Consider a system of m linear equations with n unknowns:

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + \cdots + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + \cdots + & a_{2,n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + \cdots + & a_{m,n}x_n & = & b_m \end{array}$$

Here $x_1, \ldots, x_n$ are the unknowns, $a_{i,j}$ and b_i are numbers. We associate to such a system two matrices, the **coefficient matrix** A and the **augmented matrix** B:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{bmatrix} \qquad B = \left[\begin{array}{cccc:c} a_{1,1} & a_{1,2} & \ldots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} & b_m \end{array}\right]$$

(the vertical lione separating the last column of B is just for our convenience).

Our goal is to solve systems of linear equations. We call system **consistent** if it has at least one solution, otherwise - when there are no solutions - we call it **inconsistent**.

When we attempt to solve such a system of equations, the following three types of manipulations are useful: add a multiple of one equation to another equation; switch two of the equations; multiply an equation by a non-zero constant. Each such manipulation produces a new system of equations, however both systems have the same sets of solutions (clearly any solution to the original system is also a solution to the new system; since the manipulations are reversible, the converse is also true). It is straightforward to see that the coefficient and augmented matrices of the new system are obtained from the corresponding matrices of the original system by performing an elementary row operation (this observation is one of the motivations for considering matrices and elementary row operations). We say that two systems of linear equations (in the same unknowns) are **equivalent** if they have the same sets of solutions. Our discussion so far justifies the following result:

Proposition. If two systems of linear equations (with the same number of unknowns) have row equivalent augmented matrices then the systems are equivalent.

Remark. The converse is also true for consistent systems, i.e. equivalent consistent systems have row equivalent augmented matrices (this makes sense only if both systems have the same number of equations; we can always assume this by adding several "obvious" equations: $0 \cdot x_1 + 0 \cdot x_2 + \ldots + 0 \cdot x_n = 0$).

Consider now a system of m linear equations with n unknowns as above. Let B be the augmented matrix of this system. We know that B is row equivalent to a matrix D in a reduced row-echelon form. According to our proposition above, solving our original system is equivalent to solving the system with augmented matrix D. It turns out that systems with augmented matrix in a reduced row-echelon form are easy to solve.

Suppose first that the last column of D is a pivot column. This means that the last non-zero row of D is $0, 0, \ldots, 0, 1$. The equation corresponding to this row is $0 \cdot x_1 + 0 \cdot x_2 + \ldots + 0 \cdot x_n = 1$ and this equation clearly has no solutions. It follows that the system is inconsistent (i.e. has no solutions).

Suppose now that the last column of D is not a pivot column but all other columns are pivot. Then D has the following form

$$D = \left[\begin{array}{cccccc:c} 1 & 0 & & 0 & \ldots & 0 & u_1 \\ 0 & 1 & & 0 & \ldots & 0 & u_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & & 0 & 1 & u_n \\ 0 & 0 & & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & 0 & 0 & 0 & 0 \end{array}\right]$$

(there may be no zero rows at all). The corresponding system of equations has evidently unique solution $x_1 = u_1, \ldots, x_n = u_n$

Finally, suppose that the last column of D is not a pivot column and there are other non-pivot columns. It is convenient to introduce the following terminology: unknowns corresponding to non-pivot columns are called **free variables**, and the remaining unknowns are called **dependent** variables. It is not hard to see that each non-zero row of D yields an equation which expresses one of the dependent variables in terms of the free variables. It follows that we can specify arbitrary values to the free variables and then uniquely compute the dependent variables yielding a solution to our system. Thus, the system has infinitely many solutions parametrized by t parameters, where t is the number of non pivot columns of D, **not counting the last column** (i.e. t is the number of non-pivot columns of the coefficient matrix, i.e. $t = n - \text{rank}(D)$). Let us illustrate this by the following example:

Suppose that D is the following matrix:

$$D = \left[\begin{array}{ccccccc:c} 1 & 2 & 0 & -1 & 4 & 0 & -2 & 1 \\ 0 & 0 & 1 & 3 & 5 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The first, third, and sixth columns are pivot columns (and the last column is not a pivot column), so x_1, x_3, x_6 are the dependent variables and x_2, x_4, x_5, x_7 are the free variables. The first equation is $x_1 + 2x_2 - x_4 + 4x_5 - 2x_7 = 1$, i.e. $x_1 = -2x_2 + x_4 - 4x_5 + 2x_7 + 1$. Similarly, the second equations is $x_3 + 3x_4 + 5x_5 + 3x_7 = 2$, i.e. $x_3 = -3x_4 - 5x_5 - 3x_7 + 2$. Finally, the third equation is $x_6 + 2x_7 = 3$, i.e. $x_6 = -2x_7 + 3$. The solutions to the system are given in terms of 4 parameters (free variables) x_2, x_4, x_5, x_7 by the formulas

$$\begin{aligned} x_1 &= -2x_2 + x_4 - 4x_5 + 2x_7 + 1 \\ x_3 &= - 3x_4 - 5x_5 - 3x_7 + 2 \\ x_6 &= - 2x_7 + 3 \end{aligned}$$

We may choose any values for the free variables and compute the dependent variables using the above formulas to get a solution. For example, choosing $x_2 = 1$, $x_4 = -1$, $x_5 = 0$, $x_7 = 2$ gives the solution $x_1 = -10$, $x_2 = 1$, $x_3 = -1$, $x_4 = -1$, $x_5 = 0$, $x_6 = -1$, $x_7 = 2$. From this point of view, perhaps the simplest solution is obtained by choosing all the free variables to be 0.

