Consider a finite dimensional vector space V. Let $n = \dim(V)$. We know that any basis of V consists of n vectors. Let us choose one such basis $b_1, b_2, \ldots, b_n$. We will denote this basis by B, so B will stand for the list $b_1, b_2, \ldots, b_n$ (note that there are many choices for a basis of V). Any vector v in V can be expressed in a unique way as a linear combination of vectors in B:

$$v = s_1 b_1 + s_2 b_2 + \ldots + s_n b_n$$

for unique scalars $s_1, \ldots, s_n$. The numbers $s_1, \ldots, s_n$ are called **the coordinates of v in the basis B**. This way each vector v can be identified with a string of numbers $(s_1, \ldots, s_n)$, and conversely any string of numbers $(s_1, \ldots, s_n)$ corresponds to unique vector, namely $s_1 b_1 + s_2 b_2 + \ldots + s_n b_n$.

There is a very useful way of thinking about coordinates in terms of linear transformations. Consider the linear transformation $K_B : \mathbb{R}^n \longrightarrow V$ given by $K_B(x_1, \ldots, x_n) = x_1 b_1 + x_2 b_2 + \ldots + x_n b_n$. Since B is a basis, K_B is a bijection. The inverse transformation K_B^{-1} assigns to each vector v its coordinates $K_B^{-1}(v)$ (considered as an element of $\mathbb{R}^n$). Conversely, if $L : \mathbb{R}^n \longrightarrow V$ is a linear transformation and a bijection then the vectors $b_1 = L(e_1)$, ..., $b_n = L(e_n)$ form a basis B of V and $L = K_B$.

Definition. A linear transformation $L : V \longrightarrow W$ is called an **isomorphism** if it is a bijection. Two vector spaces are called **isomorphic** if there is an isomorphism mapping one of the spaces onto the other.

Thus, choosing a basis of a vector space V of dimension n is essentially the same as choosing an isomorphism between V and $\mathbb{R}^n$. In particular, we have the following result:

Theorem. Two finite dimensional vector spaces are isomorphic if and only if they have the same dimension.

Indeed, if $b_1, \ldots, b_n$ is a basis B of a vector space V and $d_1, d_2, \ldots, d_n$ is a basis D of a vector space W then the linear transformation $K_D K_B^{-1} : V \longrightarrow W$ is an isomorphism.

We have seen some time ago that if W is a vector space and $w_1, \ldots w_n$ are arbitrary vectors in W then there is unique liner transformation $T : \mathbb{R}^n \longrightarrow W$ such that $T(e_i) = w_i$ for $i = 1, 2 \ldots, n$. The following rather simple but very useful observation extends this result to arbitrary vector spaces.

Theorem. Let $b_1, b_2, \ldots, b_n$ be a basis B of a vector space V. If $w_1, \ldots, w_n$ are arbitrary vectors in a vector space W then there is a unique linear transformation $T : V \longrightarrow W$ such that $T(b_i) = w_i$ for $i = 1, \ldots, n$.

Indeed, suppose first that T exists. For any $v \in V$ we can write $v = x_1 b_1 + \ldots + x_n b_n$ and then

$$T(v) = x_1 T(b_1) + \ldots + x_n T(b_n) = x_1 w_1 + \ldots + x_n w_n.$$

Thus T is uniquely determined if it exists at all. To see that T exists, consider the linear transformation $L : \mathbb{R}^n \longrightarrow W$ given by the formula $L(x_1, ..., x_n) = x_1 w_1 + \ldots + x_n w_n$. The composition $L \circ K_B^{-1}$ maps b_i to w_i for $i = 1, \ldots, n$, so it has the required property.

Suppose now that W is another vector space with a basis $D : d_1, \ldots, d_m$ and that $T : V \longrightarrow W$ is a linear transformation. Then $K_D^{-1} T K_B$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, so it is equal to L_M for some $m \times n$ matrix M. The i-th column of M is $K_D^{-1} T K_B(e_i)$. Since $K_B(e_i) = b_i$, we have

$$K_D^{-1} T K_B(e_i) = K_D^{-1} T(b_i) = \text{ coordinates of } T(b_i) \text{ in the basis } D.$$

This prompts the following definition

Let $T : V \longrightarrow W$ be a linear transformation. Let B be a basis $b_1, \ldots, b_n$ of V and let D be a basis $d_1, \ldots, d_m$ of W. The **matrix of the linear transformation T in the bases B, D** is the $m \times n$ matrix ${}_DT_B$, whose i-th column consists of coordinates of the vector $T(b_i)$ in the basis D.

The matrix ${}_DT_B$ represents the linear transformation $K_D^{-1}TK_B : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. In other words, we have the following important fact.

