In the last note we came up with a conjectural formulas for inverting square matrices. Let us recall our conjecture.

Definition. For any $n \times n$ matrix A define $A_{i,j}$ to be the matrix obtained from A by removing its $i-$th row and $j-$th column.

We predict that for every n there is a function Δ_n which to any $n \times n$ matrix A assigns a scalar $\Delta_n(A)$, called the **determinant** of A, which has the following properties:

- $\Delta_1([a]) = a$.
- $n \times n$ matrix A is invertible if and only if $\Delta_n(A) \neq 0$.
- For any $n \times n$ matrix A define the matrix A^D to be the $n \times n$ matrix whose i,j-entry is equal to $(-1)^{i+j}\Delta_{n-1}(A_{j,i})$, for $1 \leq i,j \leq n$. Then $AA^D = \Delta_n(A)I = A^DA$. In particular, if $\Delta_n(A) \neq 0$ then
$$A^{-1} = \frac{1}{\Delta_n(A)}A^D.$$

Comparing the $1,1$-entries on both sides of the equality $AA^D = \Delta_n(A)I$ we see that if Δ_n with the above property exist for all n then they must be given by the following recursive formula:

$$\Delta_1([a]) = a,$$

$$\Delta_n(A) = a_{1,1}\Delta_{n-1}(A_{1,1}) - a_{1,2}\Delta_{n-1}(A_{1,2}) + \ldots + (-1)^{n+1}a_{1,n}\Delta_{n-1}(A_{1,n}) = \sum_{j=1}^{n}(-1)^{1+j}a_{1,j}\Delta_{n-1}(A_{1,j})$$

where $A = [a_{i,j}]$. For example, for $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ we have

$$\Delta_2(A) = a_{1,1}\Delta_1([a_{2,2}]) - a_{1,2}\Delta_1([a_{2,1}]) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.$$

Now, for a 3×3 matrix $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$ we have

$$\Delta_3(A) = a_{1,1}\Delta_2\left(\begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}\right) - a_{1,2}\Delta_2\left(\begin{bmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{bmatrix}\right) + a_{1,3}\Delta_2\left(\begin{bmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix}\right) =$$
$$= a_{1,1}(a_{2,2}a_{3,3} - a_{2,3}a_{3,2}) - a_{1,2}(a_{2,1}a_{3,3} - a_{2,3}a_{3,1}) + a_{1,3}(a_{2,1}a_{3,2} - a_{2,2}a_{3,1}) =$$
$$= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1}.$$

And so on. What is left to show is that this unique possible Δ_n's given by the recursive formula indeed have the expected properties. In the previous note we have proved this for $n = 2, 3$. Proving this for all n requires a substantial effort. It turns out that the determinant has many other interesting properties.

After all this discussion let us one more time state our recursive definition of Δ_n

Definition. To any $n \times n$ matrix $A = [a_{i,j}]$ we assign a scalar $\Delta_n(A)$, called the **determinant** of A, as follows:

$$\Delta_1([a]) = a,$$

$$\Delta_n(A) = a_{1,1}\Delta_{n-1}(A_{1,1}) - a_{1,2}\Delta_{n-1}(A_{1,2}) + \ldots + (-1)^{n+1}a_{1,n}\Delta_{n-1}(A_{1,n}) = \sum_{j=1}^{n}(-1)^{1+j}a_{1,j}\Delta_{n-1}(A_{1,j}).$$

Theorem. The determinant has the following properties.

1. $\Delta_n(I) = 1$;
2. $\Delta_n(A) = 0$ if two consecutive rows of A are equal.
3. We write $\Delta_n(r_1, ..., r_n)$ for $\Delta_n(A)$ when the $i-$th row of A is r_i for $i = 1, 2, ..., n$. Then, we have

   $$\Delta_n(r_1, ..., ar_i' + br_i'', ..., r_n) = a\Delta_n(r_1, ..., r_i', ..., r_n) + b\Delta_n(r_1, ..., r_i'', ..., r_n)$$

