In the last two notes we have introduced several fundamental concepts. Now we will discuss methods to deal with these concepts in practice. We will be working with vectors in $\mathbb{R}^m$. Suppose we have given vectors $v_1, \ldots, v_n, v$ in $\mathbb{R}^m$. We may ask the following questions:

1. is v in the span of $v_1, \ldots v_n$? If yes, how to express v as a linear combination of $v_1, \ldots, v_n$?
2. are the vectors $v_1, \ldots v_n$ linearly independent? If not, how to choose among them a subsequence which is a basis of of the subspace $\mathrm{span}\{v_1, \ldots, v_n\}$?
3. what is the dimension of $\mathrm{span}\{v_1, \ldots, v_n\}$?

It turns out that these questions are equivalent to some questions we have already learned how to answer and a rather simple technique which we have already used many times can be used again to answer all these questions.

Consider the linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ which sends e_1 to v_1, e_2 to v_2,..., e_n to v_n. We have seen that $\mathrm{span}\{v_1, \ldots, v_n\}$ is the same as the image of L. So our first question is equivalent to: is v in the image of L? If yes, find $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $L(x) = v$. We also know that $v_1, \ldots, v_n$ are linearly independent if and only if L is one-to-one. Now $L = L_A$, where A is the $m \times n$ matrix whose columns are the vectors $v_1, \ldots, v_n$. So all we need to do is to consider the augmented matrix $M = [A|v]$ (i.e. we add v as the last column) and v is in the span if and only if the last column of M is not a pivot column. Any solutions $x = (x_1, \ldots, x_n)$ to the system of linear equations with augmented matrix M will give us expression of v as linear combination of $v_1, \ldots, v_n$: $v = x_1v_1 + \ldots + x_nv_n$. Furthermore, vectors $v_1, \ldots, v_n$ are linearly independent if and only if the rank of A is equal to n.

In order to answer the last two of our questions, we need to understand which subsequences of $v_1, \ldots, v_n$ are linearly dependent and which are linearly independent. Let S be a subset of the set $\{1, 2, \ldots, n\}$. If we consider only the vectors v_i whose number i belongs to S, i.e. $i \in S$, then these vectors are linearly dependent if and only if there are numbers x_i for $i \in S$ which are not all zero and such that the sum $\sum_{i \in S} x_iv_i = 0$ (this means that we only use the numbers from S in the summation). This is the same as having a vector $x = (x_1, \ldots, x_n)$ such that not all x_i are zero, but $x_i = 0$ whenever i is not in S, and $L_A(x) = 0$. For example, v_2, v_4, v_5 are linearly dependent iff there is a non-zero vector x of the form $(0, x_2, 0, x_4, x_5, 0, \ldots, 0)$ such that $L_A(x) = 0$. Let us write down this observation explicitly:

Let A be an $m \times n$ matrix and let S be a subset of $\{1, 2, \ldots, n\}$. The columns of A with numbers in the subset S are linearly dependent if and only if there is a non-zero vector $x = (x_1, \ldots, x_n)$ such that $x_i = 0$ for all i **not in the subset** S and $L_A(x) = 0$.

Suppose now that B is an $m \times n$ matrix row-equivalent to A. This is the same as saying that $B = EA$ for some invertible $m \times m$ matrix E (since every invertible matrix is a product of elementary matrices and multiplication on the left by an elementary matrix acts as elementary row operation). Thus $L_B = L_E \circ L_A$. Since L_E is a bijection, we have $L_B(x) = 0$ if and only if $L_A(x) = 0$. Thus columns of A with numbers in a subset S are linearly dependent if and only if columns of B with numbers in S are linearly dependent and a linear dependencies among columns of A are the same as linear dependencies among columns of B and vice versa. Let us apply this observation to the case when B is the reduced row-echelon form row equivalent to A. Suppose s is the rank of B (which is the same as rank of A), i.e. the number if pivot columns. Note that the first pivot column of B is $e_1 = (1, 0, \ldots, 0)$, the second pivot column is e_2, ..., the last pivot columns is e_s. Thus pivot columns of B are linearly independent. It follows that pivot columns of A are linearly independent. Any other column of B will look like $(a_1, \ldots, a_s, 0, \ldots, 0)$ (when written as a row) so it will be a linear combination of the pivot columns:

