k) If $A^2 = I$ then $A = I$ or $A = -I$. Here A is a matrix and I is the identity matrix.

Problem 5. Let $v_1, ..., v_n$ be a basis of V, $n > 1$. Is the set $v_1 + v_2$, $v_2 + v_3, ...$, $v_{n-1} + v_n$, $v_n + v_1$ a basis for V? How about the converse ?

Problem 6. Let $C(\mathbb{R})$ be the space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $\sin(x)$ and $\cos(x)$ are linearly independent and that the vector space spanned by sin and cos is contained in the solutions to the differential equation $y'' + y = 0$.

Problem 7. Let W and U be non-trivial, proper subspaces of a vector space V. Show that there exists v in V which does not belong to U nor to W.