MATH 304 - Linear Algebra

We continue the discussion of systems of linear equations. Consider a system of m linear equations with n unknowns:

$$\begin{array}{ccccccccc}
a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\
a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & b_m
\end{array}$$

Suppose that the coefficient matrix of the system is fixed but the numbers $b_1, \ldots, b_m$ can vary. We would like to know for which choices of the b_i's the system is consistent. Our technique of row reduction can be used to solve this problem. Let us look at a concrete example:

$$\begin{array}{rcl}
3x_1 - x_2 + 2x_3 + 3x_4 - x_5 & = & b_1 \\
x_1 - x_2 + 2x_3 + 3x_4 + x_5 & = & b_2 \\
2x_1 - 3x_2 + 6x_3 + 9x_4 + 4x_5 & = & b_3 \\
7x_1 - 2x_2 + 4x_3 + 6x_4 - 3x_5 & = & b_4
\end{array}$$

The augmented matrix of this system is

$$\left[\begin{array}{rrrrr:l}
3 & -1 & 2 & 3 & -1 & b_1 \\
1 & -1 & 2 & 3 & 5 & b_2 \\
2 & -3 & 6 & 9 & 4 & b_3 \\
7 & -2 & 4 & 6 & -3 & b_4
\end{array}\right].$$

We perform $S_{1,2}$ followed by $S_{2,3}$ to get

$$\left[\begin{array}{rrrrr:l}
1 & -1 & 2 & 3 & 5 & b_2 \\
2 & -3 & 6 & 9 & 4 & b_3 \\
3 & -1 & 2 & 3 & -1 & b_1 \\
7 & -2 & 4 & 6 & -3 & b_4
\end{array}\right].$$

Then we perform $E_{2,1}(-2)$, $E_{3,1}(-3)$, $E_{4,1}(-7)$ and get

$$\left[\begin{array}{rrrrr:r}
1 & -1 & 2 & 3 & 5 & b_2 \\
0 & -1 & 2 & 3 & 2 & b_3 - 2b_2 \\
0 & 2 & -4 & -6 & -4 & b_1 - 3b_2 \\
0 & 5 & -10 & -15 & -10 & b_4 - 7b_2
\end{array}\right].$$

Next we do $E_{3,2}(2)$, $E_{4,2}(5)$ followed by $D_2(-1)$ and get

$$\left[\begin{array}{rrrrr:r}
1 & -1 & 2 & 3 & 5 & b_2 \\
0 & 1 & -2 & -3 & -2 & 2b_2 - b_3 \\
0 & 0 & 0 & & 0 & b_1 - 7b_2 + 2b_3 \\
0 & 0 & 0 & 0 & 0 & -17b_2 + 5b_3 + b_4
\end{array}\right].$$

At this point the coefficient part of our matrix is in row-echelon form. If both $b_1 - 7b_2 + 2b_3$ and $-17b_2 + 5b_3 + b_4$ are zero then the last column is not a pivot column and our system is consistent (with free variables x_3, x_4, x_5). On the other hand, if at least one of $b_1 - 7b_2 + 2b_3$ and $-17b_2 + 5b_3 + b_4$ is non-zero, then the last column is a pivot column and the system is inconsistent. We see that our system of equations is consistent if and only if $b_1 - 7b_2 + 2b_3 = 0$ and $-17b_2 + 5b_3 + b_4 = 0$.

The above method works in general: we perform elementary row operations on the augmented matrix so that we get a matrix whose coefficient part is in row-echelon form. The system is consistent if and only if no row of this matrix has its first non-zero entry in the last column. In other words, if in some row all entries but the last are 0 then the last entry must also be 0. This condition leads to a homogeneous system of equations with unknowns $b_1, \ldots, b_m$ and our original system is consistent iff $b_1, \ldots, b_m$ form a solution to this homogeneous system. Note that it may happen that this homogeneous system of equations is empty (i.e. we do not get any equations at all).

Let us ask now which $m \times n$ matrices A have the property that every system of equations with A as a coefficient matrix is consistent. If a matrix in row-echelon form row equivalent to A has no zero rows, then in the above discussion the last column can not be a pivot column no matter what $b_1, \ldots, b_m$ are.

In other words, if $\mathrm{rank}(A) = m$ then every system of equations with coefficient matrix A is consistent. Conversely, suppose that the last row of a matrix D in row-echelon form which is row equivalent to A is a zero row. Let D_1 be the matrix obtained from D be adding one more column, which has all entries zero except the last one (say the bottom entry is 1). The system with augmented matrix D_1 is clearly inconsistent. There is a sequence of elementary row operations which transforms D to A and these sequence transforms D_1 into some matrix B. The system with augmented matrix B is inconsistent and has coefficient matrix A. We see that every system of equations with coefficient matrix A is consistent if and only if $\mathrm{rank}(A) = m$.

Let us ask another question: which $m \times n$ matrices A have the property that every system of equations with A as a coefficient matrix has at most one solution. In other words, the system should be either inconsistent or have a unique solution. This will happen if and only if the homogeneous system with coefficient matrix A has a unique solution. This means that this system should not have any free variables, i.e. every column of A must be a pivot column. In other words, every system of equations with A as a coefficient matrix has at most one solutions if and only if $\mathrm{rank}(A) = n$.

Putting the above two possibilities together we conclude that every system of equations with A as a coefficient matrix has exactly one solution if and only if $\mathrm{rank}(A) = n = m$.

Let us collect the observations made above in the following theorem.

Theorem. Let $A = [a_{i,j}]$ be an $m \times n$ matrix. Then

- every system of linear equations with coefficient matrix A is consistent if and only if $\mathrm{rank}(A) = m$ (so $m \leq n$).
- every system of linear equations with coefficient matrix A has at most one solution if and only if $\mathrm{rank}(A) = n$ (so $n \leq m$).
- every system of linear equations with coefficient matrix A has a unique solution if and only if $\mathrm{rank}(A) = m = n$.

In general, there is a homogeneous system of equations such that $b_1, ..., b_m$ is a solution to this system if and only if the system

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + \cdots + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + \cdots + & a_{2,n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + \cdots + & a_{m,n}x_n & = & b_m \end{array}$$

is consistent. Such a homogeneous system can be found by the method outlined at the beginning of this note.