We continue the discussion of systems of linear equations. Consider a system of m linear equations with n unknowns:

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & b_m \end{array}$$

We call such a system **homogeneous** if $b_1 = \ldots = b_m = 0$. In other words, a system is homogeneous if the last column of the augmented matrix of the system is a zero column. Thus, to each system of linear equations there is associated a unique homogeneous system which has the same coefficient matrix.

We will see that there is a nice relation between solutions of a consistent system of linear equations and solutions to the associated homogeneous system. In order to speak efficiently about solutions we will use the following terminology: we say that an n-tuple of numbers $(u_1, u_2, \ldots, u_n)$ is a solution to a system of equations in the unknowns $x_1, \ldots, x_n$ if setting $x_1 = u_1, x_2 = u_2, \ldots, x_n = u_n$ yields a solution to the system.

Remark. Homogeneous systems are always consistent: $(0, \ldots, 0)$ is a solution to any such system.

Our goal is to find a relation between solutions to a consistent system of linear equations (on the left) and the associated homogeneous system (on the right):

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & b_m \end{array} \qquad \begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & 0 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & 0 \end{array}$$

Chose a solution $(s_1, \ldots, s_n)$ to our system. This means that

$$\begin{array}{ccccccccc} a_{1,1}s_1 & + & a_{1,2}s_2 & + & \cdots & + & a_{1,n}s_n & = & b_1 \\ a_{2,1}s_1 & + & a_{2,2}s_2 & + & \cdots & + & a_{2,n}s_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m,1}s_1 & + & a_{m,2}s_2 & + & \cdots & + & a_{m,n}s_n & = & b_m \end{array}.$$

Consider now another solution $(t_1, \ldots, t_n)$ to our system, so

$$\begin{array}{ccccccccc} a_{1,1}t_1 & + & a_{1,2}t_2 & + & \cdots & + & a_{1,n}t_n & = & b_1 \\ a_{2,1}t_1 & + & a_{2,2}t_2 & + & \cdots & + & a_{2,n}t_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m,1}t_1 & + & a_{m,2}t_2 & + & \cdots & + & a_{m,n}t_n & = & b_m \end{array}$$

Subtracting i-th equations of each system yields:

$$\begin{array}{ccccccccc} a_{1,1}(t_1 - s_1) & + & a_{1,2}(t_2 - s_2) & + & \cdots & + & a_{1,n}(t_n - s_n) & = & 0 \\ a_{2,1}(t_1 - s_1) & + & a_{2,2}(t_2 - s_2) & + & \cdots & + & a_{2,n}(t_n - s_n) & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m,1}(t_1 - s_1) & + & a_{m,2}(t_2 - s_2) & + & \cdots & + & a_{m,n}(t_n - s_n) & = & 0 \end{array}.$$

This means that $(t_1 - s_1, t_2 - s_2, \ldots, t_n - s_n)$ is a solution to the associated homogeneous system. This observation allows us to define a function T from the set of solutions to our system to the set of solutions to the associated homogeneous system as follows: T takes a solution $(t_1, \ldots, t_n)$ of our system and sends it to the solution $(t_1 - s_1, t_2 - s_2, \ldots, t_n - s_n)$ to the associated homogeneous system. In other words,

$$T(t_1, \ldots, t_n) = (t_1 - s_1, t_2 - s_2, \ldots, t_n - s_n).$$

It is clear that the function is one-to-one: if $T(t_1, \ldots, t_n) = T(h_1, \ldots, h_n)$, i.e. $(t_1 - s_1, t_2 - s_2, \ldots, t_n - s_n) = (h_1 - s_1, h_2 - s_2, \ldots, h_n - s_n)$, then clearly $t_i = h_i$ for each i.

Suppose now that $(u_1, u_n, \ldots, u_n)$ is a solution to the associated homogeneous system. Then, for $i = 1, 2, \ldots, m$, we have

$$a_{i,1}(u_1 + s_1) + a_{i,2}(u_2 + s_2) + \ldots + a_{i,n}(u_n + s_n) =$$

$$= (a_{i,1}u_1 + a_{i,2}u_2 + \ldots + a_{i,n}u_n) + (a_{i,1}s_1 + a_{i,2}s_2 + \ldots + a_{i,n}s_n) = 0 + b_i = b_i.$$

We see that $(u_1 + s_1, u_2 + s_2, \ldots, u_n + s_n)$ is a solution to our system and $T(u_1 + s_1, \ldots, u_n + s_n) = (u_1, u_2, \ldots, u_n)$. This shows that T is onto. Being both one-to-one and onto, T is a bijection. The inverse function T^{-1} takes a solution $(u_1, u_n, \ldots, u_n)$ to the associated homogeneous system and sends it to $(u_1 + s_1, u_2 + s_2, \ldots, u_n + s_n)$, i.e. $T^{-1}(u_1, u_n, \ldots, u_n) = (u_1 + s_1, u_2 + s_2, \ldots, u_n + s_n)$.

We summarize the above discussion in the following theorem.

Theorem. If a system of linear equations is consistent and $(s_1, ..., s_n)$ is a solution to this system then the assignment

$$(u_1, u_n, \ldots, u_n) \mapsto (u_1 + s_1, u_2 + s_2, \ldots, u_n + s_n)$$

defines a bijection between the solutions to the associated homogeneous system and solutions to the given system. In particular, the number of solutions of a consistent system is the same as the number of solutions to the associated homogeneous system.

Let us apply the above discussion to the case when the given system is already homogeneous. We see that given any two solutions $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$ to the homogeneous system, the n-tuple $(s_1 + u_1, \ldots, s_n + u_n)$ is also a solution. Furthermore, it is straightforward to see that for any number c, the n-tuple $(cs_1, \ldots, cs_n)$ is again a solution. It is this property which makes homogeneous systems of special interest to us, as the goal of the course is to study structures which are closed by addition and multiplication by numbers (whatever it means).

The discussion below is optional. It outlines a reason why two consistent systems of equations in the same unknowns have row equivalent augmented matrices.

Consider two consistent systems of linear equations in the unknowns $x_1, \ldots, x_n$ and assume that they are equivalent (i.e. have the same sets of solutions). Adding equations of the form $0 \cdot x_1 + 0 \cdot x_2 + \ldots + 0 \cdot x_n = 0$ if necessary, we may assume that each system has the same number of equations. The relation between solutions to a system and to the associated homogeneous system established above shows that the associated homogeneous systems are also equivalent. Let A_1, A_2 be the coefficient matrices of the systems and let B_1, B_2 be the augmented matrices.

In the second set of notes we established the following result: if D_1, D_2 are matrices in reduced row-echelon form and if the homogeneous systems of equations with coefficient matrices D_1 and D_2 are equivalent, then $D_1 = D_2$ (we have seen how to recover D_i from the set of solutions).

Let us apply this to the case when D_i is row equivalent to A_i. As the four homogeneous systems with coefficient matrices D_1, A_1, A_2, D_2 all have the same solutions, we conclude that $D_1 = D_2$. Thus A_1 and A_2 are row equivalent. Consider the sequence of elementary row operations which transforms A_1 to A_2. Applying these operations to B_1 we obtain a matrix B all of whose columns except possibly the last one coincide with the corresponding columns of B_2. Now the systems of equations with augmented matrices B, B_1, and B_2 are all equivalent. Since the coefficient parts of B and B_2 coincide and since the systems with augmented matrices B and B_2 have a common solution, we see that the last columns of B and B_2 must also be the same. In other words, $B = B_2$ and consequently B_1 and B_2 are row equivalent.