Quizzes for Math 304

QUIZ 1. A system of linear equations has augmented matrix

$$A = \left[\begin{array}{rrrr:r} 2 & 4 & 1 & 1 & 4 \\ -1 & -2 & 0 & -1 & -1 \\ 2 & 4 & 3 & -1 & 5 \\ 1 & 2 & -1 & 1 & -1 \end{array}\right]$$

a) Write down this system of equations;

b) Find the reduced row-echelon form of A;

c) What are the pivot columns of A the rank of A?

d) what are the free variables of the system found in a).

e) Solve the system of equations found in a).

Solution. a) The system of linear equations with augmented matrix A is

$$\begin{aligned} 2x_1 + 4x_2 \quad + x_3 + 4x_4 &= 4 \\ -x_1 - 2x_2 \qquad\quad - \ x_4 &= -1 \\ 2x_1 + 4x_2 \quad +3x_3 - \ x_4 &= 5 \\ x_1 + 2x_2 \quad -x_3 + \ x_4 &= -1 \end{aligned}.$$

b) For convenience, we perform the operation $S_{1,4}$ to have 1 in the first entry of the first column (this way we avoid fractions). We get the matrix

$$\left[\begin{array}{rrrr:r} 1 & 2 & -1 & 1 & -1 \\ -1 & -2 & 0 & -1 & -1 \\ 2 & 4 & 3 & -1 & 5 \\ 2 & 4 & 1 & 1 & 4 \end{array}\right].$$

Now we perform operations $E_{2,1}(1)$, $E_{3,1}(-2)$, $E_{4,1}(-2)$ to get

$$\left[\begin{array}{cccc:c} 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 5 & -3 & 7 \\ 0 & 0 & 3 & -1 & 6 \end{array}\right].$$

Next we perform $E_{3,2}(5)$, $E_{4,2}(3)$ to get

$$\left[\begin{array}{cccc:c} 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right].$$

We perform $D_3(-1/3)$ and get

$$\left[\begin{array}{cccc:c} 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right].$$

The we perform $E_{4,3}(1)$ and get

$$\left[\begin{array}{cccc:c} 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

The last matrix is in a row-echelon form. Already at this stage we could read off the rank, free variables, and the number of solutions to the system. To get to a reduced row-echelon form we perform $E_{1,4}(1)$, $E_{2,4}(2)$, $E_{3,4}(-1)$ to get

$$\left[\begin{array}{cccc:c} 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Then we do $E_{1,3}(-1)$ to get

$$\left[\begin{array}{cccc:c} 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Next we do $D_2(-1)$ and then $E_{1,2}(1)$ to finally get the reduced row-echelon form

$$\left[\begin{array}{cccc:c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

c) We now see that the first, third, fourth, and fifth columns of A are pivot columns. It follows that the rank of A is 4.

d) The free variables correspond to the non-pivot columns, so x_2 is the only free variable.

e) Since the last column of A is a pivot column, the system is inconsistent, i.e. has no solutions.

QUIZ 2. a) Is the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix}$$

one-to-one? Explain your answer.

b) Is the matrix

$$B = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

onto? Is it one-to-one? Explain your answer.

c) State a definition of a linear transformation.

Solution. a) Recall that an $m \times n$ matrix A is one-to-one if and only if $\text{rank}(A) = n$. Recall also that rank is always smaller or equal than each m and n. In our case, $m = 3$, $n = 4$ so $\text{rank}(A) \leq 3$ and A is not one-to-one.

b) We need to compute the rank of B. We perform elementary row operations $E_{2,1}(-1)$, $E_{3,1}(-1)$, and $E_{4,1}(-1)$ to get

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

Now we perform $E_{3,2}(-1)$ and get

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

Finally, after performing $E_{4,3}(-1)$ we get a matrix in row-echelon form:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

We see that all rows are non-zero (all columns are pivot) so the rank of B is 4. Since B is a 4×4 matrix of rank 4, it is both one-to-one and onto.

c) A **linear transformation** from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ which satisfies the following conditions:

1. $L(u + w) = L(u) + L(w)$ for any vectors u, w in $\mathbb{R}^n$.
2. $L(cu) = cL(u)$ for any $u \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

QUIZ 3. a) Compute the product

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}.$$

Is C one-to-one? Answer this question without any computations.

b) The function $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $T(a, b, c) = (a-b+c, 2a+c)$ is a linear transformation. What is the matrix of T, i.e what is the matrix A such that $T = L_A$?

c) Write down the 4×4 matrix $E_{2,3}(-2)$.

d) Find the inverse of the matrix $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$.