   for any i, $1 \leq i \leq n$.
4. $\Delta_n(B) = -\Delta_n(A)$ if B is obtained from A by switching two consecutive rows.
5. $\Delta_n(B) = -\Delta_n(A)$ if B is obtained from A by switching any two rows. In other words, $\Delta_n(S_{i,j}A) = -\Delta_n(A)$.
6. $\Delta_n(A) = 0$ if two rows of A are equal.
7. $\Delta_n(E_{i,j}(a)A) = \Delta_n(A)$
8. $\Delta_n(D_i(a)A) = a\Delta_n(A)$
9. $\Delta_n(A) = 0$ if A has a zero row.
10. $\Delta_n(E_{i,j}(a)) = 1$, $\Delta_n(D_i(a)) = a$ and $\Delta_n(S_{i,j}) = -1$.
11. $\Delta_n(EB) = \Delta_n(E)\Delta_n(B)$ for any elementary matrix E and any matrix B.
12. If A is invertible then $\Delta_n(A) \neq 0$ and $\Delta_n(AB) = \Delta_n(A)\Delta_n(B)$ for any B;
13. A is singular if and only if $\Delta_n(A) = 0$. In other words, A is invertible if and only if $\Delta_n(A) \neq 0$.
14. $\Delta_n(AB) = \Delta_n(A)\Delta_n(B)$ for any A, B;
15. $\Delta_n(A^t) = \Delta_n(A)$ (here A^t is the transpose of A)
16. Laplace expansion by the i-th row ($i = 1, 2, ..., n$):

    $$\Delta_n(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \Delta_{n-1}(A_{i,j}).$$

17. If $1 \leq i, k \leq n$ and $k \neq i$ then

    $$0 = \sum_{j=1}^{n} (-1)^{i+j} a_{k,j} \Delta_{n-1}(A_{i,j}).$$

18. $$AA^D = \Delta_n(A)I = A^D A.$$

19. Laplace expansion by the j-th column ($j = 1, 2, ..., n$) :

    $$\Delta_n(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \Delta_{n-1}(A_{i,j}), \ j = 1, 2, ..., n,$$

20. If $1 \leq j, k \leq n$ and $k \neq j$ then

    $$0 = \sum_{i=1}^{n} (-1)^{i+k} a_{i,j} \Delta_{n-1}(A_{i,k}).$$

Proof: The proof will consist of several steps. First, we will establish properties 1,2,3. Then we will derive properties 4-15 from properties 1-3. Finally, we will establish properties 16-20.

Property 1. Note that if I is $n \times n$ identity matrix then $I_{1,1}$ is the $(n-1) \times (n-1)$ identity matrix. Thus our definition of Δ_n yields $\Delta_n(I) = \Delta_{n-1}(I)$. Repeating this reasoning we see that

$$\Delta_n(I) = \Delta_{n-1}(I) = \ldots = \Delta_1(I) = \Delta_1([1]) = 1.$$

(Note that above we use I to denote the identity matrix of any size; the appropriate size follows from the context.)

Property 2. This is the hardest part of the proof. We will first obtain a formula for the determinant, which is of interests in its own right. In what follows, we assume that n is at least 3. Recall the defining formula

$$\Delta_n(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1,j} \Delta_{n-1}(A_{1,j})$$

Now we apply the defining formula to each $\Delta_{n-1}(A_{1,j})$. For $1 \le s < t \le n$, let $B(s,t)$ be the matrix obtained from A by removing its first two rows and both s-th and t-th columns. It is easy to see that $(A_{1,j})_{1,k} = B(k,j)$ if $k < j$ and $(A_{1,j})_{1,k} = B(j,k+1)$ if $k \ge j$. Also, the $1,k$-entry of $A_{1,j}$ is $a_{2,k}$ if $k < j$ and it is $a_{2,k+1}$ if $k \ge j$. Thus

$$\Delta_{n-1}(A_{1,j}) = \sum_{1 \le k < j} a_{2,k} \Delta_{n-2}(B(k,j))(-1)^{1+k} + \sum_{j \le k < n} a_{2,k+1} \Delta_{n-2}(B(j,k+1))(-1)^{1+k}.$$

