In the last note we defined determinant of a square matrix and established a long list of properties of the determinant. We ere denoting the determinant of an $n \times n$ matrix A by $\Delta_n(A)$. From now on we will use the notation $\det A$ for the determinant of A. Another common notation for the determinant of A is $|A|$. This notation is particularly useful when applied to explicit matrices. For example, the determinant of a 4×4 matrix $\begin{bmatrix} 0 & 3 & 2 & 0 \\ -2 & -5 & 0 & 2 \\ -3 & 0 & 5 & 3 \\ 0 & -3 & -2 & 0 \end{bmatrix}$ is denoted by $\begin{vmatrix} 0 & 3 & 2 & 0 \\ -2 & -5 & 0 & 2 \\ -3 & 0 & 5 & 3 \\ 0 & -3 & -2 & 0 \end{vmatrix}$, which is simpler and more convenient than $\det \begin{bmatrix} 0 & 3 & 2 & 0 \\ -2 & -5 & 0 & 2 \\ -3 & 0 & 5 & 3 \\ 0 & -3 & -2 & 0 \end{bmatrix}$. We will use both notations.

Some of the 20 properties of the determinant established in the last note are special cases of the other properties. We stated these properties the way we did to structure our justification of the properties. Now we can summarize the properties in a shorter list:

1. If A, B, C are $n \times n$ matrices such that k-th rows of A, B, C coincide for $k \neq i$ and the i-th row of C is equal to the sum of a times the i-th row of A and b times the i-th row of B then $\det C = a \det A + b \det B$.
2. $\det A = 0$ if and only if A is singular.
3. $\det(AB) = \det A \det B$ for any square matrices A, B of the same size.
4. $\det E_{i,j}(a) = 1$, $\det D_i(a) = a$, $\det S_{i,j} = -1$.
5. $\det A = \det A^t$ for any square matrix A.
6. $AA^D = (\det A)I = A^D A$.
7. Laplace expansion by the i-th row ($i = 1, 2, ..., n$): if $A = [a_{i,j}]$ then
$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det A_{i,j}.$$
8. Laplace expansion by the j-th column ($j = 1, 2, ..., n$): if $A = [a_{i,j}]$ then
$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det A_{i,j}.$$

These properties provide several tools to compute determinants of concrete matrices. For example, properties 3 and 4 tell us that performing a row or column operation of the form $E_{i,j}(a)$ (i.e. adding a multiple of one row (column) to another row (column)) does not change the value of the determinant and switching 2 rows changes sign of the determinant. Laplace expansions allow us to use any row or column to expand the determinant and reduces the computation to computations of determinants of smaller size.

One very useful property of determinants is the following.

> If $A = [a_{i,j}]$ is lower-triangular or upper-triangular then $\det A = a_{1,1}a_{2,2}\dots a_{n,n}$ is the product of diagonal entries of A.

Indeed, suppose that A is lower triangular of size n. Thus $a_{1,2} = \ldots = a_{1,n} = 0$. Laplace expansion by the first row yields then $\det A = a_{1,1} \det A_{1,1}$. Note that $A_{1,1}$ is also lower-triangular with diagonal entries $a_{2,2}, \dots, a_{n,n}$. Now we can repeat the process (or use induction), and then repeat again and so on and eventually arrive at $\det A = a_{1,1}a_{2,2}\dots a_{n,n}$.

When A is upper triangular then A^t is lower lower-triangular with the same diagonal entries as A, hence the result follows from property 5.

Exercise. Let M be a matrix of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, where A is a $k \times k$ matrix, C is a $m \times m$ matrix, B is a $k \times m$ matrix and 0 is a $m \times k$ matrix whose all entries are 0. Prove that $\det M = \det A \det C$.

Example. Computation of determinants of 2×2 matrices is straightforward using the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. For example,

$$\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} = 15 - 16 = -1.$$

Example. Computation of 3×3 determinants is still relatively simple. One method is to use Laplace expansion in some row or column to reduce to 2×2 determinants. We look for a row or column with as many zero entries as possible. For example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 2 & 3 & 1 \end{vmatrix} = (-1)\cdot 2 \cdot \begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix} + (-1)\cdot 3 \cdot \begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix} = (-2)(4-10) - 3(5-12) = 33.$$

We used Laplace expansion in the second column.

Alternatively, we could first do column operations $E_{1,2}(-2)$ (add -2 times the first column to the second column) and $E_{1,3}(-3)$, which do not change the value of the determinant and then use Laplace expansion in the first row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & -8 & -7 \\ 2 & -1 & -5 \end{vmatrix} = \begin{vmatrix} -8 & -7 \\ -1 & -5 \end{vmatrix} = 40 - 7 = 33.$$

