So far we have been manipulating with rows of a matrix to transform it into a particularly convenient form and then deduce from it interesting information related to the original matrix. Forgetting for a moment about the actual context in which this has been done, there does not seem to be any reason why row manipulations should be more interesting than analogous column manipulations. We could try to repeat many of our reasonings in the context of column manipulations. However, there is a very simple operation on matrices which changes rows into columns and vice versa.

Definition. Let $A = [a_{i,j}]$ be an $m \times n$ matrix. The **transpose matrix** A^T of A is an $n \times m$ matrix whose first column is the first row of A, second column is the second row of A, ..., m-th column is the m-th row of A. Equivalently, for any $1 \le i \le n$ and $1 \le j \le m$, the i,j-entry of A^T is equal to the j,i-entry of A

Example. When $A = \begin{bmatrix} 3 & -1 & 2 & 3 & -1 \\ 1 & -1 & 2 & 3 & 5 \\ 2 & -3 & 6 & 9 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} 3 & 1 & 2 \\ -1 & -1 & -3 \\ 2 & 2 & 6 \\ 3 & 3 & 9 \\ -1 & 5 & 4 \end{bmatrix}$.

The following properties of transposition of matrices should be straightforward from the definition.

If A, B are matrices of the same size and c is a number then

$$(A+B)^T = A^T + B^T, \quad (cA)^T = cA^T, \quad (A^T)^T = A.$$

Suppose now that A is an $k \times m$ matrix and B is an $m \times n$ matrix, so the product AB is defined. We would like to see if there is any relation between the matrices $(AB)^T$, A^T, B^T. Well, recall that the i,j-entry of $(AB)^T$ is equal to the j,i entry of AB which, in turn, is equal to the dot product of j-th row of A and i-th column of B. However, the j-th row of A is the same as the j-th column of A^T, and the i-th column of B is the same as the i-th row of B^T. Since the dot product is commutative, we can summarize the above reasoning into the following observation: the i,j-entry of $(AB)^T$ is equal to the dot product of the i-th row of B^T and the j-th column of A^T. Note that the same is true for the i,j-entry of the matrix B^TA^T. In other words, the matrices $(AB)^T$ and B^TA^T have identical entries, hence they are equal:

$$(AB)^T = B^TA^T$$

In other words, the transpose of a product is the product of the transposes of the factors but **in the reversed order.** This is true for products of more than two matrices as well.

Recall now that we defined square matrices $E_{i,j}(a)$ such that $E_{i,j}(a)A$ is obtained from A by adding to the i-th row of A the j-th row multiplied by a. This is the same as saying that $(E_{i,j}(a)A)^T$ is obtained from A^T by adding to the i-th column of A^T the j-th column multiplied by a. On the other hand, we know that

$$(E_{i,j}(a)A)^T = A^TE_{i,j}(a)^T.$$

Recall now that $E_{i,j}(a)$ has a as its i,j-entry, has 1 at all entries on the main diagonal and has 0 everywhere else. It should be clear from this description that $E_{i,j}(a)^T$ has a as its j,i-entry, has 1 at all entries on the main diagonal and has 0 everywhere else. In other words, $E_{i,j}(a)^T = E_{j,i}(a)$. Putting this together, we have the following result:

$E_{i,j}(a)^T = E_{j,i}(a)$. The matrix $BE_{i,j}(a)$ is obtained from B by adding to the j-th column of B the i-th column multiplied by a.

In plain words, multiplication by $E_{i,j}(a)$ on the right is equivalent to certain column operation on B. Note however that there is a slight but important difference between the row operation when you multiply by by $E_{i,j}(a)$ on the left and the column operation when you multiply by $E_{i,j}(a)$ on the right. For example:

$$E_{2,3}(2)A \text{ adds 2 times the third row of } A \text{ to the second row of } A,$$

but

$$AE_{2,3}(2) \text{ adds 2 times the second column of } A \text{ to the third column of } A.$$

It is natural to ask now how the remaining two types row operations, $D_i(a)$ and $S_{i,j}$ fit into the operation of transposition. For $D_i(a)$ the answer is very simple. Recall that $D_i(a)$ is a diagonal matrix. It is straightforward to see the transpose of any diagonal matrix is the same matrix: $A = A^T$ when A is diagonal (note that transposing a square matrix simply reflexes all its entries with respect to the main diagonal). From this we get

$D_i(a)^T = D_i(a)$. The matrix $BD_i(a)$ is obtained from B by multiplying the i-th column of B by a.

Recall now that $S_{i,j}$ is a square matrix obtained from the identity matrix by swapping its i-th and j-th rows. Thus $S_{i,j}$ has 1 both in its i, j-entry and in its j, i-entry and all other entries outside the main diagonal are 0. Thus, when reflected in its main diagonal, $S_{i,j}$ will not change, i.e. $S_{i,j}^T = S_{i,j}$. It follows that:

$S_{i,j}^T = S_{i,j}$. The matrix $BS_{i,j}$ is obtained from B by swapping the i-th and j-th column of B.

The moral of our discussion so far is that performing elementary column operations is the same as multiplication the appropriate elementary matrix on the right. Also, by taking transpose, we can reduce discussion of elementary column operations back to elementary row operations.

We end this note with the following definition.

Definition. A square matrix A is called **symmetric** if it is equal to its transpose: $A = A^T$.

Thus, diagonal matrices and the matrices $S_{i,j}$ are symmetric.