Inserting these formulas into our formula for Δ_n, we get then the following formula:

$$\Delta_n(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1,j} \sum_{1 \le k < j} a_{2,k} \Delta_{n-2}(B(k,j))(-1)^{1+k} + \sum_{j=1}^{n} (-1)^{1+j} a_{1,j} \sum_{j \le k < n} a_{2,k+1} \Delta_{n-2}(B(j,k+1))(-1)^{1+k}$$

$$= \sum_{1 \le k < j \le n} (-1)^{2+j+k} a_{1,j} a_{2,k} \Delta_{n-2}(B(k,j)) + \sum_{1 \le j \le k < n} (-1)^{2+j+k} a_{1,j} a_{2,k+1} \Delta_{n-2}(B(j,k+1)).$$

The last sum can be written in terms of $s = k+1$ as follows:

$$\sum_{1 \le j < s \le n} (-1)^{1+j+s} a_{1,j} a_{2,s} \Delta_{n-2}(B(j,s))$$

and changing the name of j into k and s into j it is

$$\sum_{1 \le k < j \le n} (-1)^{1+k+j} a_{1,k} a_{2,j} \Delta_{n-2}(B(k,j)).$$

Thus we get the promised formula for Δ_n:

$$\Delta_n(A) = \sum_{1 \le k < j \le n} (-1)^{2+j+k} a_{1,j} a_{2,k} \Delta_{n-2}(B(k,j)) + \sum_{1 \le k < j \le n} (-1)^{1+k+j} a_{1,k} a_{2,j} \Delta_{n-2}(B(k,j)) =$$

$$= \sum_{1 \le k < j \le n} (-1)^{1+j+k} (a_{1,k} a_{2,j} - a_{1,j} a_{2,k}) \Delta_{n-2}(B(k,j)).$$

Suppose now that the first two rows of A coincide: $a_{1,j} = a_{2,j}$ for $j = 1, \ldots, n$. Then each term $a_{1,j} a_{2,k} - a_{1,k} a_{2,j}$ in the above formula is 0 and consequently $\Delta_n(A) = 0$. Thus we proved that if the first two rows of A coincide then $\Delta_n(A) = 0$ (we did it for $n \ge 3$; for $n = 2$ this is straightforward).

Now we can prove the full Property 2. We already know that it holds for $n = 2$. Suppose we have established the property for all sizes smaller than n. Let A be an $n \times n$ matrix with two consecutive rows equal. If the first 2 rows of A are equal, we already proved that $\Delta_n(A) = 0$. If rows k and $k+1$ of A are equal for some $k > 1$, then each matrix $A_{1,j}$ has two equal rows. Since the size of $A_{1,j}$ is smaller than n, we know that $\Delta_n(A_{1,j}) = 0$ for $j = 1, \ldots, n$. Our defining formula for the determinant implies that $\Delta_n(A) = 0$. This completes our argument.

Property 3. We establish this property using technique similar to the one employed at the end of our proof of Property 2. Namely, it is clear that Property 3 holds for 1×1 matrices. Now suppose that we already established Property 3 for matrices of size less than n. We will show that the property holds for matrices of size n (the method we use is called **mathematical induction**).

Let A be the matrix with rows $r_1, ..., ar'_i + br''_i, ..., r_n$, let B be the matrix with rows $r_1, ..., r'_i, ..., r_n$, and let C be the matrix with rows $r_1, ..., r''_i, ..., r_n$. Thus we need to show that $\Delta_n(A) = a\Delta_n(B) + b\Delta_n(C)$
We consider two cases.

<u>**case 1:** $i = 1$.</u> In this case, we have $A_{1,j} = B_{1,j} = C_{1,j}$ for $j = 1, \dots, n$. Let $r'_1 = (a'_{1,1}, \dots, a'_{1,n})$, $r''_1 = (a''_{1,1}, \dots, a''_{1,n})$, so $r_1 = (aa'_{1,1} + ba''_{1,1}, \dots, aa'_{1,n} + ba''_{1,n})$. Using our definition of the determinant, we have

$$\Delta_n(A) = \sum_{j=1}^{n} (-1)^{1+j} (aa'_{1,1} + ba''_{1,1}) \Delta_{n-1}(A_{1,j}) = a \sum_{j=1}^{n} (-1)^{1+j} a'_{1,1} \Delta_{n-1}(B_{1,j}) + b \sum_{j=1}^{n} (-1)^{1+j} a'_{1,1} \Delta_{n-1}(C_{1,j}) =$$

$$= a\Delta_n(B) + b\Delta_n(C),$$

so Property 3 is true in this case.