Example. For the 4×4 determinant $\begin{vmatrix} 0 & 3 & -2 & 0 \\ -2 & -5 & 0 & 2 \\ -3 & 0 & 5 & 3 \\ 0 & -3 & -2 & 2 \end{vmatrix}$ we could use Laplace expansion in the first row or column, but this would require computing two 3×3 determinants. We could use elementary row operation $E_{3,2}(-3/2)$ to make one more entry in the first column 0, but this would introduce fractions. So we first multiply second row by -3, third row by 2 and then do the row operation $E_{3,2}(1)$. We need to remember that multiplying a row or column of a matrix A by a produces a matrix whose determinant is $a \det A$, so we need to divide by a to compensate:

$$\begin{vmatrix} 0 & 3 & -2 & 0 \\ -2 & -5 & 0 & 2 \\ -3 & 0 & 5 & 3 \\ 0 & -3 & -2 & 2 \end{vmatrix} = \frac{1}{-3}\begin{vmatrix} 0 & 3 & -2 & 0 \\ 6 & 15 & 0 & -6 \\ -3 & 0 & 5 & 3 \\ 0 & -3 & -2 & 2 \end{vmatrix} = \frac{1}{-3}\cdot\frac{1}{2}\begin{vmatrix} 0 & 3 & -2 & 0 \\ 6 & 15 & 0 & -6 \\ -6 & 0 & 10 & 6 \\ 0 & -3 & -2 & 2 \end{vmatrix} = \frac{-1}{6}\begin{vmatrix} 0 & 3 & -2 & 0 \\ 6 & 15 & 0 & -6 \\ 0 & 15 & 10 & 0 \\ 0 & -3 & -2 & 2 \end{vmatrix} =$$

Now we do Laplace expansion in the first column

$$= \frac{-1}{6}\cdot(-1)\cdot 6 \begin{vmatrix} 3 & -2 & 0 \\ 15 & 10 & 0 \\ -3 & -2 & 2 \end{vmatrix} = 2\begin{vmatrix} 3 & -2 \\ 15 & 10 \end{vmatrix} = 120.$$

We are now ready to discuss a method of finding eigenvalues of a given matrix A. Recall that λ is an eigenvalue of A if and only if $A - \lambda I$ is singular, which happens if and only if $\det(A - \lambda I) = 0$. This prompts the following important definition.

Definition. Let A be an $n \times n$ matrix. The function $p_A(t) = \det(A - tI)$ is called the **characteristic polynomial** of the matrix A.

It is not hard to see that $p_A(t)$ is indeed a polynomial in t of degree n with leading coefficient $(-1)^n$. This justifies the name.

The following proposition is immediate from our discussion so far.

Proposition. Let A be an $n \times n$ matrix. λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial $p_A(t)$ of the matrix A.

Our strategy for finding the eigenvalues of A is now straightforward: compute the characteristic polynomial of A and then find its roots.

Example. Finding the characteristic polynomial of a 2×2 matrix is quite simple. Indeed, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$p_A(t) = \begin{vmatrix} a-t & b \\ c & d-t \end{vmatrix} = (a-t)(d-t) - bc = t^2 - (a+d)t + ad - bc.$$

For example, in note 17 we considered the matrix $A = \begin{bmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{bmatrix}$. Thus $p_A(t) = t^2 - \frac{5}{6}t - \frac{1}{6}$. It is easy to see that the roots of this polynomial are 1 and $-1/6$.

Example. In note 17 we also looked at the matrix $\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$. The characteristic polynomial of this matrix is

$$p(t) = \begin{vmatrix} 4-t & 0 & 1 \\ 2 & 3-t & 2 \\ 1 & 0 & 4-t \end{vmatrix} = (3-t) = \begin{vmatrix} 4-t & 1 \\ 1 & 4-t \end{vmatrix} = (3-t)((4-t)^2-1) = (3-4)(4-t-1)(4-t+1) = (3-t)^2(5-t).$$

Thus 3 and 5 are the roots of $p(t)$, i.e. they are the eigenvalues of our matrix (as claimed in note 17).

Example. In note 18 we worked with the matrix $M = \begin{bmatrix} 0 & 3 & 2 & 0 \\ -2 & -5 & 0 & 2 \\ -3 & 0 & 5 & 3 \\ 0 & -3 & -2 & 0 \end{bmatrix}$. The characteristic polynomial of M is

$$p_M(t) = \begin{vmatrix} -t & 3 & 2 & 0 \\ -2 & -5-t & 0 & 2 \\ -3 & 0 & 5-t & 3 \\ 0 & -3 & -2 & -t \end{vmatrix} = \begin{vmatrix} -t & 3 & 2 & -t \\ -2 & -5-t & 0 & 0 \\ -3 & 0 & 5-t & 0 \\ 0 & -3 & -2 & -t \end{vmatrix} =$$

(we added the first column to the last column; now we use Laplace expansion in the last column)

$$= (-1)(-t) \begin{vmatrix} -2 & -5-t & 0 \\ -3 & 0 & 5-t \\ 0 & -3 & -2 \end{vmatrix} + (-t) \begin{vmatrix} -t & 3 & 2 \\ -2 & -5-t & 0 \\ -3 & 0 & 5-t \end{vmatrix} =$$