Solution. a) We have

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 3 \\ -2 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}.$$

The matrix C is not one-to-one. To justify this, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}$. Thus $C = AB$ and therefore $L_C = L_A \circ L_B$. Now $L_B : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is not one-to-one (since $3 > 2$). This means that $L_B(u) = L_B(w)$ for two different vectors u, w. It follows that $L_C(u) = L_A(L_B(u)) = L_A(L_B(w)) = L_C(w)$, so L_C is not one-to-one.

b) Recall that if A is an $m \times n$ matrix then the first column of A is $L_A(e_1)$, the second column of A is $L_A(e_2)$, etc. In our case, $T(e_1) = T(1,0,0) = (1,2)$, $T(e_2) = T(0,1,0) = (-1,0)$, and $T(e_3) = T(0,0,1) = (1,1)$. Thus the matrix of T is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

c)The 4×4 matrix $E_{2,3}(-2)$ is

$$E_{2,3}(-2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

d) We need to find the matrix in reduced row-echelon form row equivalent to the matrix

$$\left[\begin{array}{cc:cc} 3 & 4 & 1 & 0 \\ 5 & 7 & 0 & 1 \end{array}\right].$$

We do $D_1(1/3)$ to get

$$\left[\begin{array}{cc:cc} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 5 & 7 & 0 & 1 \end{array}\right].$$

Next we do $E_{2,1}(-5)$ and get

$$\left[\begin{array}{cc:cc} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{-5}{3} & 1 \end{array}\right].$$

Now we do $D_2(3)$ and get

$$\left[\begin{array}{cc:cc} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -5 & 3 \end{array}\right].$$

Finally, we do $E_{1,2}(-4/3)$ and get

$$\left[\begin{array}{cc:cc} 1 & 0 & 7 & -4 \\ 0 & 1 & -5 & 3 \end{array}\right].$$

Thus

$$\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}.$$

We can also conclude that

$$\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix} = E_{1,2}(-4/3)D_2(3)E_{2,1}(-5)D_1(1/3)$$

and

$$\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} = (E_{1,2}(-4/3)D_2(3)E_{2,1}(-5)D_1(1/3))^{-1} = D_1(3)E_{2,1}(5)D_2(1/3)E_{1,2}(4/3).$$

QUIZ 4. Let $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 1 \\ 5 & 3 & 2 \end{bmatrix}$.

a) Find A^{-1}.

b) Express A as a product of elementary matrices.

c) Suppose that B is a matrix such that $AB = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Is B invertible? Explain your answer.

Solution. a) We start with the matrix

$$\left[\begin{array}{ccc:ccc} 3 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 5 & 3 & 2 & 0 & 0 & 1 \end{array}\right].$$

Do $E_{1,2}(-1)$ to get

$$\left[\begin{array}{ccc:ccc} 1 & 1 & -1 & 1 & -1 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 5 & 3 & 2 & 0 & 0 & 1 \end{array}\right].$$

Then do $E_{2,1}(-2)$, $E_{3,1}(-5)$ and get

$$\left[\begin{array}{ccc:ccc} 1 & 1 & -1 & 1 & -1 & 0 \\ 0 & -1 & 3 & -2 & 3 & 0 \\ 0 & -2 & 7 & -5 & 5 & 1 \end{array}\right].$$

Now do $E_{1,2}(1)$, $E_{3,2}(-2)$ which results in

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 2 & -1 & 2 & 0 \\ 0 & -1 & 3 & -2 & 3 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right].$$

Next do $E_{2,3}(-3)$, $E_{1,3}(-2)$ and get

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 4 & -2 \\ 0 & -1 & 0 & 1 & 6 & -3 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right].$$

Finally, do $D_2(-1)$ and get

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 4 & -2 \\ 0 & 1 & 0 & -1 & -6 & 3 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right].$$

Thus

$$A^{-1} = \begin{bmatrix} 1 & 4 & -2 \\ -1 & -6 & 3 \\ -1 & -1 & 1 \end{bmatrix}.$$

b) From part a) we have

$$A^{-1} = D_2(-1)E_{2,3}(-3)E_{1,3}(-2)E_{1,2}(1)E_{3,2}(-2)E_{2,1}(-2)E_{3,1}(-5)E_{1,2}(-1).$$

Thus

$$A = (A^{-1})^{-1} = E_{1,2}(1)E_{3,1}(5)E_{2,1}(2)E_{3,2}(2)E_{1,2}(-1)E_{1,3}(2)E_{2,3}(3)D_2(-1).$$

c) We know from a) that A is invertible. If B were invertible too, then AB would be invertible as well. However, AB is a 3×3 matrix of rank 2 (note that it is in reduced row-echelon form), so it is not invertible. Thus B is not invertible.