<u>**case 2:** $i > 1$.</u> In this case, for $j = 1, \dots, n$, the $i-1$st row of $A_{1,j}$ (which corresponds to i-th row of A) is equal to the sum of a times the $i-1$st row of $B_{1,j}$ and b times the $i-1$st row of $C_{1,j}$. Moreover, for $k \neq i$ the k-th rows of $A_{1,j}$, $B_{1,j}$, $C_{1,j}$ coincide. Since we assume that Property 3 holds for matrices of size less than n, we get

$$\Delta_{n-1}(A_{1,j}) = a\Delta_{n-1}(B_{1,j}) + b\Delta_{n-1}(C_{1,j}) \quad \text{for } j = 1, \dots, n.$$

Since A, B, C have the same first row $(a_{1,1}, \dots, a_{1,n})$, we get

$$\Delta_n(A) = \sum_{j=1}^{n} (-1)^{1+j} \Delta_{n-1}(A_{1,j}) = \sum_{j=1}^{n} (-1)^{1+j} (a\Delta_{n-1}(B_{1,j}) + b\Delta_{n-1}(C_{1,j})) =$$

$$a \sum_{j=1}^{n} (-1)^{1+j} \Delta_{n-1}(B_{1,j}) + b \sum_{j=1}^{n} (-1)^{1+j} \Delta_{n-1}(C_{1,j}) = a\Delta_n(B) + b\Delta_n(C),$$

so Property 3 is true in this case as well.

This completes our first step of establishing properties 1,2,3. Now we will see that these three properties imply properties 1-15.

Property 4. Let the rows of A be $r_1, \dots, r_n$ suppose that B is obteined from A by switching rows k and $k+1$. Thus,

$$\Delta_n(A) = \Delta_n(r_1, \dots, r_k, r_{k+1}, \dots, r_n) \quad \text{and} \quad \Delta_n(B) = \Delta_n(r_1, \dots, r_{k+1}, r_k, \dots, r_n).$$

From property 2 we get that $\Delta_n(r_1, \dots, r_k + r_{k+1}, r_k + r_{k+1}, \dots, r_n) = 0$ (since rows k and $k+1$ coincide). On the other hand, using property 3 we get

$$\Delta_n(r_1, \dots, r_k + r_{k+1}, r_k + r_{k+1}, \dots, r_n) = \Delta_n(r_1, \dots, r_k, r_k + r_{k+1}, \dots, r_n) + \Delta_n(r_1, \dots, r_{k+1}, r_k + r_{k+1}, \dots, r_n) =$$

$$= \Delta_n(r_1, \dots, r_k, r_k, \dots, r_n) + \Delta_n(r_1, \dots, r_k, r_{k+1}, \dots, r_n) + \Delta_n(r_1, \dots, r_{k+1}, r_k, \dots, r_n) + \Delta_n(r_1, \dots, r_{k+1}, r_{k+1}, \dots, r_n).$$

Since $\Delta_n(r_1, \dots, r_k, r_k, \dots, r_n) = 0 = \Delta_n(r_1, \dots, r_{k+1}, r_{k+1}, \dots, r_n)$, we conclude that

$$0 = \Delta_n(r_1, \dots, r_k, r_{k+1}, \dots, r_n) + \Delta_n(r_1, \dots, r_{k+1}, r_k, \dots, r_n) = \Delta_n(A) + \Delta_n(B)$$

i.e. $\Delta_n(B) = -\Delta_n(A)$.

Property 5. Suppose that B is obtained from A by switching rows k and l with $k < l$. This can be achieved by several switches of consecutive rows, namely switch rows k and $k+1$, then $k+1, k+2$,..., then $l-1, l$ then $l-2, l-1$,..., then finally $k+1, k$. We performed a total of $(l-k) + (l-k-1) = 2(l-k) - 1$ switches of consecutive rows. By property 4, each times we switch two consecutive rows, the determinant changes sign. Thus, to get $\Delta_n(B)$ we need to switch the sign of $\Delta_n(A)$ an odd number of times, so $\Delta_n(B) = -\Delta_n(A)$.

