In the last note we outlined a procedure to determine if a given linear transformation is diagonalizable. In order to make this procedure work we need a method to find all eigenvalues of a given matrix M. Recall that λ is an eigenvalue of M if and only if the matrix $M - \lambda I$ is singular, i.e. it is not invertible. This brings us back to the problem of inverting a given matrix A. Even though we have an efficient algorithm to invert a given matrix (or show that it is not invertible), this algorithm gives us no insight on how the entries of A^{-1} depend on the entries of A. It is our goal now to understand how A^{-1} is built from the entries of A. This will lead us to a discovery of a very important concept, namely the **determinant** of a square matrix. The determinant will be our main tool for finding eigenvalues of a given matrix.

Let us start with a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We assume for now that $a \neq 0$. In order to compute A^{-1} we employ the row-reduction technique. Thus we start with the matrix

$$\left[\begin{array}{cc:cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

and perform the following elementary row operations: $E_{2,1}(-c/a)$, $D_2(a)$ to get

$$\left[\begin{array}{cc:cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right]$$

We see that A is invertible iff $ad - bc \neq 0$ and if this is the case then we may perform further operations: $D_2(1/(ad-bc))$, $E_{1,2}(-b)$, $D_1(1/a)$ to get

$$\left[\begin{array}{cc:cc} 1 & 0 & d/(ad-bc) & -b/(ad-bc) \\ 0 & 1 & -c/(ad-bc) & a/(ad-bc) \end{array}\right].$$

Thus $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. This formula was obtained under the assumption that $a \neq 0$, but it is straightforward to see that it works as well for $a = 0$. Set $\Delta_2(A) = ad - bc$ and for any 1×1 matrix $B = [b]$ define $\Delta_1(B) = b$. Hence we proved that a 2×2 matrix A is invertible iff $\Delta_2(A) \neq 0$ and then

$$A^{-1} = \frac{1}{\Delta_2(A)} \begin{bmatrix} \Delta_1([d]) & -\Delta_1([b]) \\ -\Delta_1([c]) & \Delta_1([a]) \end{bmatrix}$$

The reason why we write the inverse of A in such a strange form using Δ_1 will become clear soon.

Now we try to perform similar computations for a 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

To start, we assume again that $a \neq 0$ and consider the matrix

$$\left[\begin{array}{ccc:ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right]$$

We perform on this matrix elementary operations $E_{2,1}(-d/a)$, $E_{3,1}(-g/a)$, $D_2(a)$, $D_3(a)$ and get

$$\left[\begin{array}{ccc:ccc} a & b & c & 1 & 0 & 0 \\ 0 & ea-bd & fa-dc & -d & a & 0 \\ 0 & ha-gb & ia-gc & -g & 0 & a \end{array}\right]$$

We see that A is invertible iff the matrix $\begin{bmatrix} ea-bd & fa-dc \\ ha-gb & ia-gc \end{bmatrix}$ is invertible (i.e. has rank 2) and we already know that this is equivalent to $(ea-bd)(ia-gc)-(fa-dc)(ha-gb) \neq 0$, i.e. to $a(aei+bfg+cdh-ceg-afh-bdi) \neq 0$. Since we assumed that $a \neq 0$, A is invertible iff $aei+bfg+cdh-ceg-afh-bdi \neq 0$. Set $\Delta = aei+bfg+cdh-ceg-afh-bdi$ and assume that $\Delta \neq 0$. Now we could perform further elementary row operations, but we will be more clever here and note that the product $D_3(a)D_2(a)E_{3,1}(-g/a)E_{2,1}(-d/a)$ of elementary matrices representing row operations performed so far equals

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g/a & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -d/a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -d & a & 0 \\ -g & 0 & a \end{bmatrix}$$

This means that

$$\begin{bmatrix} 1 & 0 & 0 \\ -d & a & 0 \\ -g & 0 & a \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 0 & ea-bd & fa-dc \\ 0 & ha-gb & ia-gc \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \begin{bmatrix} a & b & c \\ 0 & ea-bd & fa-dc \\ 0 & ha-gb & ia-gc \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ -d & a & 0 \\ -g & 0 & a \end{bmatrix}.$$