$$(a_1, \ldots, a_s, 0, \ldots, 0) = a_1e_1 + \ldots + a_se_s.$$

It follows that the corresponding column of A is the same linear combination of the pivot column of A. Thus we get the following important result.

The pivot columns of a matrix A form a basis of the subspace spanned by all columns of A. In other words, the pivot columns of A form a basis of the image of the linear transformation L_A. In particular, the dimension of the subspace spanned by all columns of A (i.e the dimension of the image of L_A) is equal to the rank of A. Moreover, a column of the reduced row-echelon form row equivalent to A provides coefficients to express the corresponding column of A as a linear combination of the pivot columns of A.

This result tells us how to answer the last two of our questions. The dimension of $\operatorname{span}\{v_1, \dots, v_n\}$ is equal to the rank of the matrix A. And the pivot columns of A form a basis of $\operatorname{span}\{v_1, \dots, v_n\}$.

Let us illustrate the above discussion with a concrete example which shows how the method works.

Example. Let $v_1 = (1, 2, -1, 0, 3)$, $v_2 = (1, 1, -1, 2, 1)$, $v_3 = (2, 3, -2, 2, 4)$, $v_4 = (0, 1, 0, -2, 2)$, $v_5 = (1, 1, 0, 0, 1)$, $u = (1, 0, 0, 0, 0)$, $w = (1, 2, 0, -2, 3)$. Determine whether u or w belongs to $\operatorname{span}\{v_1, \dots, v_5\}$. Among the vectors $v_1, \dots, v_5$ find a basis of $\operatorname{span}\{v_1, \dots, v_5\}$ and express all the other vectors as linear combinations of the vectors in the basis found.

Solution. We start with the matrix

$$\left[\begin{array}{rrrrr|rr} 1 & 1 & 2 & 0 & 1 & 1 & 1 \\ 2 & 1 & 3 & 1 & 1 & 0 & 2 \\ -1 & -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & -2 & 0 & 0 & -2 \\ 3 & 1 & 4 & 2 & 1 & 0 & 3 \end{array}\right].$$

We list the vectors $v_1, \dots, v_5$ spanning our subspace as columns to the left of the dividing line (this part is the matrix A in our discussion above) and to the right of the dividing line we list as columns the vectors u, w (these are the vectors about which we want to know if they belong to the span or not).

Next we perform elementary row operations to bring the part to the left of the dividing line to the reduced row-echelon form. We start with $E_{2,1}(-2)$, $E_{3,1}(1)$, $E_{5,1}(-3)$ and get

$$\left[\begin{array}{rrrrr|rr} 1 & 1 & 2 & 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & -2 & 0 & 0 & -2 \\ 0 & -2 & -2 & 2 & -2 & -3 & 0 \end{array}\right].$$

Then we perform $E_{1,2}(1)$, $E_{4,2}(2)$, $E_{5,2}(-2)$ and get

$$\left[\begin{array}{rrrrr|rr} 1 & 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Finally, we do $E_{4,3}(2)$, $E_{2,3}(1)$ followed by $D_2(-1)$ to get

$$\left[\begin{array}{rrrrr|rr} 1 & 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

At this point the part to the left of the dividing line is in reduced row-echelon form (i.e. this is the reduced row echelon form row equivalent to A). Now we look at the part to the left and any particular column to the right of the dividing line. If the column to the right has a non-zero entry in some row whose part to the left of the dividing line has only zeros, the vector corresponding to this column does not belong to the span (the system is inconsistent). Otherwise the vector belongs to the span. In our case the, the first column to the right of the dividing line has a non zero entry in the fourth row, and all entries to the left of the dividing line in the 4th row are zero. This means that u is not in $\operatorname{span}\{v_1, \dots, v_5\}$. On the other hand, the second column to the right of the division line has no non-zero entry in rows 4 or 5. Thus w is in $\operatorname{span}\{v_1, \dots, v_5\}$.