QUIZ 5. a) Define the kernel of a liner transformation $T : V \longrightarrow W$.

b) Finish the sentence: The vectors $v_1, ..., v_n$ are linearly independent if and only if ... .

c) Let $v_1 = (1, 1, 1, 1)$, $v_2 = (0, 3, 1, 3)$, $v_3 = (-1, 2, 0, 3)$.

1. Is $v = (-1, 8, 2, 11)$ in the span of v_1, v_2, v_3? If yes, express v as a linear combination of v_1, v_2, v_3.

2. Are v_1, v_2, v_3 linearly independent?

Solution. a) The **kernel** $\ker(T)$ of the linear transformation $T : V \longrightarrow W$ is the set $\{v \in V : T(v) = 0\}$, i.e. the set of all vectors in V which are mapped to 0 by T. The kernel is a subspace of V.

b) The vectors $v_1, ..., v_n$ are **linearly independent** if and only if none of them is 0 and none is a linear combination of the other vectors.

Equivalently:

The vectors $v_1, ..., v_n$ are **linearly independent** if and only if the only numbers $a_1, \ldots, a_n$ such that $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$ are $a_1 = a_2 = \ldots = a_n = 0$.

c) The first question asks if there are solutions to $x_1 v_1 + x_2 v_2 + x_3 v_3 = v$. Second question asks if there is a non-trivial solution to $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$. We answer both questions by finding reduced row echelon form row-equivalent to the matrix

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & -1 \\ 1 & 3 & 2 & 8 \\ 1 & 1 & 0 & 2 \\ 1 & 3 & 3 & 11 \end{array}\right].$$

Do $E_{2,1}(-1)$, $E_{3,1}(-1)$, $E_{4,1}(-1)$ to get

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & -1 \\ 0 & 3 & 3 & 9 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 4 & 12 \end{array}\right].$$

Then do $S_{2,3}$ followed by $E_{3,2}(-3)$ and $E_{4,2}(-3)$ and get

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Finally, do $S_{3,4}$ followed by $E_{2,3}(-1)$ and $E_{1,3}(1)$ to get

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We see that the corresponding system of linear equations is consistent (the last column is not pivot) and has unique solution $x_1 = 2$, $x_2 = 0$, $x_3 = 3$. This means that $v = 2v_1 + 0 \cdot v_2 + 3v_3$ is in the span of v_1, v_2, v_3.

For the second question, we ask if the associated homogeneous system has non-trivial solutions. Since all the columns of the coefficient matrix are pivot, the homogeneous system has only trivial solution, so v_1, v_2, v_3 are linearly independent.

QUIZ 6. a) What does it mean that a vector space V is finite dimensional?

b) Define a basis of a finite dimensional vector space?

c) Let $v_1 = (1, 2, 1, 1)$, $v_2 = (2, 4, 2, 2)$, $v_3 = (0, 1, -1, 2)$, $v_4 = (1, 4 - 1, 5)$. Let $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be the linear transformation such that $T(e_1) = v_1$, $T(e_2) = v_2$, $T(e_3) = v_3$, $T(e_4) = v_4$.

1. Is $(1, 1, 1, 1)$ in the image of T?
2. Find a basis of the image of T and express each v_i as a linear combination of the vectors in this basis.
3. Find a basis of the kernel of T.

Solution. a) A vector space V is finite dimensional if $V = \text{span}\{v_1, \ldots, v_n\}$ for some finite sequence $v_1, \ldots, v_n$ of vectors in V.

b) A basis of a finite dimensional vector space V is a sequence $v_1, \ldots, v_n$ of vectors in V which is linearly independent and such that $V = \text{span}\{v_1, \ldots, v_n\}$.

c) The matrix of T is

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 1 & 2 & -1 & -1 \\ 1 & 2 & 2 & 5 \end{bmatrix}.$$

We start with the matrix

$$\left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 4 & 1 \\ 1 & 2 & -1 & -1 & 1 \\ 1 & 2 & 2 & 5 & 1 \end{array}\right]$$

and perform elementary row operations to bring the part to the left of the dividing line into reduced row-echelon form. We get

$$\left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right].$$

Since the last column is a pivot column, we conclude that $(1,1,1,1)$ is not in the image of T. This answers part 1.