Property 6. If rows k and l of A are equal, $k < l$, then let B be obtained from A by switching rows $k+1$ and l. By property 5, $\Delta_n(B) = -\Delta_n(A)$. Since B has two consecutive rows equal, we get $\Delta_n(B) = 0$ by property 2. Thus $\Delta_n(A) = 0$.

Property 7. Let $r_1, \ldots, r_n$ be the rows of A. Then $E_{i,j}(a)A$ has i-th row equal to $r_i + ar_j$ and k-th rows of A and $E_{i,j}(a)A$ coincide for $k \neq i$. By property 3, we have

$$\Delta_n(E_{i,j}(a)A) = \Delta_n(r_1, \ldots, r_{i-1}, r_i + ar_j, \ldots, r_n) = \Delta_n(r_1, \ldots, r_{i-1}, r_i, \ldots, r_n) + a\Delta_n(r_1, \ldots, r_{i-1}, r_j, \ldots, r_n).$$

Note that $\Delta_n(r_1, \ldots, r_{i-1}, r_i, \ldots, r_n) = \Delta_n(A)$ and, by property 6, $\Delta_n(r_1, \ldots, r_{i-1}, r_j, \ldots, r_n) = 0$, since the ith and jth rows are the same. Thus $\Delta_n(E_{i,j}(a)A) = \Delta_n(A)$.

Property 8. This property is a special case of property 3, with $b = 0$ (recall that $D_i(a)A$ is obtained from A by multiplying the i-th row of A by a).

Property 9. Suppose that the i-th row of A is zero. Then $D_i(2)A = A$. Thus, by property 8,

$$2\Delta_n(A) = \Delta_n(D_i(2)A) = \Delta_n(A)$$

which clearly implies that $\Delta_n(A) = 0$.

Property 10. This follows immediately from property 1 and properties 7,8,5 respectively applied to the matrix $A = I$.

Property 11. This follows immediately from property 10 and property 7 if $E = E_{i,j}(a)$, property 8 if $A = D_i(a)$, and property 5 if $E = S_{i,j}$.

Property 12. If A is invertible then $A = E_1 \ldots E_k$ for some elementary matrices $E_1, \ldots, E_k$. By property 11, for any matrix B we have

$$\Delta_n(AB) = \Delta_n(E_1 \ldots E_k B) = \Delta_n(E_1)\Delta_n(E_2 \ldots E_k B) = \ldots = \Delta_n(E_1)\Delta_n(E_2) \ldots \Delta_n(E_k)\Delta_n(B).$$

Using $B = I$ we get $\Delta_n(A) = \Delta_n(E_1)\Delta_n(E_2) \ldots \Delta_n(E_k) \neq 0$ since each $\Delta_n(E_i) \neq 0$ by property 10. Returning to arbitrary B, we get

$$\Delta_n(AB) = \Delta_n(E_1)\Delta_n(E_2) \ldots \Delta_n(E_k)\Delta_n(B) = \Delta_n(A)\Delta_n(B).$$

Property 13. If A is not singular than A is invertible and $\Delta_n(A) \neq 0$ by property 12.

Suppose now that A is singular and let B be the reduced row echelon form of A. Since A is singular, the last row of B is zero. Thus $\Delta_n(B) = 0$ by property 9. There is and invertible matrix M such that $A = MB$ (since A, B are row equivalent). Thus, using property 12,

$$\Delta_n(A) = \Delta_n(MB) = \Delta_n(M)\Delta_n(B) = 0.$$

Property 14. If A is invertible, the result follows from property 12. If A is singular then AB is singular too. By property 13, we have $\Delta_n(A) = 0 = \Delta_n(AB)$, so the results holds in this case as well.

