Solutions to Exam I
MATH 304, section 6

YOU MUST SHOW ALL WORK TO GET CREDIT.

Problem 1. Let $A = \begin{bmatrix} 1 & 2 & 5 & -6 \\ 2 & 0 & -1 & 3 \\ 3 & 1 & -2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 5 & -6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -2 \\ 1 & 1 \\ -1 & 3 \\ -2 & 3 \end{bmatrix}$, $I = I_4 = 4{\times}4$ identity matrix.

Compute the following or explain why it does not make sense (2 points each):

a) AC b)BA c) B^{-1} d) $E_{3,2}(-2)C$ e) $A + 2B$ f) $2I - B^T$.

Solution. a)

$$AC = \begin{bmatrix} 1 & 2 & 5 & -6 \\ 2 & 0 & -1 & 3 \\ 3 & 1 & -2 & 6 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & 1 \\ -1 & 3 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ 1 & 2 \\ 0 & 7 \end{bmatrix}$$

b) BA does not make sense since the number of columns of B is not the same as the number of rows of A.

c) B^{-1} does not make sense since the matrix B is not invertible. Indeed, B is a 4×4 matrix in reduced row-echelon form. It has rank 3 and invertible 4×4 matrices have rank 4. Alternatively, B has a zero row, hence can not be invertible.

d) $E_{3,2}(-2)C$ is obtained from C by adding the second row of C multiplied by -2 to the third row. Thus

$$E_{3,2}(-2)C = \begin{bmatrix} 3 & -2 \\ 1 & 1 \\ -3 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & 1 \\ -1 & 3 \\ -2 & 3 \end{bmatrix}.$$

e) $A + 2B$ does not make sense since the matrices A and B have different sizes.

f)

$$2I - B^T = 2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 5 & -6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ -6 & -3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ -5 & -1 & 2 & 0 \\ 6 & 3 & -1 & 2 \end{bmatrix}.$$

Problem 2. Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 4 & 4 & 4 & 8 & 0 \\ 4 & 1 & 0 & 5 & 1 \\ 6 & 3 & 2 & 9 & 1 \end{bmatrix}$.

a) (5 points) Find the reduced row-echelon form row-equivalent to A. Name all the elementary row operations performed.

b) (2 points) What are the pivot columns of A?

c) (2 points) What is the rank of A?

d) (5 points) State the domain and codomain of the linear transformation L_A. Is L_A one-to-one? onto? Explain your answers.

Solution. a) First we perform $S_{1,2}$ followed by $D_1(1/4)$ and get

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 4 & 1 & 0 & 5 & 1 \\ 6 & 3 & 2 & 9 & 1 \end{bmatrix}.$$

Next we do $E_{3,1}(-4)$ and $E_{4,1}(-6)$ to get

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -3 & -4 & -3 & 1 \\ 0 & -3 & -4 & -3 & 1 \end{bmatrix}.$$

Now we perform $E_{4,3}(-1)$ followed by $E_{1,2}(-1)$ and $E_{3,2}(3)$ and get

$$\begin{bmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Next we do $D_3(1/2)$ followed by $E_{2,3}(-2)$ and $E_{1,3}(1)$ and get

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is in reduced row-echelon form.

b) From the reduced row echelon form we see that the pivot columns of A are the first, second, and third columns of A.

c) The rank of A is the number of pivot columns of A (or, equivalently, the number of non-zero rows in the reduced row-echelon form row equivalent to A), so $\operatorname{rank}(A) = 3$.

d) Since A is a 4×5 matrix, the linear transformation L_A maps $\mathbb{R}^5$ to $\mathbb{R}^4$. Thus the domain of L_A is $\mathbb{R}^5$ and the codomain is $\mathbb{R}^4$.

The linear transformation L_A is one-to-one iff the rank of A is equal to the number of columns of A, which is not true in our case. Thus L_A is not one-to-one.

The linear transformation L_A is onto iff the rank of A is equal to the number of rows of A, which is not true in our case. Thus L_A is not onto.

