Given a linear transformation $T : V \longrightarrow W$, we often want to understand the two subspaces associated with T: the kernel of T and the image of T. We would like to know the dimensions of these subspaces and a basis (as we will see soon, choosing a basis of a vector space allows us to identify it with $\mathbb{R}^n$, so we can apply our techniques of computations with matrices to answer various questions about the vector space).

Suppose then that we are given a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. We know that $T = L_A$ for unique $m \times n$ matrix A. In the previous note we learned that the pivot columns of A form a basis of the image of L_A. Thus, the image of L_A (which is the same as the column space of A) had dimension equal to the rank of A. Our goal now is to gain a better understanding of the kernel of L_A. Clearly, the kernel of L_A is the set of solutions of the homogeneous system of linear equations with coefficient matrix A: $Ax = 0$. We already know how to solve this system and we now translate our method of solving this system into the language of bases and linear combinations. To this end, let P be a reduced row-echelon form of A. From P we know the free variables (which correspond to non-pivot columns of P) and how to express the dependent variables in terms of the free variables. Let s be the rank of A so we have $n - s$ free variables. For each $1 \leq i \leq n - s$ there is unique solution $v_i = (v_{1,i}, v_{2,i}, \ldots, v_{n,i})$ corresponding to choosing 1 for the i-th free variable and 0 for all the other free variables. We claim that the vectors $v_1, \ldots, v_{n-s}$ are linearly independent. Indeed, suppose that the i-th free variable has number m_i, so $1 \leq m_1 < m_2 < \ldots < m_{n-s} \leq n$. Suppose that $a_1v_1 + \ldots + a_{n-s}v_{n-s} = 0$. Note that the m_i-th coordinate of v_j is 0 when $j \neq i$ and it is 1 when $j = i$. Thus the m_i-th coordinate of $a_1v_1 + \ldots + a_{n-s}v_{n-s}$ is equal to a_i, so $a_i = 0$ for all i. This shows the linear independence of $v_1, \ldots, v_{n-s}$. Furthermore, we know that for any choice $x_{m_1} = t_1, \ldots, x_{m_{n-s}} = t_{n-s}$ of values of the free variables there is a unique solution $x = (x_1, x_2, \ldots, x_n)$ to our system with these values of the free variables. On the other hand, $t_1v_1 + \ldots + t_{n-s}v_{n-s}$ is a solution with m_i-th coordinate equal to t_i for $i = 1, \ldots, n - s$. It follows that $x = t_1v_1 + \ldots + t_{n-s}v_{n-s}$. In other words, the vectors $v_1, \ldots, v_{n-s}$ span the space of all solutions to $Ax = 0$. Putting all these together we get the following result.

Theorem. Let A be an $m \times n$ matrix. Consider the homogeneous system of linear equations $Ax = 0$. Let s be the rank of A. For each $1 \leq i \leq n - s$ let $v_i = (v_{1,i}, v_{2,i}, \ldots, v_{n,i})$ be the unique solution corresponding to choosing 1 for the i-th free variable and 0 for all the other free variables. Then $v_1, v_2, \ldots, v_{n-s}$ is a basis of the space of all solutions to the system, i.e. it is a basis of the kernel of the linear transformation L_A. In particular, the dimension of the kernel of L_A is equal to $n - s$:

$$\dim(\ker(L_A)) = n - \operatorname{rank}(A).$$

It is convenient to make the following definition.

Definition. Let A be an $m \times n$. The **nullity** of A, denoted by $\operatorname{nul}(A)$ is the dimension of the space of solutions to the homogeneous system of linear equations $Ax = 0$. In other words, $\operatorname{nul}(A) = \dim(\ker(L_A))$.

As an immediate corollary we get the following important theorem.

Rank-nullity Theorem. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then

$$\dim(\ker(T)) + \dim(\operatorname{im}(T)) = n$$

where $\operatorname{im}(T)$ denotes the image of T. Equivalently, for any $m \times n$ matrix A we have

$$\operatorname{rank}(A) + \operatorname{nul}(A) = n.$$

Example. Find bases of the kernel and the image of the linear transformation L_A, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 3 & 2 & 3 \\ 3 & 2 & 3 & 2 & 3 & 2 & 3 \\ 2 & -1 & 2 & -1 & 2 & -1 & 2 \end{bmatrix}.$$

