Solutions to Exam 3
MATH 304, Section 6

1. a) (8 points) The space $\mathbb{P}_3$ of all polynomials $f(x)$ of degree ≤ 3 has a basis B consisting of polynomials $1, 1+x, 1+x+x^2, 1+x+x^2+x^3$ and a basis D consisting of polynomials $1, x, x^2, x^3$. Consider the linear transformation $T : \mathbb{P}_3 \longrightarrow \mathbb{P}_3$ defined by $T(f(x)) = xf'(x) - f(x)$. Find the matrix ${}_DT_B$.

   b) (8 points) Find the transition matrix from the basis $(1,0,1), (1,1,0), (0,1,1)$ to the basis $(1,0,0), (1,1,0), (1,1,1)$ of $\mathbb{R}^3$. What are the coordinates of a vector v in the second basis if its coordinates in the first basis are $(1,-1,2)$?

   c) (8 points) A linear transformation $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is represented in the standard bases by the matrix $\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix}$. What is the matrix representation of L in the bases $(1,1,0), (1,-1,0), (1,1,1)$ of $\mathbb{R}^3$ and $(2,1), (3,2)$ of $\mathbb{R}^2$?

Solution.a) In order to find the matrix ${}_BT_D$ of the linear transformation $T : \mathbb{P}_3 \longrightarrow \mathbb{P}_3$ defined by $T(f(x)) = xf'(x) - f(x)$ we need to find coordinates of each of the polynomials $T(1)$, $T(1+x)$, $T(1+x+x^2)$ and $T(1+x+x^2+x^3)$ in the basis $1, x, x^2, x^3$. This is quite easy:

$T(1) = x \cdot 0 - 1 = -1 = -1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$

$T(1+x) = x \cdot 1 - (x+1) = -1 = -1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$

$T(1+x+x^2) = x \cdot (2x+1) - (1+x+x^2) = x^2 - 1 = -1 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3$

$T(1+x+x^2+x^3) = x \cdot (3x^2+2x+1) - (1+x+x^2+x^3) = 2x^3+x^2-1 = -1 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 2 \cdot x^3$

It follows that the matrix of T in the given bases is ${}_DT_B = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

b) In order to find the change of basis matrix from the basis $(1,0,1), (1,1,0), (0,1,1)$ to the basis $(1,0,0), (1,1,0), (1,1,1)$ we need to express each vector of the basis $(1,0,1), (1,1,0), (0,1,1)$ as a linear combination of the vectors $(1,0,0), (1,1,0), (1,1,1)$. This is achieved by finding the reduced row-echelon form of the matrix

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right].$$

This requires only two elementary row operations: $E_{1,2}(-1)$ and $E_{2,3}(-1)$. We get the matrix

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

Hence the desired change of basis matrix equals $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. The coordinates of v in the second basis are equal to

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

c) A linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is represented in the standard bases by the matrix $\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix}$. We would like to find the matrix of T in the bases $(1,1,0), (1,-1,0), (1,1,1)$ of $\mathbb{R}^3$ and $(2,1), (3,2)$ of $\mathbb{R}^2$. Recall the following fundamental fact: if a linear transformation

$T : V \longrightarrow W$ is given by a matrix ${}_{\mathbf{d}}T_{\mathbf{b}}$ in bases $\mathbf{b}$ of V and $\mathbf{d}$ of W then the matrix of T in some other bases $\mathbf{b}'$ of V and $\mathbf{d}'$ of W is given by

$${}_{\mathbf{d}'}T_{\mathbf{b}'} = {}_{\mathbf{d}'}\mathrm{I}_{\mathbf{d}} \cdot {}_{\mathbf{d}}T_{\mathbf{b}} \cdot {}_{\mathbf{b}}\mathrm{I}_{\mathbf{b}'},$$

where ${}_{\mathbf{d}'}\mathrm{I}_{\mathbf{d}}$ is the transition matrix from the basis $\mathbf{d}$ to the basis $\mathbf{d}'$ and similarly ${}_{\mathbf{b}}\mathrm{I}_{\mathbf{b}'}$ is the transition matrix from the basis $\mathbf{b}'$ to the basis $\mathbf{b}$.