Problem 3. The augmented matrix of a system of linear equations is A and R is reduced row-echelon form row-equivalent to A:

$$A = \left[\begin{array}{cccccc:c} 2 & 4 & 1 & 4 & 0 & 8 & 4 \\ 3 & 6 & 1 & 5 & 1 & 10 & 8 \\ 2 & 4 & 0 & 2 & 1 & 5 & 5 \end{array}\right], \qquad R = \left[\begin{array}{cccccc:c} 1 & 2 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \end{array}\right]$$

1. (3 points) Write down this system of equations.
2. (2 points) What are the independent variables?
3. (2 points) What is the rank of the coefficient matrix?
4. (5 points) Solve this system of linear equations.

Solution. a) The system of equations with augmented matrix A is:

$$\begin{aligned} 2x_1 + 4x_2 + x_3 + 4x_4 + 0x_5 + 8x_6 &= 4 \\ 3x_1 + 6x_2 + x_3 + 5x_4 + x_5 + 10x_6 &= 8 \\ 2x_1 + 4x_2 + 0x_3 + 2x_4 + x_5 + 5x_6 &= 5 \end{aligned}$$

b) Since B is in reduced row-echelon form and it is row-equivalent to A, the independent (free) variables correspond to non-pivot columns of B. Thus x_2, x_4, x_6 are the free variables.

c) The rank of A is the same as the rank of B, which is the number of non-zero rows of B, which is 3. Thus A has rank 3.

d) We know that the systems with augmented matrices A and R have the same solutions. Thus the solutions to our system are:

$x_1 = -2x_2 - x_4 - 3x_6 + 1$, $x_2 =$ anything, $x_3 = -2x_4 - 2x_6 + 2$, $x_4 =$ anything, $x_5 = x_6 + 3$, $x_6 =$ anything.

Problem 4. Let $A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 4 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

1. (7 points) Find the inverse of A. Verify your answer by performing appropriate multiplication.
2. (3 points) Express A as a product of elementary matrices.
3. (4 points) What is the inverse of A^T? Explain your answer.

Solution. a) In order to invert A we start with the matrix

$$\left[\begin{array}{ccc:ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 3 & 4 & 3 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

We perform $S_{1,3}$ to get

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 3 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 & 0 & 0 \end{array}\right]$$

Then we do $E_{2,1}(-3)$, $E_{3,1}(-2)$ and get

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & -3 \\ 0 & 1 & -4 & 1 & 0 & -2 \end{array}\right]$$

Next we do $E_{1,2}(-1)$, $E_{3,2}(-1)$ and get

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 5 & 0 & -1 & 4 \\ 0 & 1 & -3 & 0 & 1 & -3 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right]$$

Next we do $D_3(-1)$ and get

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 5 & 0 & -1 & 4 \\ 0 & 1 & -3 & 0 & 1 & -3 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right]$$

Finally, we do $E_{2,3}(3)$, $E_{1,3}(-5)$ and get

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 5 & -6 & 9 \\ 0 & 1 & 0 & -3 & 4 & -6 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right]$$

Since the left part is now the identity matrix, the right part is the inverse of A, i.e.

$$A^{-1} = \begin{bmatrix} 5 & -6 & 9 \\ -3 & 4 & -6 \\ -1 & 1 & -1 \end{bmatrix}$$

To verify our answer we check that $AA^{-1} = I$:

$$\begin{bmatrix} 2 & 3 & 0 \\ 3 & 4 & 3 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & -6 & 9 \\ -3 & 4 & -6 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The above computations tells us that

$$A^{-1} = E_{1,3}(-5)E_{2,3}(3)D_3(-1)E_{3,2}(-1)E_{1,2}(-1)E_{3,1}(-2)E_{2,1}(-3)S_{1,3}.$$

Inverting the product, we get

$$A = S_{1,3}E_{2,1}(3)E_{3,1}(2)E_{1,2}(1)E_{3,2}(1)D_3(-1)E_{2,3}(-3)E_{1,3}(5).$$

To answer the last question note that transposing the equation $AA^{-1} = I$ we get

$$(AA^{-1})^T = (A^{-1})^T A^T = I^T = I.$$

It follows that the inverse of A^T is $(A^{-1})^T$, i.e.

