The results obtained in the last note give us the following strategy to determine if a linear transformation $T : V \longrightarrow V$ is diagonalizable and to find a basis in which the matrix of T is diagonal.

- **Step 1.** Represent T in some basis B by the matrix $M = {}_B T_B$. Let B consist of vectors $v_1, \ldots, v_n$ (so $n = \dim(V)$). If $V = \mathbb{R}^n$ then usually we take the standard basis for B.

- **Step 2.** Find all the eigenvalues $\lambda_1, \ldots, \lambda_k$ of the matrix M (we know that $k \leq n$). At the moment we do not have a systematic method for this step. We will learn one soon.

- **Step 3.** For each eigenvalue λ_i find the dimension and a basis of the subspace of $\mathbb{R}^n$ consisting of solutions to the homogeneous system of linear equations $(M - \lambda_i I)x = 0$. Let d_i be the dimension of this space and let $z_{i,1}, z_{i,2}, \ldots, z_{i,d_i}$ be a basis. Let $w_{i,j}$ be the vector in V whose coordinates in the basis B are given by $z_{i,j}$ (thus, if $z_{i,j} = (a_1, a_2, \ldots, a_n)$ then $w_{i,j} = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$). Then $w_{i,1}, \ldots, w_{i,d_i}$ is a basis of the eigenspace $V(\lambda_i)$ corresponding to the eigenvalue λ_i.

  If $d_1 + \ldots + d_k < n$ then T is not diagonalizable. Otherwise, we have $d_1 + \ldots + d_k = n$ and T is diagonalizable.

- **Step 4.** Suppose that $d_1 + \ldots + d_k = n$. Then the vectors

  $$w_{1,1}, \ldots, w_{1,d_1}, w_{2,1}, \ldots, w_{2,d_2}, \ldots, w_{k,1}, \ldots, w_{k,d_k}$$

  form a basis D of V consisting of eigenvectors of T. The matrix $N = {}_D T_D$ of T in the basis D is the diagonal matrix whose diagonal entries are as follows: λ_1 repeated d_1 times followed by λ_2 repeated d_2 times, followed by λ_3 repeated d_3 times,..., followed by λ_k repeated d_k times. We have $N = A^{-1}MA$, where A is the transition matrix from the basis D to the basis B. The columns of the matrix A are the vectors

  $$z_{1,1}, \ldots, z_{1,d_1}, z_{2,1}, \ldots, z_{2,d_2}, \ldots, z_{k,1}, \ldots, z_{k,d_k}$$

  found in step 3.

Example. Let V be the space $M_2(\mathbb{R})$ of all 2×2 matrices Let $P = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$. Define $T : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$ by $T(X) = PX - XP$. It is easy to see that T is a linear transformation. We would like to know whether T is diagonalizable.

Step 1. The matrices $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $P_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis of $M_2(\mathbb{R})$. This is our basis B. We have

$$T(P_1) = PP_1 - P_1P = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -3 & 0 \end{bmatrix} = -2P_2 - 3P_3,$$

$$T(P_2) = PP_2 - P_2P = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 0 & -3 \end{bmatrix} = 3P_1 - 5P_2 - 3P_4,$$

$$T(P_3) = PP_3 - P_3P = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 5 & -2 \end{bmatrix} = 2P_1 + 5P_3 - 2P_4,$$

$$T(P_4) = PP_4 - P_4P = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} = 2P_2 + 3P_3.$$

It follows that the matrix $M = {}_B T_B$ representing T in the basis B is

$$M = \begin{bmatrix} 0 & 3 & 2 & 0 \\ -2 & -5 & 0 & 2 \\ -3 & 0 & 5 & 3 \\ 0 & -3 & -2 & 0 \end{bmatrix}$$

Step 2. Now we need to find all the eigenvalues of M. We do not have yet a systematic method to do so. Let us then take for granted that the eigenvalues of M are $0, 1, -1$.