Solution. The reduced row-echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The free variables of the system $Ax = 0$ are x_3, x_5, x_6 and x_7 and the system is equivalent to:

$$\begin{aligned} x_1 &= -x_3 \quad -x_5 \qquad\qquad -x_7 \\ x_2 &= \qquad\quad 2x_5 + x_6 \quad +2x_7 \\ x_3 &= \ \ x_3 \\ x_4 &= \qquad -2x_5 - 2x_6 \quad -2x_7 \\ x_5 &= \qquad\quad x_5 \\ x_6 &= \qquad\qquad\quad x_6 \\ x_7 &= \qquad\qquad\qquad\qquad x_7 \end{aligned}$$

The choice $x_3 = 1, x_5 = x_6 = x_7 = 0$ yields the solution $v_1 = (-1, 0, 1, 0, 0, 0, 0)$.
The choice $x_5 = 1, x_3 = x_6 = x_7 = 0$ yields the solution $v_2 = (-1, 2, 0, -2, 1, 0, 0)$.
The choice $x_6 = 1, x_3 = x_5 = x_7 = 0$ yields the solution $v_3 = (0, 1, 0, -2, 0, 1, 0)$.
The choice $x_7 = 1, x_3 = x_5 = x_6 = 0$ yields the solution $v_4 = (-1, 2, 0, -2, 0, 0, 1)$.

Thus the kernel of L_A has dimension 4 and a basis v_1, v_2, v_3, v_4. Every vector in the kernel (i.e every solution of $Ax = 0$) can be expressed as $t_1v_1 + t_2v_2 + t_3v_3 + t_4v_4$ for unique choice of the parameters t_1, t_2, t_3, t_4.

The image of L_A has dimension 3 and the pivot columns of A form a basis of the image, i.e. $(1, 1, 3, 2), (1, 0, 2, -1), (1, 1, 2, -1)$ is a basis of $\mathrm{im}(L_A)$.

From now on, when we ask to solve a system of linear equations $Ax = b$, we mean to express all solutions in terms of a basis of the space of solutions to the homogeneous system $Ax = 0$. Namely we find a basis $v_1, \ldots v_r$ of the space of solutions to $Ax = 0$ and we find a particular solution y to the system $Ax = b$. Then every solution to $Ax = b$ can be expressed as $t_1v_1 + \ldots + t_rv_r + y$ for unique choice of the parameters $t_1, \ldots, t_r$. For y it is convenient to choose the solution corresponding to the choice of all the free variables to be 0.

Example. Solve the following system of linear equations:

$$\begin{aligned} x_1 + x_2 \quad +x_3 + x_4 \quad +x_5 + x_6 \quad +x_7 &= 1 \\ x_1 \qquad\quad +x_3 + x_4 \ +3x_5 + 2x_6 + 3x_7 &= 2 \\ 3x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 + 2x_6 + 3x_7 &= 2 \\ 2x_1 - x_2 \ +2x_3 - x_4 \ +2x_5 - x_6 \ +2x_7 &= -1 \end{aligned}$$

Solution: The augmented matrix of this system is

$$\left[\begin{array}{ccccccc:c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 3 & 2 & 3 & 2 \\ 3 & 2 & 3 & 2 & 3 & 2 & 3 & 2 \\ 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 \end{array}\right]$$

The reduced row-echelon form of this matrix is

$$\left[\begin{array}{ccccccc:c} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & -1 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last column is not a pivot column so the system is consistent. The free variables are x_3, x_5, x_6 and x_7 and the system takes form:

$$\begin{aligned} x_1 &= -x_3 \; -x_5 \qquad\quad -x_7 \\ x_2 &= \qquad\; 2x_5 + x_6 \; +2x_7 - 1 \\ x_3 &= \; x_3 \\ x_4 &= \qquad -2x_5 - 2x_6 - 2x_7 - 2 \\ x_5 &= \qquad\quad x_5 \\ x_6 &= \qquad\qquad\quad x_6 \\ x_7 &= \qquad\qquad\qquad\quad x_7 \end{aligned}$$

We see that a basis of solutions to the associated homogeneous system is $\mathbf{v}_1 = (-1, 0, 1, 0, 0, 0, 0)$, $\mathbf{v}_2 = (-1, 2, 0, -2, 1, 0, 0)$, $\mathbf{v}_3 = (0, 1, 0, -2, 0, 1, 0)$, $\mathbf{v}_4 = (-1, 2, 0, -2, 0, 0, 1)$. A particular solution to the non-homogeneous system is obtained by setting $x_3 = x_5 = x_6 = x_7 = 0$. We get $\mathbf{y} = (0, -1, 0, 2, 0, 0, 0)$.

The general solution to the system can be written as:

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (0, -1, 0, 2, 0, 0, 0) + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3 + t_4\mathbf{v}_4$$

where t_1, t_2, t_3, t_4 are independent parameters.