$$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 5 & -6 & 9 \\ -3 & 4 & -6 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 5 & -3 & -1 \\ -6 & 4 & 1 \\ 9 & -6 & -1 \end{bmatrix}.$$

Problem 5. Consider the linear transformation $L_A : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by the matrix $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -2 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 1 & 0 \end{bmatrix}$.

(10 points) Find a vector (a, b, c, d) which is not in the image of L_A.

Solution. Vector (a, b, c, d) is in the image of L_A if and only if the system of linear equations with augmented matrix

$$\left[\begin{array}{cccc:c} 1 & 1 & 0 & -1 & a \\ 0 & 1 & -1 & -2 & b \\ 2 & 0 & 2 & 2 & c \\ 2 & 1 & 1 & 0 & d \end{array}\right]$$

has a solution. This will be the case if and only if the last column of the augmented matrix is not a pivot column. We perform the elementary row operations $E_{3,1}(-2)$, $E_{4,1}(-2)$ and get

$$\left[\begin{array}{cccc:c} 1 & 1 & 0 & -1 & a \\ 0 & 1 & -1 & -2 & b \\ 0 & -2 & 2 & 4 & c-2a \\ 0 & -1 & 1 & 2 & d-2a \end{array}\right].$$

Now we do $E_{3,2}(2)$, $E_{4,2}(1)$ and get

$$\left[\begin{array}{cccc:c} 1 & 1 & 0 & -1 & a \\ 0 & 1 & -1 & -2 & b \\ 0 & 0 & 0 & 0 & c-2a+2b \\ 0 & 0 & 0 & 0 & d-2a+b \end{array}\right].$$

The part to the left of the dividing line is now in row-echelon form. The last column is not a pivot column if and only if $c-2a+2b = 0$ and $d-2a+b = 0$. Thus (a, b, c, d) is in the image of L_A if and only if both $c-2a+2b = 0$ and $d-2a+b = 0$. In other words, if at least one of $c-2a+2b$ or $d-2a+b$ is not zero, then (a, b, c, d) is not in the image. We can take $a = b = c = d = 1$ and then $c - 2a + 2b = 1 \neq 0$, so $(1, 1, 1, 1)$ is not in the image of L_A.

Problem 6. (10 points) Find a matrix X such that $XA = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}$ knowing that $A^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

Solution. Note that if $XA = B$ and A is invertible, then

$$BA^{-1} = (XA)A^{-1} = X(AA^{-1}) = XI = X.$$

Thus, in our case, we have

$$X = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Problem 7. (10 points) A linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ satisfies

$$T(1,1,2) = (1,1,4), \quad T(0,-1,0) = (0,-2,-3), \quad T(0,0,1) = (-1,-1,0).$$

Find the matrix representing T. What is $T(2,3,4)$?

Solution. Recall that the columns of the matrix representing T are $T(e_1)$, $T(e_2)$, $T(e_3)$. We know that $T(e_3) = T(0,0,1) = (-1,-1,0)$. Since $T(-e_2) = T(0,-1,0) = (0,-2,-3)$, we see that $T(e_2) = -T(-e_2) = (0,2,3)$. It remains to find $T(e_1)$. Note that

$$(1,1,4) = T(1,1,2) = T(e_1+e_2+2e_3) = T(e_1)+T(e_2)+2T(e_3) = T(e_1)+(0,2,3)+2(-1,-1,0) = T(e_1)+(-2,0,3).$$

It follows that $T(e_1) = (1,1,4) - (-2,0,3) = (3,1,1)$. Thus the matrix of T is

$$\begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 3 & 0 \end{bmatrix}.$$

This means that

$$T(a,b,c) = (3a - c, a + 2b - c, a + 3b).$$

In particular, $T(2,3,4) = (2,4,11)$.

