MATH 304 - Linear Algebra

An $m \times n$ **matrix** is a rectangular table with m rows and n columns. The i, j-entry of a matrix is the entry in the i-th row and j-th column. For example, here is a 4×5 matrix:

$$M = \begin{bmatrix} 2 & 6 & 1 & 4 & 4 \\ 1 & 3 & 1 & 3 & 3 \\ 1 & 3 & 2 & 5 & 5 \\ 1 & 3 & 3 & 7 & 7 \end{bmatrix}$$

and its $3, 2$-entry is 3. In this course, the entries of matrices will be numbers.

Elementary row operations. We perform the following operations on rows of matrices.

- $E_{i,j}(a)$ denotes the operation "add a times the j-th row to the i-th row". Here $i \neq j$ and a is any number.

- $S_{i,j}$ denotes the operation "switch i-th and j-th rows".

- $D_i(a)$ denotes the operation "multiply the i-th row by a". Here a can be any $\mathbf{non-zero}$ number (i.e. 0 is not allowed for a).

For example, applying each of the operations $E_{3,2}(-1)$, $S_{1,4}$, $D_2(\pi)$ to the matrix M above yields

$$E_{2,3}(-1)M = \begin{bmatrix} 2 & 6 & 1 & 4 & 4 \\ 0 & 0 & -1 & -2 & -2 \\ 1 & 3 & 2 & 5 & 5 \\ 1 & 3 & 3 & 7 & 7 \end{bmatrix}, \quad S_{1,4}M = \begin{bmatrix} 1 & 3 & 3 & 7 & 7 \\ 1 & 3 & 1 & 3 & 3 \\ 1 & 3 & 2 & 5 & 5 \\ 2 & 6 & 1 & 4 & 4 \end{bmatrix}, \quad D_2(\pi)M = \begin{bmatrix} 2 & 6 & 1 & 4 & 4 \\ \pi & 3\pi & \pi & 3\pi & 3\pi \\ 1 & 3 & 2 & 5 & 5 \\ 1 & 3 & 3 & 7 & 7 \end{bmatrix}$$

We say that a matrix M is **row equivalent** to a matrix N if there is a sequence of elementary row operations which transforms M into N. Note that each elementary row operation is reversible: $E_{i,j}(-a)E_{i,j}(a)M = M$, $S_{i,j}S_{i,j}M = M$, $D_i(\frac{1}{a})D_i(a)M = M$ for any matrix M. It follows that if M is row equivalent to N then N is row equivalent to M, and that any matrix is row equivalent to itself. It is clear that if M is row equivalent to N and N is row equivalent to P then M is row equivalent to P.

Note that, in general, elementary row operations do not commute with each other, so the order in which they are applied matters. However, for a fixed j, the operations $E_{1,j}(a_1), E_{2,j}(a_2), E_{3,j}(a_3), \ldots$ commute with each other. Similarly, for a fixed i, the operations $E_{i,1}(a_1), E_{i,2}(a_2), E_{i,3}(a_3), \ldots$ commute with each other.

Exercise. Verify that $E_{i,j}(a)E_{i,j}(b)M = E_{i,j}(a+b)M$ and $D_j(1/a)E_{i,j}(1)D_j(a)M = E_{i,j}(a)M$ for every matrix M.

Exercise. Verify that $D_j(-1)E_{i,j}(1)E_{j,i}(-1)E_{i,j}(1)M = S_{i,j}M$ for any matrix M. Also, $E_{j,i}(1)E_{i,j}(-1)E_{j,i}(1)E_{i,j}(1/a)E_{j,i}(-a)E_{i,j}(1/a)M = D_i(a)D_j(1/a)M$.

The moral of the last exercise is that the operations $S_{i,j}$ are redundant, but having them among our tools often reduces the number of operations one needs to perform.

The goal of the elementary row operations is to transform a given matrix to a more convenient form. To make this more precise we need some terminology. We say that a row (column) of a matrix is a **zero** row (column), if all the entries in this row (column) are zero.

We say that a matrix N is in a **row-echelon form** if:

1. every non-zero row of N is above any zero row of N;

2. the first non-zero entry in any non-zero row is to the right of the first non-zero entry in the row above it.

We say that N is in a **reduced row-echelon form** if it is in a row-echelon form and, in addition, satisfies the following property:

3. In every non-zero row the first non-zero entry is equal to 1 and all other entries in its column are 0.

The key result, on which almost all computations in this class will be based, is the following theorem.

Theorem 0.1 *Any matrix is row-equivalent to a unique matrix in a reduced row-echelon form.*

The uniqueness part of the theorem is not obvious and we will sketch an argument for it after we discuss systems of linear equations. The fact that there is always a sequence of elementary row operations which transforms a given matrix into a matrix in a reduced row echelon form follows from an algorithm producing such a sequence. First we transform our matrix into a row-echelon form using the following procedure:
Look at the first non-zero column of a given matrix. Performing an elementary row operation if

necessary make the top entry in this column non-zero (operation of the type $E_{1,i}(1)$ will always work for this step, but it is often easier to do a switch of the form $S_{1,i}$). Then eliminate (make equal to 0) all the other entries in this column by performing operations of the type $E_{i,1}(a)$ for appropriate a (often, especially when doing this by hand, it is more convenient to use more operations in order to avoid complicated fractions).