It is quite easy to see that the inverse of a matrix of the form

$$M = \begin{bmatrix} a & b & c \\ 0 & x & y \\ 0 & z & t \end{bmatrix}$$

must be of this form too, i.e.

$$M^{-1} = \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & u & v \\ 0 & w & s \end{bmatrix}.$$

The equality $MM^{-1} = I$ tells us now that $\begin{bmatrix} u & v \\ w & s \end{bmatrix}$ is the inverse to $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ and $\alpha = 1/a$, $\beta = -(bu+cw)/a$, $\gamma = -(bv+cs)/a$. Since we already know how to invert a 2×2 matrix, we apply it to invert $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ to find that

$$\begin{bmatrix} a & b & c \\ 0 & ea-bd & fa-dc \\ 0 & ha-gb & ia-gc \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & (ch-bi)/a\Delta & (bf-ce)/a\Delta \\ 0 & (ia-gc)/a\Delta & (-fa+dc)/a\Delta \\ 0 & (-ha+gb)/a\Delta & (ae-bd)/a\Delta \end{bmatrix}.$$

Thus

$$A^{-1} = \begin{bmatrix} 1/a & (ch-bi)/a\Delta & (bf-ce)/a\Delta \\ 0 & (ia-gc)/a\Delta & (-fa+dc)/a\Delta \\ 0 & (-ha+gb)/a\Delta & (ae-bd)/a\Delta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -d & a & 0 \\ -g & 0 & a \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} (ei-fh) & -(bi-ch) & (bf-ce) \\ -(di-fg) & (ai-cg) & -(af-cd) \\ (dh-eg) & -(ah-bg) & (ae-bd) \end{bmatrix}$$

Letting $\Delta_3(A) = \Delta = aei+bfg+cdh-ceg-afh-bdi$, the last formula can be written as follows:

$$A^{-1} = \frac{1}{\Delta_3(A)} \begin{bmatrix} \Delta_2\left(\left[\begin{smallmatrix} e & f \\ h & i \end{smallmatrix}\right]\right) & -\Delta_2\left(\left[\begin{smallmatrix} b & c \\ h & i \end{smallmatrix}\right]\right) & \Delta_2\left(\left[\begin{smallmatrix} b & c \\ e & f \end{smallmatrix}\right]\right) \\ -\Delta_2\left(\left[\begin{smallmatrix} d & f \\ g & i \end{smallmatrix}\right]\right) & \Delta_2\left(\left[\begin{smallmatrix} a & c \\ g & i \end{smallmatrix}\right]\right) & -\Delta_2\left(\left[\begin{smallmatrix} a & c \\ d & f \end{smallmatrix}\right]\right) \\ \Delta_2\left(\left[\begin{smallmatrix} d & e \\ g & h \end{smallmatrix}\right]\right) & -\Delta_2\left(\left[\begin{smallmatrix} a & b \\ g & h \end{smallmatrix}\right]\right) & \Delta_2\left(\left[\begin{smallmatrix} a & b \\ d & e \end{smallmatrix}\right]\right) \end{bmatrix}.$$

We proved this formula under the assumption that $a \neq 0$ but it is not hard to see that it is still true when $a = 0$. Thus a 3×3 matrix A is invertible if and only if $\Delta_3(A) \neq 0$ and then the inverse of A is given by the above formula.

It is now time to analyze our results and try to find a pattern in our formulas for A^{-1}. What is the (i, j) entry of the matrix on the right in our formulas? It is not that hard to come up with the following answer: the (i, j) entry equals ± 1 times the appropriate Δ applied to a matrix obtained from A by removing its $j-$th row and $i-$th column. Also, the sign ± 1 is not that hard to understand: it is equal to $(-1)^{i+j}$. This suggest the following definition:

Definition. For any $n \times n$ matrix A define $A_{i,j}$ to be the matrix obtained from A by removing its $i-$th row and $j-$th column.