Thus we need to compute the transition matrices from the basis $(1,1,0), (1,-1,0), (1,1,1)$ to the standard basis of $\mathbb{R}^3$ and from the standard basis of $\mathbb{R}^2$ to the basis $(2,1), (3,2)$. In order to get the first transition matrix we need to express the vectors $(1,1,0), (1,-1,0), (1,1,1)$ as linear combination of the vectors in the standard basis. This does not require any work and we see that the transition matrix from the basis $(1,1,0), (1,-1,0), (1,1,1)$ to the standard basis of $\mathbb{R}^3$ is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. The transition matrix from the standard basis of $\mathbb{R}^2$ to the basis $(2,1), (3,2)$ is a little more difficult to compute, but its inverse, i.e. the transition matrix from the basis $(2,1), (3,2)$ to the standard basis of $\mathbb{R}^2$ is very easy to compute; it equals $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. We just need to invert this matrix and we see that the transition matrix from the standard basis of $\mathbb{R}^2$ to the basis $(2,1), (3,2)$ equals $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. We are ready to compute the matrix of T in the bases $(1,1,0), (1,-1,0), (1,1,1)$ of $\mathbb{R}^3$ and $(2,1), (3,2)$ of $\mathbb{R}^2$; it equals

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -2 & -8 \\ -4 & 2 & 6 \end{bmatrix}.$$

Second method. There is another method of solving this problem. We first compute the images

$T(1,1,0) = \begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = (4,0);$

$T(1,-1,0) = \begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = (2,2);$

$T(1,1,1) = \begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (2,4);$ Now we want to express each of these 3 vectors of $\mathbb{R}^2$ as a linear combination of the vectors $(2,1), (3,2)$. We do that by finding the reduced row-echelon form of the matrix

$$\left[\begin{array}{cc:ccc} 2 & 3 & 4 & 2 & 2 \\ 1 & 2 & 0 & 2 & 4 \end{array}\right]$$

which is

$$\left[\begin{array}{cc:ccc} 1 & 0 & 8 & -2 & -8 \\ 0 & 1 & -4 & 2 & 6 \end{array}\right]$$

The right part of this matrix is the desired matrix representation of T.

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2. Let $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$.

   a) (6 points) Compute the determinant of A. Carefully explain each step of your computations.

   b) (4 points) Write down the Laplace (cofactor) expansion of $\det A$ along the third column.

Solution. a) We have

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{vmatrix} = (-2)\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix} = (-2)(-1)1\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2(-1-1) = -4.$$

First we performed the the elementary row operation $E_{2,1}(-1)$, then we took the Laplace expansion in the second row, then we took the Laplace expansion in the second column, finally we calculated the 2×2 determinant.

A different method. We perform on A the elementary row operations $S_{1,3}$, $E_{4,1}(-1)$, $E_{3,2}(-1)$, $S_{3,4}$ to get the matrix $B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. This means that $S_{3,4}E_{3,2}(-1)E_{4,1}(-1)S_{1,3}A = B$.Since $\det S_{i,j} = -1$ and $\det E_{i,j}(a) = 1$ and det of a product is the product of det's, we see that $(-1) \cdot 1 \cdot 1 \cdot (-1) \det A = \det B$, i.e. $\det A = \det B$. Since B is upper triangular, $\det B = 1 \cdot 1 \cdot (-2) \cdot 2 = -4$. Thus $\det A = -4$.

b) The Laplace expansion in the third column is

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{vmatrix} = (-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} + (-1)^{2+3} \cdot 1 \cdot \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} + (-1)^{3+3} \cdot 1 \cdot \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} +$$

$$+(-1)^{4+3} \cdot (-1) \cdot \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

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3. a) (8 points) Find the characteristic polynomial and the eigenvalues of the matrix $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Is M diagonalizable?

   b) (8 points) The numbers 2 and 3 are the only eigenvalues of the matrix $B = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{bmatrix}$. Find bases of the corresponding eigenspaces. Is B diagonalizable? Explain your answer.

   c) (8 points) A 3×3 matrix C has eigenvectors $v_1 = (1, 1, 1)$ with eigenvalue 1, $v_2 = (1, 1, 0)$ with eigenvalue -1 and $v_3 = (1, 0, 0)$ with eigenvalue 0. It follows that C is diagonalizable, i.e. there exist a matrix P and a diagonal matrix D such that $C = PDP^{-1}$. Find P, D and then C.

Solution. a) The characteristic polynomial of $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ equals

$$p(\lambda) = \det(M - \lambda \mathrm{I}) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} = (-\lambda)^3 + 1 \cdot 1 \cdot 0 + 0 \cdot 1 \cdot 1 - 0 \cdot (-\lambda) \cdot 0 - 1 \cdot 1 \cdot (-\lambda) - (-\lambda) \cdot 1 \cdot 1 =$$

$$= -\lambda^3 + 2\lambda = \lambda(2 - \lambda^2) = \lambda(\sqrt{2} - \lambda)(\sqrt{2} + \lambda)$$

The eigenvalues of M are the roots of $p(\lambda)$ so they are 0, $\sqrt{2}$ and $-\sqrt{2}$. Since M has three distinct eigenvalues, M is diagonalizable (we proved that $n \times n$ matrix with n distinct eigenvalues is diagonalizable).

b) The numbers 2 and 3 are the only eigenvalues of the matrix $B = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{bmatrix}$.