If $s_1, \ldots, s_n$ are the coordinates of a vector $v \in V$ in the basis B, then ${}_DT_B \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$ are the coordinates of $T(v)$ in the basis D.

An important special case of the above discussion is the case when $V = W$ and $T = I$ is the identity transformation. The matrix ${}_DI_B$ is called the **transition matrix from the basis B to the basis D**.

Let B be a basis $b_1, \ldots, b_n$ of V and let D be another basis $d_1, \ldots, d_n$ of V. The **transition matrix from basis B to basis D** is the $n \times n$ matrix ${}_DI_B$, whose i-th column consists of coordinates of the vector b_i in the basis D.

In particular, we have the following property which justifies the name "transition matrix":

If $s_1, \ldots, s_n$ are the coordinates of a vector $v \in V$ in the basis B, then ${}_DI_B \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$ are the coordinates of v in the basis D.

Suppose now that we have two linear transformations: $T : V \longrightarrow W$ and $S : W \longrightarrow U$. Let B be a basis $b_1, b_2, \ldots, b_n$ of V, let D be a basis $d_1, d_2, \ldots, d_n$ of W, and let G be a basis $g_1, \ldots, g_k$ of U. Te matrix ${}_DT_B$ represents the linear transformation $K_D^{-1}TK_B : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Similarly, the matrix ${}_GS_D$ represents the linear transformation $K_G^{-1}SK_D : \mathbb{R}^m \longrightarrow \mathbb{R}^k$. It follows that the product $({}_GS_D) \cdot ({}_DT_B)$ represents the composition $(K_G^{-1}SK_D)(K_D^{-1}TK_B) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$. Using the associativity of composition and the fact that $K_DK_D^{-1} = I$ is the identity, we see that $(K_G^{-1}TK_D)(K_D^{-1}TK_B) = K_G^{-1}(ST)K_B$. On the other hand, the linear transformation $K_G^{-1}(ST)K_B$ is represented by the matrix ${}_G(ST)_B$, so we have ${}_G(ST)_B = ({}_GS_D) \cdot ({}_DT_B)$. Thus we have the following important result.

Theorem. Let $T : V \longrightarrow W$ and $S : W \longrightarrow U$ be linear transformations. Let B be a basis $b_1, b_2, \ldots, b_n$ of V, let D be a basis $d_1, d_2, \ldots, d_n$ of W, and let G be a basis $g_1, \ldots, g_k$ of U. Then ${}_G(ST)_B = ({}_GS_D) \cdot ({}_DT_B)$.

Note that for any basis D of a vector space V the matrix ${}_DI_D$ is the identity matrix. In fact, if D consists of vectors $d_1, \ldots, d_n$ then the coordinates of d_i in the basis D are all zero except the i-th coordinate, which is 1.

This observation paired with our last theorem tell us that for any two bases B, D of a vector space V we have ${}_DI_B\ {}_BI_D ={}_D I_D = I$. In other words:

Transition matrices are invertible and $({}_DI_B)^{-1} ={}_B I_D$.

The following straightforward corollary of our last theorem allows us to compare matrices of the same linear transformation in different bases.

Let $T : V \longrightarrow W$ be a linear transformation. Let B and B_1 be two bases of V and let D and D_1 be two bases of W. Then

$${}_{D_1}T_{B_1} ={}_{D_1} I_D\ {}_DT_B\ {}_BI_{B_1} ={}_{D_1} I_D\ {}_DT_B\ {}_{B_1}I_B^{-1}$$

Example. Let $\mathbb{P}_3$ be the space of all polynomials of degree at most 3. It is a vector space of dimension 4. The polynomials $1, x, x^2, x^3$ for a basis B. Polynomials $1, x-1, (x-1)^2, (x-1)^3$ for a basis D. The function $T : \mathbb{P}_3 \longrightarrow \mathbb{P}_3$ defined by $T(f) = f - 2(x-1)f'$ is a linear transformation (f' is the derivative of f).

a) Find the transition matrix from the basis B to the basis D.

b) Find the matrix ${}_BT_B$ of T in the bases B of the domain and B of the codomain.

c) Find the matrix ${}_DT_D$ of T in the bases D of the domain and D of the codomain.

d) Find the matrix ${}_DT_B$ of T in the bases B of the domain and D of the codomain.

Solution. a) In order to find the transition matrix ${}_DI_B$ we need to express each vector from basis B as a linear combination of vectors from basis D. It is easier first to find the transition matrix ${}_BI_D$ form the basis D to the basis B, as this amounts to expressing vectors from basis D as linear combinations of vectors from the basis B. After doing so, we will use the fact that ${}_DI_B ={}_B I_D^{-1}$.