Since columns 1,2,5 are the pivot columns of the part to the left of the dividing line, vectors v_1, v_2, v_5 form a basis of $\operatorname{span}\{v_1, \dots, v_5\}$. Moreover, from the reduced row-echelon form we see that $v_3 = v_1 + v_2$, $v_4 = v_1 - v_2$, $w = v_1 - v_2 + v_5$ (the coefficients are given by the corresponding columns of the reduced matrix).

Let A be an $m \times n$ matrix. In our discussion above we were led to consider the subspace spanned by the columns of A. This motivates the following definition.

Definition. Let A be an $m \times n$ matrix. The subspace of $\mathbb{R}^m$ spanned by the columns of A is called the **column space of** A. Similarly, the subspace of $\mathbb{R}^n$ spanned by the rows of A is called the **row space of** A.

We have seen that the dimension of the column space of a matrix A coincides with the rank of A. What can we say about the row space?

The following simple observation, which is an immediate consequence of the way matrices are multiplied, is very useful for questions about column or row spaces.

Let A be an $m \times n$ matrix, let B be $n \times k$ matrix, and let $C = AB$.

- The i-th column of C is the linear combination of the columns of A with coefficients provided by the i-th column of B. In particular, the column space of C is contained in the column space of A.
- The i-th row of C is the linear combination of the rows of B with coefficients provided by the i-th row of A. In particular, the row space of C is contained in the row space of B.

Example. Consider the product

$$AB = \begin{bmatrix} 3 & -1 & 2 & 3 & -1 \\ 1 & -1 & 2 & 3 & 5 \\ 2 & -3 & 6 & 9 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & 2 \\ 7 & -2 \\ -1 & 3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 11 \\ 26 & -5 \\ 35 & 5 \end{bmatrix} = C.$$

We see that the second row of C is

$$(26, -5) = (-1, 2) - (4, 2) + 2(7, -2) + 3(-1, 3) + 5(4, -2),$$

i.e. it is a linear combination of the rows of B with coefficients provided by the 2nd row of A. Similarly, the second column of C is

$$(11, -5, 5) = 2(3, 1, 2) + 2(-1, -1, -3) - 2(2, 2, 6) + 3(3, 3, 9) - 2(-1, 5, 4),$$

i.e. it is a linear combination of the columns of A with coefficients provided by the 2nd column B.

An immediate corollary from the last observation is the following result.

Let A and B be $m \times n$ matrices which are row equivalent. Then the row spaces of A and B coincide.

Indeed, there is an invertible $m \times m$ matrix M such that $B = MA$. Thus the row space of B is contained in the row space of A. Since M is invertible, we have $A = M^{-1}B$, so the row space of A is contained in the row space of B. It follows that the row spaces of A and B are equal.

We can apply this to the case when B is the reduced row-echelon form row equivalent to A. It is easy to see that the non-zero rows of of a matrix in reduced row-echelon form are linearly independent. In fact, consider the i-th row of B. Let us look at the i-th pivot column of of B, say it is column j. It has 1 in the i-th row and zeros everywhere else. Thus all rows of B except the i-th row have zero in the j-th place. Thus the i-th row can not be a linear combination of the other rows. This proves that the non-zero rows of B are linearly independent. Thus we get the following theorem.

> Let A be an $m \times n$ matrix. Then non zero rows of the reduced row-echelon form row equivalent to A form a basis of the row space of A. Thus the dimension of the row space of A is equal to the rank of A.

As a corollary we get the following theorem.