The results obtained so far can be now formulated as follows:

- A 2×2 matrix A is invertible if and only of $\Delta_2(A) \neq 0$ and then

$$A^{-1} = \frac{1}{\Delta_2(A)} \begin{bmatrix} \Delta_1(A_{1,1}) & -\Delta_1(A_{2,1}) \\ -\Delta_1(A_{1,2}) & \Delta_1(A_{2,2}) \end{bmatrix}$$

- A 3×3 matrix A is invertible if and only of $\Delta_3(A) \neq 0$ and then

$$A^{-1} = \frac{1}{\Delta_3(A)} \begin{bmatrix} \Delta_2(A_{1,1}) & -\Delta_2(A_{2,1}) & \Delta_2(A_{3,1}) \\ -\Delta_2(A_{1,2}) & \Delta_2(A_{2,2}) & -\Delta_2(A_{3,2} \\ \Delta_2(A_{1,3}) & -\Delta_2(A_{2,3}) & \Delta_2(A_{3,3}) \end{bmatrix}.$$

This leads us to the following prediction: for each n there is a function Δ_n which to each $n \times n$ matrix A assigns a scalar $\Delta_n(A)$ such that A is invertible iff $\Delta_n(A) \neq 0$. Moreover, if $\Delta_n(A) \neq 0$ then the inverse of A should be equal to $\dfrac{1}{\Delta_n(A)} A^D$, where $A^D = (d_{i,j})$ is an $n \times n$ matrix whose i, j-entry $d_{i,j}$ is given by the formula

$$d_{i,j} = (-1)^{i+j} \Delta_{n-1}(A_{j,i}).$$

We have already seen that this is true for $n = 2, 3$. Since we know the formula for Δ_3, we can now compute A^D for a 4×4 matrix A. So for a 4×4 matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{4,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}$$

we compute A^D using our formula for Δ_3 and then multiply A and A^D. We do not provide the actual computations here, but we encourage the reader to perform such computations (it may take some time and one has to be extremely careful not to make any mistakes). As expected, we see that $AA^D = \Delta_4(A)I$, where

$$\Delta_4(A) = a_{1,1}a_{2,2}a_{3,3}a_{4,4} + a_{1,2}a_{2,1}a_{3,4}a_{4,3} + a_{1,3}a_{2,4}a_{3,1}a_{4,2} + a_{1,4}a_{2,3}a_{3,2}a_{4,1} +$$

$$a_{1,2}a_{2,3}a_{3,1}a_{4,4} + a_{1,2}a_{2,4}a_{3,3}a_{4,1} + a_{1,3}a_{2,2}a_{3,4}a_{4,1} + a_{1,1}a_{2,3}a_{3,4}a_{4,2} +$$

$$a_{1,3}a_{2,1}a_{3,2}a_{4,4} + a_{1,4}a_{2,1}a_{3,3}a_{4,2} + a_{1,4}a_{2,2}a_{3,1}a_{4,3} + a_{1,1}a_{2,4}a_{3,2}a_{4,3} -$$

$$a_{1,2}a_{2,1}a_{3,3}a_{4,4} - a_{1,3}a_{2,2}a_{3,1}a_{4,4} - a_{1,4}a_{2,2}a_{3,3}a_{4,1} - a_{1,1}a_{2,3}a_{3,2}a_{4,4} -$$
$$a_{1,1}a_{2,4}a_{3,3}a_{4,2} - a_{1,1}a_{2,2}a_{3,4}a_{4,3} - a_{1,2}a_{2,3}a_{3,4}a_{4,1} - a_{1,2}a_{2,4}a_{3,1}a_{4,3} -$$
$$a_{1,3}a_{2,4}a_{3,2}a_{4,1} - a_{1,3}a_{2,1}a_{3,2}a_{4,2} - a_{1,4}a_{2,3}a_{3,1}a_{4,2} - a_{1,4}a_{2,1}a_{3,2}a_{4,3}.$$

These computations imply that if $\Delta_4(A) \neq 0$ then A is invertible and our predicted formula for A^{-1} holds. It is not clear at this point that if $\Delta_4(A) = 0$ then A is not invertible (theoretically, it could happen that A^D is the zero matrix), but this can be done with some extra work. Having Δ_4 we can in a similar way compute Δ_5 and so on and verify that we get formulas for the inverse of A as predicted. This gives a strong evidence that our prediction is true and even tells us an inductive method to construct Δ_n. In fact, let A be an $n \times n$ matrix. From our prediction that $A^{-1} = \dfrac{1}{\Delta_n(A)} A^D$ we see that $AA^D = \Delta_n(A)I$. The $(1,1)$ entry of AA^D is given by $a_{1,1}d_{1,1} + a_{1,2}d_{2,1} + ... + a_{1,n}d_{n,1} = a_{1,1}\Delta_{n-1}(A_{1,1}) - a_{1,2}\Delta_{n-1}(A_{1,2}) + ... + (-1)^{n+1}a_{1,n}\Delta_{n-1}(A_{1,n})$. Since the $(1,1)$ entry of $\Delta_n(A)I$ is $\Delta_n(A)$, we see that there is no choice for $\Delta_n(A)$; if it exists, it must be given by the formula