Let us summarize our discussion in the following theorem

Theorem. Let B be the augmented matrix of a system of linear equations with n unknowns. Then

- the system is inconsistent (has no solutions) if and only if the last column of B is a pivot column.
- the system has unique solution if and only if all columns of B except the last one are pivot columns.
- the system has infinitely many solutions if and only if the last column and at least one other column are not pivot columns of B. In this case the solutions are given in terms of t parameters (free variables), where $t = n - \text{rank}(B)$.

The discussion below is optional. It outlines a reason why the matrix in reduced row-echelon form row equivalent to a given matrix is unique.

Let A be a matrix in a reduced row-echelon form. Consider the system of linear equations with coefficient matrix A and augmented matrix B having the last column with only 0's (in other words, in the notation at the beginning of this note, we set $b_1 = \ldots = b_m = 0$). It turns out that we can recover A from the set of all solutions to this system. Indeed, let $x_1, \ldots, x_n$ be the unknowns. For each k let us asked the following question: is there a solution to the system such that $x_k = 1$ and $x_{k+1} = \ldots = x_n = 0$? If x_k is a dependent variable (i.e. the k-th column of A is a pivot column) then x_k can be expressed in terms of the variables $x_{k+1}, \ldots, x_n$ which are all 0, hence x_k must be zero as well. Thus the answer is "no". If x_k is a free variable, then we can set $x_k = 1$ and all the other free variables to be 0 and there will be a solutions corresponding to this choice of parameters which will satisfy our requirement, so the answer is "yes". We see that which columns of A are pivot and which are not is uniquely determined by the properties of the solution set to our system. Since the first pivot column of A has 1 in the first row and 0 everywhere else, this column is determined by the solutions. Similarly, the second pivot column has 1 in the second row and 0 everywhere else, so it is determined as well, and similarly for all other pivot columns. What about the non-pivot columns? Suppose that the k-th column of A is not a pivot column, so x_k is a free variable. Consider the unique solution with $x_k = 1$ and all other free variables equal to 0. It is easy to see that the value of the first dependent variable in this solution is equal to the negative of the first entry in the k-th column. Similarly, the value of the second dependent variable in this solution is equal to the negative of the second entry in the k-th column, etc. Thus the k-th column is determined by the set of solutions to our system. This completes our claim that A is determined by the set of solutions to the system. Suppose now that A_1 is another matrix in a reduced row-echelon form row equivalent to A. Thus the corresponding system of equations will have the same solutions as the one for A. Therefore the columns of A_1 will be the same as the columns of A, as they are determined by the set of solutions and therefore $A = A_1$.

Let us illustrate the above reasoning by a concrete example. Suppose that

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 & 4 & 0 & -2 \\ 0 & 0 & 1 & 3 & 5 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding system of equations has augmented matrix

$$B = \left[\begin{array}{ccccccc:c} 1 & 2 & 0 & -1 & 4 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 5 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The solutions to this system are given by the formulas

$$\begin{aligned} x_1 &= -2x_2 + x_4 - 4x_5 + 2x_7 \\ x_3 &= \quad\quad - 3x_4 - 5x_5 - 3x_7 \\ x_6 &= \quad\quad\quad\quad\quad\quad - 2x_7 \end{aligned}$$

We ask the questions:

is there a solution with $x_1 = 1$ and $x_2 = \ldots = x_7 = 0$? Clearly no.

is there a solution with $x_2 = 1$ and $x_3 = \ldots = x_7 = 0$? Yes, $x_1 = -2$.

is there a solution with $x_3 = 1$ and $x_4 = \ldots = x_7 = 0$? Clearly no.

is there a solution with $x_4 = 1$ and $x_5 = \ldots = x_7 = 0$? Yes, for example $x_3 = -3$, $x_2 = 0$, $x_1 = 1$.

is there a solution with $x_5 = 1$ and $x_6 = x_7 = 0$? Yes, for example $x_4 = 0$, $x_3 = -5$, $x_2 = 0$, $x_1 = -4$.

is there a solution with $x_6 = 1$ and $x_7 = 0$? Clearly no.

is there a solution with $x_7 = 1$? Yes, for example $x_6 = -2$, $x_5 = 0$, $x_4 = 0$, $x_3 = -3$, $x_2 = 0$, $x_1 = 2$.

We see that the k-th question has positive answer if and only if x_k is a free variable. Thus we can determine which columns of A are pivot and which are not form the properties of the set of solutions. In other words, any matrix in a reduced row echelon-form for which the corresponding system has the same solutions will have the first, the third, and the sixth columns as pivot columns and the remaining columns as non-pivot columns. Now the system has unique solution with $x_2 = 1$, $x_4 = x_5 = x_7 = 0$. The dependent variables in this solution are $x_1 = -2$, $x_3 = 0$, $x_6 = 0$, which are the negatives of the entries in the second column. Similarly, the system has unique solution with $x_4 = 1$, $x_2 = x_5 = x_7 = 0$. The dependent variables in this solution are $x_1 = 1$, $x_3 = -3$, $x_6 = 0$, which are the negatives of the entries in the 4-th column. The system has unique solution with $x_5 = 1$, $x_2 = x_4 = x_7 = 0$. The dependent variables in this solution are $x_1 = -4$, $x_3 = -5$, $x_6 = 0$, which are the negatives of the entries in the 5-th column. Finally, the system has unique solution with $x_7 = 1$, $x_2 = x_4 = x_5 = 0$. The dependent variables in this solution are $x_1 = 2$, $x_3 = -3$, $x_6 = -2$, which are the negatives of the entries in the 4-th column. This illustrates how we recover A from the properties of the set of solutions to the associated system of equations.