The eigenspace corresponding to the eigenvalue 2 is the kernel of the matrix $B - 2\mathrm{I} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{bmatrix}$. In order to find a basis of this eigenspace we transform our matrix to its reduced row-echelon form by performing the row operations $E_{2,1}(1)$, $E_{3,1}(1)$, $E_{1,2}(1)$, $S_2(-1)$. The reduced row-echelon form is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$. The only free variable is the one corresponding to the third column, so we set $x_3 = 1$ and compute $x_1 = -1$ and $x_2 = 3$. Thus the eigenspace corresponding to 2 is one-dimensional with basis $(-1, 3, 1)$.

In order to find the eigenspace corresponding to the eigenvalue 3 we find the reduced row-echelon form of the matrix $B - 3\mathrm{I} = \begin{bmatrix} 0 & 1 & -2 \\ -1 & -3 & 5 \\ -1 & -1 & 1 \end{bmatrix}$ by performing the elementary row operations $T_{1,3}$, $E_{2,1}(-1)$, $S_1(-1)$, $S_2(-1/2)$, $E_{3,2}(-1)$, $E_{1,2}(-1)$. The reduced row-echelon form is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. The only free variable is the one corresponding to the third column, so we set $x_3 = 1$ and compute $x_1 = -1$ and $x_2 = 2$. Thus the eigenspace corresponding to 3 is also one-dimensional with basis $(-1, 2, 1)$.

Since the sum of the dimensions of all eigenspaces is less than 3, the matrix B is not diagonalizable.

c) A 3×3 matrix C has eigenvectors $v_1 = (1, 1, 1)$ with eigenvalue 1, $v_2 = (1, 1, 0)$ with eigenvalue -1 and $v_3 = (1, 0, 0)$ with eigenvalue 0. It follows that C is diagonalizable, i.e. there exist a matrix P and a diagonal matrix D such that $C = PDP^{-1}$. We know that we may take $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and P is the change of basis matrix from v_1, v_2, v_3 to the standard basis. Thus $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. We apply the algorithm for inverting matrices to P and find that $P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$. Thus

$$C = PDP^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

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4. a) (4 points) B is an upper triangular 5×5 matrix with diagonal entries $1, 2, -1, 1, 3$. We know that $E_{2,3}(-4)S_{2,3}D_2(3)E_{3,1}(-2)A = B$. Compute the determinant of A.
   b) (3 points) A matrix A satisfies $AA^t = I$. Prove that $\det A = \pm 1$.
   c) (4 points) S, T are linear transformations from V to V and v is an eigenvector for both S and T. Show that v is an eigenvector for ST.

Solution. a) Since B is upper-triangular, $\det B$ is the product of the diagonal entries of B, i.e. $\det B = 1 \cdot 2 \cdot (-1) \cdot 1 \cdot 3 = -6$. On the other hand,

$\det(E_{2,3}(-4)S_{2,3}D_2(3)E_{3,1}(-2)A) = \det(E_{2,3}(-4))\det(S_{2,3})\det(D_2(3))\det(E_{3,1}(-2))\det A =$

$$= 1 \cdot (-1) \cdot 3 \cdot 1 \cdot \det A = -3 \det A.$$

Thus $-3 \det A = \det B = -6$, so $\det A = 2$.

b) We have $1 = \det(I) = \det(AA^t) = \det A \det A^t = (\det A)^2$. Thus $(\det A)^2 = 1$, which means that $\det A = \pm 1$.

c) There are scalars a, b such that $S(v) = av$ and $T(v) = bv$. Thus

$$(ST)(v) = S(T(v)) = S(bv) = bS(v) = b(av) = (ab)v.$$

This shows that v is an eigenvector of ST (with eigenvalue ab).

5. (3 points each) Complete each definition. Make sure that you write a complete meaningful sentences containing all necessary conditions.

   a) λ is an eigenvalue of a linear transformation $T : V \longrightarrow V$ if there is a vector $v \neq 0$ in V such that $T(v) = \lambda v$.
   b) The characteristic polynomial $p(t)$ of a square matrix A is the determinant $p(t) = \det(A - tI)$.
   c) A square matrix A is diagonalizable if there is an invertible matrix M such that MAM^{-1} is diagonal. In other words, A is similar to a diagonal matrix.
   d) Two matrices A, B are similar if there is an invertible matrix M such that $B = MAM^{-1}$.
   e) w is an eigenvector of a linear transformation $T : V \longrightarrow V$ if $w \neq 0$ and $T(w) = aw$ for some scalar a.