(now we use Laplace expansion in first column for the first 3×3 determinant and Laplace expansion in the third column for the second determinant)

$$= t\left((-2)\begin{vmatrix} 0 & 5-t \\ -3 & -2 \end{vmatrix} + (-1)(-3)\begin{vmatrix} -5-t & 0 \\ -3 & -2 \end{vmatrix} - 2\begin{vmatrix} -2 & -5-t \\ -3 & 0 \end{vmatrix} - (5-t)\begin{vmatrix} -t & 3 \\ -2 & -5-t \end{vmatrix}\right) =$$

$$t((-2)3(5-t)+3(-5-t)(-2)-2\cdot 3(-5-t)-(5-t)(t(5+t)+6)) = t(6t-30+6t+30+30+6t-(5-t)(t^2+5t+6)) =$$

$$= t(18t + 30 + t^3 + 5t^2 + 6t - 5t^2 - 25t - 30) = t(t^3 - t) = t^2(t^2 - 1) = t^2(t-1)(t+1).$$

We see that the roots of p_M are $0, 1, -1$ (as claimed in note 18).

Example. The characteristic polynomial of the matrix $\begin{bmatrix} 0 & 0 & -a \\ 1 & 0 & -b \\ 0 & 1 & -c \end{bmatrix}$ is

$$\begin{vmatrix} -t & 0 & -a \\ 1 & -t & -b \\ 0 & 1 & -c-t \end{vmatrix} = \begin{vmatrix} 0 & -t^2 & -a-bt \\ 1 & -t & -b \\ 0 & 1 & -c-t \end{vmatrix} = -\begin{vmatrix} -t^2 & -a-bt \\ 1 & -c-t \end{vmatrix} = -(t^3 + ct^2 + bt + a)$$

(our first step was the row operation $E_{1,2}(t)$). We see that any cubic polynomial with leading coefficient -1 can be the characteristic polynomial of a 3×3 matrix.

We reduced the question of finding eigenvalues to the problem of finding roots of a polynomial. This is not an easy problem in general, but there are formulas for roots of polynomials of degree at most 4 and there are techniques to find close approximations to the roots of a polynomial.

Challenge. Extend the last example to matrices of any size.

We end this note with some important observations about characteristic polynomials and eigenspaces.

First a simple but important to remember remark.

If C is invertible then $\det(C^{-1}) = (\det C)^{-1}$.

Indeed, since $I = CC^{-1}$, we have $1 = \det I = \det(CC^{-1}) = \det C \det C^{-1}$, i.e. $\det(C^{-1}) = (\det C)^{-1}$.

Now we have the following important result.

Proposition. Similar matrices have equal determinants and equal characteristic polynomials.

Indeed, suppose that A and B are similar. This means that $B = CAC^{-1}$ for some invertible matrix C. Thus $\det(B) = \det(CAC^{-1}) = \det(C)\det(A)\det(C^{-1}) = \det A$. Moreover, we have

$$B - tI = CAC^{-1} - tI = C(A - tI)C^{-1}$$

so $B - tI$ and $C - tI$ are similar. Hence

$$p_B(t) = \det(B - tI) = \det(A - tI) = p_A(t).$$

This proposition allows us to define the determinant and the characteristic polynomial of a linear transformation from a finite dimensional vector space to itself.

Definition. Let $T : V \longrightarrow V$ be a linear transformation (V finite dimensional). The determinant and the characteristic polynomial of the matrix ${}_BT_B$ are the same for every basis B of V and they are called the determinant and the characteristic polynomial of T respectively.

Recall now that if $f(t)$ is a polynomial and λ is a root of f then there is $m > 0$ such that $f(t) = (t - \lambda)^m g(t)$, where $g(t)$ is a polynomial and $g(\lambda) \neq 0$. The number m is called the **multiplicity** of the root λ.

Theorem. Let $T : V \longrightarrow V$ be a linear transformation. Let $p(t)$ be the characteristic polynomial of T and let λ be a root of $p(t)$, so λ is an eigenvalue of T. The dimension d of the eigenspace $V(\lambda)$ is smaller or equal than the multiplicity of λ as a root of $p(t)$.

Indeed, choose a basis $v_1, \dots, v_d$ of the eigenspace $V(\lambda)$. We know that we can extend it to a basis B of V by adding some vectors $v_{d+1}, \dots, v_n$, $n = \dim V$. The matrix of T in the basis B has the following form $\begin{bmatrix} \lambda I_d & X \\ 0 & A \end{bmatrix}$, where I_d is the identity matrix of size d, X is some $d \times (n-d)$ matrix , A is a square matrix of size $n-d$ and 0 is the $(n-d) \times d$ zero matrix. Using Laplace expansion by the first column we get

$$p(t) = \begin{vmatrix} (\lambda - t) I_d & X \\ 0 & A - tI \end{vmatrix} = (\lambda - t) \begin{vmatrix} (\lambda - t) I_{d-1} & Y \\ 0 & A - tI \end{vmatrix}$$

where Y is obtained from X by removing its first row. Repeating this another $d-1$ times we get that

$$p(t) = (t - \lambda)^d \det(A - tI).$$

Since $\det(A - tI)$ is a polynomial, it follows that the multiplicity of λ as a root of $p(t)$ is at least d as claimed.