You apply this procedure to a given matrix M and get a matrix M_1. Then you keep the first row of M_1 unchanged and apply the procedure again to the matrix consisting of the remaining rows to get a matrix M_2. Then you keep the first 2 rows of M_2 unchanged and apply the procedure again to the matrix consisting of the remaining rows, and so on It is straightforward to see that at the end you will have a matrix in a row-echelon form.

The second procedure starts with a matrix M in a row-echelon form and produces a matrix in a reduced row-echelon form as follows. You start with the last non-zero row of M, say it is the $k-$th row. If the first non-zero entry in this row is u then divide this row by u (i.e perform $D_k(1/u)$ to get the first non-zero entry in this row equal to 1. Then, perform operations of the type $E_{i,k}(a)$ for $i = 1, \dots, k-1$ to make all entries in the column of u equal to 0. Now you move to row $k-1$ and apply the same procedure and so on. At the end you will have a matrix in a reduced row echelon form.

It should be clear from the second procedure that the first non zero entry in any non-zero row of a matrix in a row-echelon form is in the same place as the first non-zero entry in the same row of the reduced row-echelon form. We make the following definitions.

A **pivot column** of a matrix M is a column such that the corresponding column in any row-echelon form row equivalent to M contains the first non-zero entry of some row.

The **rank** of a matrix M is the number of non-zero rows in any row-echelon form row equivalent to M. Equivalently, it is the number of pivot columns of M.

A straightforward, but useful, observation is that the rank of an $m \times n$ matrix is always smaller or equal than each m and n.

We illustrate the above ideas and concepts by finding the reduced row echelon form of the following matrix M:

$$M = \begin{bmatrix} 3 & 6 & 5 & -2 & 21 & 14 & 30 \\ 2 & 4 & 5 & -3 & 19 & 12 & 27 \\ 1 & 2 & 2 & -1 & 8 & 5 & 11 \\ 2 & 4 & 3 & -1 & 13 & 9 & 19 \\ 3 & 6 & 7 & -4 & 27 & 17 & 38 \end{bmatrix}.$$

The first non-zero column of M is the first column and the top entry in this column is non-zero. However this entry is equal to 3 and to avoid division by 3 we switch the first and third rows performing $S_{1,3}$

(this step is not necessary, but it is convenient). We get the matrix

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 8 & 5 & 11 \\ 2 & 4 & 5 & -3 & 19 & 12 & 27 \\ 3 & 6 & 5 & -2 & 21 & 14 & 30 \\ 2 & 4 & 3 & -1 & 13 & 9 & 19 \\ 3 & 6 & 7 & -4 & 27 & 17 & 38 \end{bmatrix}.$$

Now we perform operations $E_{2,1}(-2)$, $E_{3,1}(-3)$, $E_{4,1}(-2)$, $E_{5,1}(-3)$ (note that these operations commute so it does not matter in which order we perform them) to get

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 8 & 5 & 11 \\ 0 & 0 & 1 & -1 & 3 & 2 & 5 \\ 0 & 0 & -1 & 1 & -3 & -1 & -3 \\ 0 & 0 & -1 & 1 & -3 & -1 & -3 \\ 0 & 0 & 1 & -1 & 3 & 2 & 5 \end{bmatrix}.$$

Now we leave the first row alone and apply the procedure to the matrix consisting of rows 2,3,4,5. The first non-zero column of this matrix is the third column and the top entry in this column is already non-zero (it is in fact 1, which is convenient). We perform operations $E_{3,2}(1)$, $E_{4,2}(1)$, $E_{5,2}(-1)$ (again, these operations commute) to get

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 8 & 5 & 11 \\ 0 & 0 & 1 & -1 & 3 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we concentrate on the matrix consisting of the last three rows. The first non-zero column is column 6 and the top entry in this column is 1. We perform operation $E_{4,3}(-1)$ to get

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 8 & 5 & 11 \\ 0 & 0 & 1 & -1 & 3 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

At this point we arrive at a matrix in row-echelon form. From it we can easily see that M has three pivot columns, namely the first, third and sixth columns (these are the columns of the last matrix which contain the first non-zero entry of some row). Thus the rank of M is equal to 3.

Now we employ our second procedure to transform the last matrix into a reduced row-echelon form. The first non-zero entry of each row is already equal to 1. We start with the third row, perform $E_{2,3}(-2)$, $E_{1,3}(-5)$ to get

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 8 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we move to the second row and perform $E_{1,2}(-2)$ to get

$$K = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix K is indeed in a reduced row-echelon form. We can summarize our computations in the following equation:

$$E_{1,2}(-2)E_{2,3}(-2)E_{1,3}(-5)E_{4,3}(-1)E_{3,2}(1)E_{4,2}(1)E_{5,2}(-1)E_{2,1}(-2)E_{3,1}(-3)E_{4,1}(-2)E_{5,1}(-3)S_{1,3}M = K$$

Right now this should be interpreted as performing elementary row operations on M, recorded "from right to left". Later we will develop concepts which will allow to interpret the above line as a certain "product" of matrices.