MATH 304 - Linear Algebra, Section 6
Exam 3

YOU MUST SHOW ALL WORK AND YOU MUST WRITE COMPLETE SENTENCES EXPLAINING WHAT YOU ARE DOING TO GET FULL CREDIT. BE VERY CAREFUL WITH YOUR ARITHMETIC.

1. a) (8 points) The space $\mathbb{P}_3$ of all polynomials $f(x)$ of degree ≤ 3 has a basis B consisting of polynomials $1, 1+x, 1+x+x^2, 1+x+x^2+x^3$ and a basis D consisting of polynomials $1, x, x^2, x^3$. Consider the linear transformation $T : \mathbb{P}_3 \longrightarrow \mathbb{P}_3$ defined by $T(f(x)) = xf'(x) - f(x)$. Find the matrix ${}_DT_B$.

   b) (8 points) Find the transition matrix from the basis $(1,0,1), (1,1,0), (0,1,1)$ to the basis $(1,0,0), (1,1,0), (1,1,1)$ of $\mathbb{R}^3$. What are the coordinates of a vector v in the second basis if its coordinates in the first basis are $(1,-1,2)$?

   c) (8 points) A linear transformation $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is represented in the standard bases by the matrix $\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix}$. What is the matrix representation of L in the bases $(1,1,0), (1,-1,0), (1,1,1)$ of $\mathbb{R}^3$ and $(2,1), (3,2)$ of $\mathbb{R}^2$?

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2. Let $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$.

   a) (6 points) Compute the determinant of A. Carefully explain each step of your computations.

   b) (4 points) Write down the Laplace (cofactor) expansion of $\det A$ along the third column.

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3. a) (8 points) Find the characteristic polynomial and the eigenvalues of the matrix $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Is M diagonalizable?

   b) (8 points) The numbers 2 and 3 are the only eigenvalues of the matrix $B = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{bmatrix}$. Find bases of the corresponding eigenspaces. Is B diagonalizable? Explain your answer.

   c) (8 points) A 3×3 matrix C has eigenvectors $v_1 = (1,1,1)$ with eigenvalue 1, $v_2 = (1,1,0)$ with eigenvalue -1 and $v_3 = (1,0,0)$ with eigenvalue 0. It follows that C is diagonalizable, i.e. there exist a matrix P and a diagonal matrix D such that $C = PDP^{-1}$. Find P, D and then C.

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4. a) (4 points) B is an upper triangular 5×5 matrix with diagonal entries $1, 2, -1, 1, 3$. We know that $E_{2,3}(-4)S_{2,3}D_2(3)E_{3,1}(-2)A = B$. Compute the determinant of A.

   b) (3 points) A matrix A satisfies $AA^t = I$. Prove that $\det A = \pm 1$.

   c) (4 points) S, T are linear transformations from V to V and v is an eigenvector for both S and T. Show that v is an eigenvector for ST.

5. (3 points each) Complete each definition. Make sure that you write a complete meaningful sentences containing all necessary conditions.

   a) λ is an eigenvalue of a linear transformation $T : V \longrightarrow V$ if ... .
   b) The characteristic polynomial $p(t)$ of a square matrix A is ... .
   c) A square matrix A is diagonalizable if ... .
   d) Two matrices A, B are similar if ... .
   e) w is an eigenvector of a linear transformation $T : V \longrightarrow V$ if ... .

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6. Answer true or false (2 points each).

   a) An invertible 4×4 matrix can have characteristic polynomial $x^4 - x^2 + 2x$.
   b) There are invertible 9×9 matrices A, B such that $AB = -BA$.
   c) Any matrix with characteristic polynomial $p(t) = t(t-1)(t+1)(t-2)$ is diagonalizable.
   d) If $v \in \mathbb{R}^n$ is an eigenvector of both matrices A and B then v is an eigenvector of $A+B$.
   e) There is a 5×5 matrix of rank 2 which has eigenvalue 2 and the corresponding eigenspace has dimension 3.
   f) If A is a 3×3 matrix then one of the matrices A, $A - I$, $A + 2I$, $A + I$ is invertible.
   g) If A, B are $n \times n$ matrices then $\det(A + B) = \det(A) + \det(B)$.
   h) If two 2×2 matrices have the same characteristic polynomial then they are similar.

The following problem is optional. You can earn 12 extra points if you solve it, but work on it only if you are done with all the other problems

7. A square matrix M is such that $M^2 = M$ (such matrices are called **idempotent**). We think of M as the linear transformation $L_M : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

   a) Show that if λ is an eigenvalue of M then $\lambda = 0$ or $\lambda = 1$.
   b) Show that the kernel of M is the eigenspace corresponding to 0 and the image of M is the eigenspace corresponding to 1.
   c) Prove that M is diagonalizable.