Problem 8. Answer true or false (2 points each).

a) $T(a,b) = (ab, a-b)$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

b) Every linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^5$ is one-to-one.

c) If AB and BA are defined then A, B are square matrices.

d) If $m < n$ and A is an $m \times n$ matrix then $Ax = 0$ has infinitely many solutions.

e) If A, B are square matrices and $AB = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ then both A and B are invertible.

f) If A and B are injective matrices (i.e. L_A and L_B are one-to-one) and AB is defined then AB is injective.

g) If A is a 3×4 matrix of rank 3 and B is a 2×3 matrix of rank 2 then BA has rank 2.

h) If A is an invertible $n \times n$ matrix then AA^T has rank n.

Solution. a) This is false. For example, $T((1,1) + (1,1)) = T(2,2) = (4,0)$ which is not the same as $T(1,1) + T(1,1) = (1,0) + (1,0) = (2,0)$.

b) This is false. For example, the function which maps every vector in $\mathbb{R}^3$ to the zero vector in $\mathbb{R}^5$ is a linear transformation but it is clearly not one-to-one.

c) This is false. For any positive integers m, n if A is an $m \times n$ matrix and B is a $n \times m$ matrix then both AB and BA are defined.

d) This is true. Since A has more columns than rows, not every column of A is a pivot column. Thus the system of linear equations $Ax = 0$ has free variables, hence it has infinitely many solutions (since homogeneous system is always consistent).

e) This is false. Note that AB is not an invertible matrix (performing operation $E_{1,2}(-1)$ will produce a zero row). If both A and B were invertible, then AB would also be invertible, which is false.

f) This is true. The composition of two one-to-one functions is also one-to-one. Since $L_A \circ L_B = L_{AB}$ and both L_A and L_B are one-to-one, also L_{AB} is one-to-one.

g) This is true. Since A is a 3×4 matrix of rank 3, L_A is onto. Similarly, L_B is onto. The composition of onto functions is also onto. Thus $L_B \circ L_A = L_{BA} : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ is onto. This means that the rank of BA is 2.

h) This is true. If A is invertible, then A^T is also invertible, hence AA^T is invertible as well. Invertible $n \times n$ matrix has rank n, so AA^T has rank n.

Problem 9. (15 points) Let A be an $m \times n$ matrix. Show that there is a matrix X such that $AX = I$ if and only if $\text{rank}(A) = m$. Here I denotes $m \times m$ identity matrix.

Solution. Suppose first that there is an $n \times m$ matrix X such that $AX = I$. We will think in terms of the linear transformations associated to matrices. $L_I : \mathbb{R}^m \longrightarrow R^m$ is the identity function Since the identity function is onto, $L_I = L_{AX} = L_A \circ L_X$ is onto. It follows that L_A is onto (if a composition $f \circ g$ of two functions is onto then f is also onto). This happens if and only if $\text{rank}(A) = m$. Thus if X exists then $\text{rank}(A) = m$.

Conversely, suppose that $\text{rank}(A) = m$. Thus $L_A : \mathbb{R}^n \longrightarrow R^m$ is onto. In particular, there are vectors $u_i \in \mathbb{R}^n$ such that $L_A(u_i) = e_i$, $i = 1, 2, \ldots, m$. Consider the $n \times m$ matrix X whose i-th column is u_i for $i = 1, 2, \ldots, m$. It follows that $L_X(e_i) = u_i$ for $i = 1, 2, \ldots, m$. Consequently,

$$L_{AX}(e_i) = (L_A \circ L_X)(e_i) = L_A(L_X(e_i)) = L_A(u_i) = e_i$$

for $i = 1, 2, \ldots, m$. It follows that the i-th column of AX is e_i for $i = 1, 2, \ldots, m$, i.e. AX is the identity matrix.