Property 15. Recall that $E_{i,j}(a)^t = E_{j,i}(a)$, $D_i(a)^t = D_i(a)$ and $S_{i,j}^t = S_{i,j}$. It follows that $\Delta_n(E) = \Delta_n(E^t)$ if E is an elementary matrix. If A is singular then so is A^t and we have $\Delta_n(A) = 0 = \Delta_n(A^t)$. If A is invertible then $A = E_1 \ldots E_k$ for some elementary matrices $E_1, \ldots, E_k$. Then $A^t = E_k^t \ldots E_1^t$ and

$$\Delta_n(A) = \Delta_n(E_1)\Delta_n(E_2) \ldots \Delta_n(E_k) = \Delta_n(E_1^t)\Delta_n(E_2^t) \ldots \Delta_n(E_k^t) = \Delta_n(E_k^t) \ldots \Delta_n(E_2^t)\Delta_n(E_1^t) = \Delta_n(A^t).$$

This completes our second step (which shows that properties 4-15 are consequences of properties 1-3, i.e. any function of matrices with properties 1-3 must have properties 4-15). Our last step is to get properties 16-20.

Property 16. Let $r_1, \ldots r_n$ be the rows of A and let B be the matrix with rows $r_i, r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n$. Clearly $A_{i,j} = B_{1,j}$ for $j = 1, \ldots, n$. Note that B can be obtained from A by switching rows i and $i-1$, Then $i-1$ and $i-2$, ..., then 2 and 1. This requires $i-1$ switches. It follows that $\Delta_n(B) = (-1)^{i-1}\Delta_n(A)$. Since the first row of B is $(a_{i,1}, \ldots, a_{i,n})$, we have

$$(-1)^{i-1}\Delta_n(A) = \Delta_n(B) = \sum_{j=1}^{n}(-1)^{1+j}a_{i,j}\Delta_{n-1}(B_{1,j}) = \sum_{j=1}^{n}(-1)^{1+j}a_{i,j}\Delta_{n-1}(A_{i,j}).$$

Multiplying both sides by $(-1)^{i-1}$ we get

$$\Delta_n(A) = (-1)^{i-1}\sum_{j=1}^{n}(-1)^{1+j}a_{i,j}\Delta_{n-1}(A_{i,j}) = \sum_{j=1}^{n}(-1)^{i+j}a_{i,j}\Delta_{n-1}(A_{i,j}).$$

Property 17. Let B be the matrix obtained from A by replacing the i-th row of A by k-th row of A. Thus the i-th and k-th rows of B coincide and therefore $\Delta_n(B) = 0$ by property 6. Clearly $B_{i,j} = A_{i,j}$ for $j = 1, \ldots, n$. Since the i-th row of B is $(a_{k,1}, \ldots, a_{k,n})$, the Laplace expansion by the i-th row of B (property 16) yields

$$0 = \Delta_n(B) = \sum_{j=1}^{n}(-1)^{i+j}a_{k,j}\Delta_{n-1}(B_{i,j}) = \sum_{j=1}^{n}(-1)^{i+j}a_{k,j}\Delta_{n-1}(A_{i,j}).$$

Property 18. Note that the i,i-entry of AA^D is given by the right side of the formula in property 15. Hence the diagonal entries of AA^D are all equal to $\Delta_n(A)$. For $k \neq i$, the k,i-entry of AA^D is given by the right side of the formula in property 17. Thus the off-diagonal entries of AA^D are all zero. This proves that $AA^D = \Delta_n(A)I$.

Consider now the transpose A^t of A. It is clear that $(A^t)_{i,j} = (A_{j,i})^t$. Since $\Delta_n((A_{j,i})^t) = \Delta_n(A_{j,i})$ by property 15, we see that $(A^t)^D = (A^D)^t$. Since we know that $A^t(A^t)^D = \Delta_n(A^t)I$, we conclude that $A^t(A^D)^t = \Delta_n(A)I$. Transposing both sides of the last equality we get

$$\Delta_n(A)I = (\Delta_n(A)I)^t = (A^t(A^D)^t)^t = ((A^D)^t)^t(A^t)^t = A^D A.$$

Property 19. Note that the right hand side of the formula in property 19 is equal to the j,j-entry of $A^D A$, which is $\Delta_n(A)$ by property 18.

Property 20. Note that the right hand side of the formula in property 20 is equal to the k,j-entry of $A^D A$, which is 0 by property 18.

This completes the justification of properties 1-20.

Exercise: Show that $(AB)^D = B^D A^D$ (this is quite easy when A or B is invertible, but more subtle in the remaining case).