Exam I, February 19

MATH 304, section 6

YOU MUST SHOW ALL WORK TO GET CREDIT.

Problem 1. Let $A = \begin{bmatrix} 2 & 0 & -1 & 3 \\ 3 & 1 & -2 & 6 \\ 1 & 2 & 5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 5 & -3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -2 \\ 1 & 1 \\ -1 & 3 \\ -2 & 3 \end{bmatrix}$, $I = I_4 = 4{\times}4$ identity matrix.

Compute the following or explain why it does not make sense (2 points each):

a) AC b)BA c) B^{-1} d) $E_{1,3}(2)C$ e) $2A - B$ f) $3I + B^T$.

Problem 2. Let $A = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 \\ 4 & 4 & 8 & 4 & 0 \\ 4 & 0 & 5 & 1 & 1 \\ 6 & 2 & 9 & 3 & 1 \end{bmatrix}$.

a) (5 points) Find the reduced row-echelon form row-equivalent to A. Name all the elementary row operations performed.

b) (2 points) What are the pivot columns of A?

c) (2 points) What is the rank of A?

d) (5 points) State the domain and codomain of the linear transformation L_A. Is L_A one-to-one? onto? Explain your answers.

Problem 3. The augmented matrix of a system of linear equations is A and R is reduced row-echelon form row-equivalent to A:

$$A = \left[\begin{array}{cccccc:c} 2 & 4 & 1 & 4 & 0 & 4 & 8 \\ 3 & 6 & 1 & 5 & 1 & 8 & 10 \\ 2 & 4 & 0 & 2 & 1 & 5 & 5 \end{array}\right], \qquad R = \left[\begin{array}{cccccc:c} 1 & 2 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 & -1 \end{array}\right]$$

1. (3 points) Write down this system of equations.
2. (2 points) What are the independent variables?
3. (2 points) What is the rank of the coefficient matrix?
4. (5 points) Solve this system of linear equations.

Problem 4. Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 0 & 3 & 2 \end{bmatrix}$.

1. (7 points) Find the inverse of A. Verify your answer by performing appropriate multiplication.
2. (3 points) Express A as a product of elementary matrices.
3. (4 points) What is the inverse of A^T? Explain your answer.

Problem 5. Consider the linear transformation $L_A : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by the matrix $A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -2 & 0 & 1 & -1 \\ 2 & 2 & 0 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix}$.

(10 points) Find a vector (a, b, c, d) which is not in the image of L_A. Show all necessary work.

More problems on reverse.

Problem 6. (10 points) Find a matrix X such that $XA = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$ knowing that $A^{-1} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$.

Problem 7. (10 points) A linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ satisfies

$$T(1,2,1) = (1,-1,4), \quad T(0,1,0) = (0,1,-2), \quad T(0,0,-1) = (-1,-1,0).$$

Find the matrix representing T. What is $T(1,2,3)$?

Problem 8. Answer true or false (2 points each).

a) $T(a,b) = (a+b, ab)$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

b) Every linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^3$ is onto.

c) If $p < q$ and B is a $p \times q$ matrix then $Bx = 0$ has infinitely many solutions.

d) If MN and NM are defined then M, N are square matrices.

e) If A, B are square matrices and $BA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ then both A and B are invertible.

f) If A and B are surjective matrices (i.e. L_A and L_B are both onto) and AB is defined then AB is surjective.

g) If A is a 4×3 matrix of rank 3 and B is a 3×2 matrix of rank 2 then AB has rank 2.

h) If B is an invertible $m \times m$ matrix then $B^T B$ has rank m.

The following problem is optional. You may earn extra credit, but work on this problem only after you are done with the other problems.

Problem 9. (15 points) Let A be an $m \times n$ matrix. Show that there is a matrix X such that $AX = I$ if and only if $\text{rank}(A) = m$. Here I denotes $m \times m$ identity matrix.