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6. Answer true or false (2 points each).

   a) An invertible 4×4 matrix can have characteristic polynomial $x^4 - x^2 + 2x$.
   b) There are invertible 9×9 matrices A, B such that $AB = -BA$.
   c) Any matrix with characteristic polynomial $p(t) = t(t-1)(t+1)(t-2)$ is diagonalizable.
   d) If $v \in \mathbb{R}^n$ is an eigenvector of both matrices A and B then v is an eigenvector of $A+B$.
   e) There is a 5×5 matrix of rank 2 which has eigenvalue 2 and the corresponding eigenspace has dimension 3.
   f) If A is a 3×3 matrix then one of the matrices A, $A - I$, $A + 2I$, $A + I$ is invertible.
   g) If A, B are $n \times n$ matrices then $\det(A + B) = \det(A) + \det(B)$.
   h) If two 2×2 matrices have the same characteristic polynomial then they are similar.

Solution.

   a) FALSE. Since 0 is a root of the characteristic polynomial, 0 is an eigenvalue of our matrix A, hence $\det(A - 0 \cdot I) = \det A = 0$, i.e. A is not invertible.
   b) FALSE. Note that $\det(AB) = \det(A)\det(B)$ and $\det(-BA) = (-1)^9 \det(B)\det(A)$. Thus $\det(A)\det(B) = -\det(A)\det(B)$. If both A, B were invertible, we would get $1 = -1$, which is clearly wrong.
   c) TRUE. The matrix is a 4×4 matrix with 4 distinct eigenvalues, hence it is diagonalizable.
   d) TRUE. If $Av = av$ and $Bv = bv$ for some scalars a, b, then $(A + B)v = Av + Bv = av + bv = (a + b)v$ so v is an eigenvector for $A + B$ (with eigenvalue $a + b$).

e) FALSE. Since A has rank 2, the kernel of A has dimension 3. The kernel is an eigenspace corresponding to the eigenvalue 0. The sum of the dimensions of all the eigenspaces is at most 5, so the eigenspace corresponding to 2 has dimension at most 2.

f) TRUE. $A - aI$ is not invertible if and only if a is a root of the characteristic polynomial of A. The characteristic polynomial of A has at most 3 roots. Thus one of $0, 1, 2, -1$ is not an eigenvalue of A and then the corresponding matrix is invertible.

g) FALSE. Both matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ have determinant 0 but $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has determinant 1.

h) FALSE. Both matrices $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ have the same characteristic polynomial $p(t) = t^2$, but they are not similar (only the zero matrix is similar to the zero matrix).

The following problem is optional. You can earn 12 extra points if you solve it, but work on it only if you are done with all the other problems

7. A square matrix M is such that $M^2 = M$ (such matrices are called **idempotent**). We think of M as the linear transformation $L_M : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

a) Show that if λ is an eigenvalue of M then $\lambda = 0$ or $\lambda = 1$.

b) Show that the kernel of M is the eigenspace corresponding to 0 and the image of M is the eigenspace corresponding to 1.

c) Prove that M is diagonalizable.

Solution.

a) Suppose that λ is an eigenvalue of M. Thus there is $v \neq 0$ such that $Mv = \lambda v$. Applying M to this equality we get $M^2 v = M(\lambda v) = \lambda M v = \lambda^2 v$. But $M^2 = M$, so we conclude that $\lambda v = \lambda^2 v$. This gives $\lambda = \lambda^2$, since $v \neq 0$. It follows that $\lambda = 0$ or $\lambda = 1$.

b) By the very definition, the eigenspace corresponding to 0 is the kernel of $M - 0 \cdot \mathrm{I} = M$. Let U be the eigenspace corresponding to 1. If $v \in U$ then $Mv = v$, so $v \in \mathrm{image}(M)$. Conversely, if $v \in \mathrm{image}(M)$ then $v = Mw$ for some w. Thus $Mv = M(Mw) = M^2 w = Mw = v$, i.e. $v \in U$. This shows that $U = \mathrm{image}(M)$,

c) Recall that $\dim\ker M + \dim \mathrm{image}(M) = n$, where n is the size of M. This implies that the sum of the dimensions of the eigenspaces of M is equal to the size of M, hence M is diagonalizable.