Step 3. The eigenspace of M corresponding to eigenvalue 0 is the space of solutions to

$$(M - 0 \cdot I)x = \begin{bmatrix} 0 & 3 & 2 & 0 \\ -2 & -5 & 0 & 2 \\ -3 & 0 & 5 & 3 \\ 0 & -3 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The reduced row-echelon form of $\begin{bmatrix} 0 & 3 & 2 & 0 \\ -2 & -5 & 0 & 2 \\ -3 & 0 & 5 & 3 \\ 0 & -3 & -2 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -5/3 & -1 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The eigenspace is therefore of dimension $d_1 = 4 - 2 = 2$. x_3 and x_4 are free variables and $(5/3, -2/3, 1, 0), (1, 0, 0, 1)$ is a basis of our space. In order to avoid fractions, we choose $z_{1,1} = (5, -2, 3, 0)$, $z_{1,2} = (1, 0, 0, 1)$ to be our basis. Then $w_{1,1} = 5P_1 - 2P_2 + 3P_3 = \begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}$ and $w_{1,2} = P_1 + P_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis of the eigenspace $V(0)$.

The eigenspace of M corresponding to eigenvalue 1 is the space of solutions to

$$(M - 1 \cdot I)x = \begin{bmatrix} -1 & 3 & 2 & 0 \\ -2 & -6 & 0 & 2 \\ -3 & 0 & 4 & 3 \\ 0 & -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The reduced row-echelon form of $\begin{bmatrix} -1 & 3 & 2 & 0 \\ -2 & -6 & 0 & 2 \\ -3 & 0 & 4 & 3 \\ 0 & -3 & -2 & -1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The eigenspace is therefore of dimension $d_2 = 4 - 3 = 1$. x_4 is the free variable and $(-1, 2/3, -3/2, 1)$ is a basis of solutions . In order to avoid fractions, we choose $z_{2,1} = (-6, 4, -9, 6)$ to be our basis. Then $w_{2,1} = -6P_1 + 4P_2 - 9P_3 + 6P_4 = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix}$ is a basis of the eigenspace $V(1)$.

The eigenspace of M corresponding to eigenvalue -1 is the space of solutions to

$$(M - (-1) \cdot I)x = \begin{bmatrix} 1 & 3 & 2 & 0 \\ -2 & -4 & 0 & 2 \\ -3 & 0 & 6 & 3 \\ 0 & -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The reduced row-echelon form of $\begin{bmatrix} 1 & 3 & 2 & 0 \\ -2 & -4 & 0 & 2 \\ -3 & 0 & 6 & 3 \\ 0 & -3 & -2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The eigenspace is therefore of dimension $d_3 = 4 - 3 = 1$. x_4 is the free variable and $z_{3,1} = (-1, 1, -1, 1)$ is a basis of solutions . Thus $w_{3,1} = -P_1 + P_2 - P_3 + P_4 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ is a basis of the eigenspace $V(-1)$.

Step 4. From step 3 we have $d_1 + d_2 + d_3 = 4 = \dim M_2(\mathbb{R})$. It follows that T is diagonalizable. Moreover, the matrices $\begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ from a basis D of $M_2(\mathbb{R})$ which consists of eigenvectors.

The transition matrix $A =_B T_D$ from basis D to basis B is the matrix $A = \begin{bmatrix} 5 & 1 & -6 & -1 \\ -2 & 0 & 4 & 1 \\ 3 & 0 & -9 & -1 \\ 0 & 1 & 6 & 1 \end{bmatrix}$. The matrix

$N =_D T_D$ of T in the basis D is the diagonal matrix

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Note that the matrix N satisfies $N^3 = N$. It follows that $T = T^3 = T^5 = \dots$ and $T^2 = T^4 = T^6 = \dots$.

Challenging Exercise. For any 2×2 matrix P one can consider the linear transformation $T : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$ defined by by $T(X) = PX - XP$. Show that if P has 2 different eigenvalues λ_1 and λ_2 then T is diagonalizable and has eigenvalues $0, \lambda_1 - \lambda_2, \lambda_2 - \lambda_1$ with corresponding eigenspaces of dimension $2, 1, 1$ respectively. Can you extend this to $n \times n$ matrices? (in particular showing that if P is diagonalizable then T is diagonalizable).