$$\Delta_n(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1,j} \Delta_{n-1}(A_{1,j})$$

Starting with $\Delta_1([a]) = a$, we get a recursive definition of the Δ_n's. What remains is to find a precise proof that the Δ_n's defined in this way in fact satisfy our prediction, i.e. that $AA^D = \Delta_n(A)I = A^D A$ holds for all $n \times n$ matrices A and that A is invertible iff $\Delta_n(A) \neq 0$. Note that the equality $AA^D = \Delta_n(A)I$ is equivalent to the following identities:

$$\Delta_n(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \Delta_{n-1}(A_{i,j}), \ i = 1,2,...,n, \quad (1)$$

and

$$0 = \sum_{j=1}^{n} (-1)^{k+j} a_{i,j} \Delta_{n-1}(A_{k,j}) \text{ for } i \neq k. \quad (2)$$

Similarly, the equality $A^D A = \Delta_n(A)I$ is equivalent to the following identities:

$$\Delta_n(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \Delta_{n-1}(A_{i,j}), \ j = 1,2,...,n, \quad (3)$$

and

$$0 = \sum_{i=1}^{n} (-1)^{i+k} a_{i,j} \Delta_{n-1}(A_{i,k}) \text{ for } j \neq k. \quad (4)$$

Definition. The scalar $\Delta_n(A)$ is called the **determinant** of A and it is usually denoted by $\det A$, or sometimes by $|A|$.

The formulas (1) and (3) are called the **Laplace expansions** of the determinant by the i-th row and by the j-th column respectively. They can be used to compute determinants in a recursive way.

Example. In order to compute $\Delta_4(A)$ for

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 1 & 3 \end{bmatrix}$$

we can use Laplace expansion by the second column to get

$$\Delta_4(A) = -2\Delta_3(A_{1,2}) + \Delta_3(A_{2,2}) - 0 \cdot \Delta_3(A_{3,2}) + 0 \cdot \Delta_3(A_{4,2}) =$$
$$= -2\Delta_3\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \end{bmatrix}\right) + \Delta_3\left(\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \end{bmatrix}\right)$$

Now Laplace expansion by the first row yields

$$\Delta_3\left(\begin{bmatrix}1 & 0 & 1\\ 1 & 3 & 0\\ 1 & 1 & 3\end{bmatrix}\right) = \Delta_2\left(\begin{bmatrix}3 & 0\\ 1 & 3\end{bmatrix}\right) - 0\cdot\Delta_2\left(\begin{bmatrix}1 & 0\\ 1 & 3\end{bmatrix}\right) + \Delta_2\left(\begin{bmatrix}1 & 3\\ 1 & 1\end{bmatrix}\right) = 9 + (1-3) = 7.$$

Similarly, we use Laplace expansion by the third column to get

$$\Delta_3\left(\begin{bmatrix}1 & 1 & 2\\ 1 & 3 & 0\\ 1 & 1 & 3\end{bmatrix}\right) = 2\cdot\Delta_2\left(\begin{bmatrix}1 & 3\\ 1 & 1\end{bmatrix}\right) - 0\cdot\Delta_2\left(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right) + 3\cdot\Delta_2\left(\begin{bmatrix}1 & 1\\ 1 & 3\end{bmatrix}\right) =$$

$$= 2(1-3) + 3(3-1) = 2.$$

It follows that $\Delta_4(A) = -2\cdot 7 + 2 = -12$.

Exercise. Compute A^D and A^{-1} for the above A.

Problem: Can you predict an explicit formula for $\Delta_n(A)$ (above we have seen such formulas for $n \leq 4$).