> **Theorem.** Let A be an $m \times n$ matrix. Then rank of A, the dimension of the row space of A, and the dimension of the column space of A are all equal. In particular A and A^T have the same rank.

To justify the last part note that the row space of A is the same as the column space of A^T.

The above results can be used to give a different method of finding a basis of span$\{v_1, \ldots, v_n\}$, given the vectors $v_1, \ldots v_n$ of $\mathbb{R}^m$. Namely, we make a matrix M whose **rows** are the vectors $v_1, \ldots v_n$ and then find the matrix N in reduced row-echelon form, row equivalent to M. The non-zero rows of N form a basis of span$\{v_1, \ldots, v_n\}$. Note however that this basis is usually not a part of the vectors $v_1, \ldots v_n$. Moreover, expressing the vectors v_i as linear combinations of the vectors in this basis is much harder than in our first method.

There is however one peculiar feature of this method. Namely, no matter which spanning set we choose for a given subspace we will always get the same basis using this method. In other words, we have the following result.

> **Theorem.** Consider two collections of vectors $v_1, \ldots, v_n$ and $w_1, \ldots, w_k$ in $\mathbb{R}^m$. Let A be the $n \times m$ matrix with rows $v_1, \ldots, v_n$ and let B be the $k \times m$ matrix with rows $w_1, \ldots, w_k$. Then span$\{v_1, \ldots, v_n\} =$ span$\{w_1, \ldots, w_k\}$ if and only if the reduced row-echelon forms row equivalent to A and B have the same non-zero rows.

Indeed, if the reduced row-echelon forms row equivalent to A and B have the same non-zero rows then these rows form a basis of both span$\{v_1, \ldots, v_n\}$ and span$\{w_1, \ldots, w_k\}$, so these two subspaces are the same.

Conversely, suppose that span$\{v_1, \ldots, v_n\} =$ span$\{w_1, \ldots, w_k\}$. Consider the $(n+k) \times m$ matrix M whose first n rows are $v_1, \ldots, v_n$ and the last k rows are $w_1, \ldots, w_k$. Since $w_i = a_{1,i}v_1 + \ldots a_{n,i}v_n$ for some numbers $a_{i,1}, \ldots, a_{i,n}$, subtracting form the $(n+i)$-th row of M the first row multiplied by $a_{1,i}$, then the second row multiplied by $a_{2,i}$,..., then the n-th row multiplied by $a_{n,i}$ (we do this for $i = 1, 2, \ldots, k$) we get a matrix M_1 row equivalent to M whose rows are $v_1, \ldots, v_n$ followed by k zero rows. Now M_1 is clearly row equivalent a matrix whose non-zero rows coincide with the non-zero rows of the reduced row-echelon form of A. Thus the reduced row echelon forms of M and A have the same non-zero rows. Now let N be the matrix with rows $w_1, \ldots, w_k$ followed by rows $v_1, \ldots, v_n$. The same reasoning as above shows that the the reduced row echelon forms of N and B have the same non-zero rows. But N is obtained from M by permuting rows, so M and N are row-equivalent, hence they have the same reduced row-echelon forms.

Example. Let $v_1 = (1, 2, -1, 0, 3)$, $v_2 = (1, 1, -1, 2, 1)$, $v_3 = (2, 3, -2, 2, 4)$, $v_4 = (0, 1, 0, -2, 2)$, $v_5 = (1, 1, 0, 0, 1)$. We will use our second method to find a basis of $\text{span}\{v_1, \ldots, v_5\}$. We start with the matrix whose rows are $v_1, \ldots, v_5$:

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 1 & 1 & -1 & 2 & 1 \\ 2 & 3 & -2 & 2 & 4 \\ 0 & 1 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

We leave it as an exercise to see that the reduced row-echelon form row equivalent to this matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the vectors $(1, 0, 0, 2, -1)$, $(0, 1, 0, -2, 2)$, $(0, 0, 1, -2, 0)$ for a basis of $\text{span}\{v_1, \ldots, v_5\}$.