We have

$$1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3, \quad x-1 = -1\cdot 1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3,$$
$$(x-1)^2 = 1\cdot 1 - 2\cdot x + 1\cdot x^2 + 0\cdot x^3, \quad (x-1)^3 = -1\cdot 1 + 3\cdot x + (-3)\cdot x^2 + 1\cdot x^3.$$

Thus the transition matrix ${}_BI_D$ form the basis D to the basis B is the matrix

$${}_BI_D = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In order to invert the matrix ${}_BI_D$ we find the reduced row-echelon form of the matrix

$$\left[\begin{array}{cccc:cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

which is equal to

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

Thus

$$
{}_D I_B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Note that from the matrix ${}_D I_B$ we see that

$$
1 = 1\cdot 1 + 0\cdot(x-1) + 0\cdot(x-1)^2 + 0\cdot(x-1)^3, \quad x = 1\cdot 1 + 1\cdot(x-1) + 0\cdot(x-1)^2 + 0\cdot(x-1)^3,
$$

$$
x^2 = 1\cdot 1 + 2\cdot(x-1) + 1\cdot(x-1)^2 + 0\cdot(x-1)^3, \; x^3 = 1\cdot 1 + 3\cdot(x-1) + 3\cdot(x-1)^2 + 1\cdot(x-1)^3.
$$

b) In order to find the matrix ${}_B T_B$, we need to express each of $T(1), T(x), T(x^2), T(x^3)$ as linear combinations of vectors in the basis B. We have

$$
T(1) = 1-2(x-1)\cdot 0 = 1\cdot 1+0\cdot x+0\cdot x^2+0\cdot x^3, \quad T(x) = x-2(x-1)\cdot 1 = 2\cdot 1+(-1)\cdot x+0\cdot x^2+0\cdot x^3,
$$

$$
T(x^2) = x^2-2(x-1)\cdot 2x = 0\cdot 1+4\cdot x+(-3)\cdot x^2+0\cdot x^3, \quad T(x^3) = x^3-2(x-1)\cdot 3x^2 = 0\cdot 1+0\cdot x+6\cdot x^2+(-5)\cdot x^3.
$$

It follows that the matrix ${}_B T_B$ is equal to

$$
{}_B T_B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & -5 \end{bmatrix}.
$$

c) In order to find the matrix ${}_D T_D$ we can use the formula ${}_D T_D = {}_D I_B \; {}_B T_B \; {}_B I_D$, since the matrices ${}_D I_B$ and ${}_B I_D$ were computed in part a). We have

$$
{}_D T_D = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}.
$$

We can also compute ${}_D T_D$ directly (it turns out quite simple in this case). This amounts to expressing each of $T(1)$, $T(x-1)$, $T((x-1)^2)$, $T((x01)^3)$ as linear combinations of vectors in the basis D. We have

$$
T(1) = 1-2(x-1)\cdot 0 = 1\cdot 1 + 0\cdot(x-1) + 0\cdot(x-1)^2 + 0\cdot(x-1)^3,
$$

$$
T(x-1) = (x-1) - 2(x-1)\cdot 1 = 0\cdot 1 + (-1)\cdot(x-1) + 0\cdot(x-1)^2 + 0\cdot(x-1)^3,
$$

$$
T((x-1)^2) = (x-1)^2 - 2(x-1)\cdot 2(x-1) = 0\cdot 1 + 0\cdot(x-1) + (-3)\cdot(x-1)^2 + 0\cdot(x-1)^3,
$$

$$
T((x-1)^3) = (x-1)^3 - 2(x-1)\cdot 3(x-1)^2 = 0\cdot 1 + 0\cdot(x-1) + 0\cdot(x-1)^2 + (-5)\cdot(x-1)^3.
$$

It follows that

$$
{}_D T_D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}.
$$

c) In order to find the matrix ${}_D T_B$ we use the formula ${}_D T_B = {}_D I_B \; {}_B T_B$. Thus

$$
{}_D T_B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & -5 \end{bmatrix}.
$$

We could also use the formula ${}_D T_B = {}_D T_D \; {}_D I_B$ which yields

$$
{}_D T_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & -5 \end{bmatrix}.
$$

Let us remark that the matrix ${}_DT_B$ tells us that

$$T(1) = 1 \cdot 1 + 0 \cdot (x-1) + 0 \cdot (x-1)^2 + 0 \cdot (x-1)^3,$$

$$T(x) = 1 \cdot 1 + (-1) \cdot (x-1) + 0 \cdot (x-1)^2 + 0 \cdot (x-1)^3,$$

$$T(x^2) = 1 \cdot 1 + (-2) \cdot (x-1) + (-3) \cdot (x-1)^2 + 0 \cdot (x-1)^3,$$

$$T(x^3) = 1 \cdot 1 + (-3) \cdot (x-1) + (-9) \cdot (x-1)^2 + (-5) \cdot (x-1)^3.$$