We see that the pivot columns to the left of the dividing line are columns one and three. Thus v_1, v_3 is a basis of the image of T. Moreover, from the reduced row-echelon form of Awe see that $v_2 = 2v_1$ and $v_4 = v_1 + 2v_3$. This answers part 2.

The homogeneous system $Ax = 0$ has free varaibles x_2 and x_4. The solution corresponsing to $x_2 = 1$, $x_4 = 0$ is $(-2,1,0,0)$. The solution corrsponding to $x_2 = 0$, $x_4 = 1$ is $(-1,0,-2,1)$. Thus $(-2,1,0,0), (-1,0,-2,1)$ is a basis of the kernel of T.

QUIZ 7. a) Find the transition matrix from the basis $v_1 = (1,1,0)$, $v_2 = (1,0,1)$, $v_3 = (0,1,1)$ to the basis $u_1 = (1,0,0)$, $u_2 = (1,1,0)$, $u_3 = (1,1,1)$ of $\mathbb{R}^3$.

b) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is given by $T(a,b) = (a + 2b, -a, 0)$. What is the matrix of T in bases e_1, e_2 of $\mathbb{R}^2$ and $v_1 = (1,1,0)$, $v_2 = (1,0,1)$, $v_3 = (0,1,1)$ of $\mathbb{R}^3$.

Solution. a) We need to express each vector of the first basis as a linear combination of the vectors in the second basis. We start with the matrix whose columns are the vectors of the second basis followed by the vectors in the first basis:

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right].$$

We find the reduced row-echelon form of this matrix, which is the matrix

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right].$$

From this matrix we see that $v_1 = u_2$, $v_2 = u_1 - u_2 + u_3$, $v_3 = -u_1 + u_3$. The transition matrix from the basis v to the basis u is

$$_uI_v = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

b) We need to express each vector $T(e_1), T(e_2)$ as a linear combination of the vectors v_1, v_2, v_3. We have

$$T(e_1) = T(1,0) = (1,-1,0) \text{ and } T(e_2) = T(0,1) = (2,0,0).$$

We start with the matrix

$$\left[\begin{array}{ccc:cc} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

The reduced row-echelon form of this matrix is

$$\left[\begin{array}{ccc:cc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right].$$

This tells us that $T(e_1) = v_2 - v_3$ and $T(e_2) = v_1 + v_2 - v_3$. Thus, the matrix of T in the bases e and v is

$$_vT_e = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

QUIZ 8. a) Define an eigenvalue of a linear transformation $T : V \longrightarrow V$.

b) The matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{bmatrix}$$

has eigenvalues -1 and 1.

1. Show that A is diagonalizable.

2. Find a matrix B such that $B^{-1}AB$ is diagonal.

Solution. a) A scalar λ is called an **eigenvalue** of the linear transformation $T : V \longrightarrow V$ if there is a **non-zero** vector $v \in V$ such that $T(v) = \lambda v$. Any such v is called an **eigenvector** of T corresponding to the eigenvalue λ. In other words, an **eigenvector** of T is a non-zero vector v such that $T(v) = \lambda v$ for some scalar λ.

b) We need to find the dimension and a basis of the eigenspaces corresponding to each eigenvalue.

The eigenspace of a matrix A corrsponding to an eigenvalue λ is the space of all solutions to the homogeneous system of linear equations: $(A - \lambda I)x = 0$.

1. The eigenspace corresponding to -1 is the space of solutions to

$$(A+I)x = \left(\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The reduced row echelon form of $\begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ 2 & 2 & -2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This matrix has rank 1, so our system has $3 - 1 = 2$-dimensional space of solutions. x_2 and x_3 are the free varaibles and the vectors $v_1 = (-1, 1, 0)$, $v_2 = (1, 0, 1)$ form a basis of the eigenspace corresponding to -1.

2. The eigenspace corresponding to 1 is the space of solutions to

$$(A-I)x = \left(\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ 2 & 0 & -2 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The reduced row echelon form of $\begin{bmatrix} 0 & 2 & -2 \\ 2 & 0 & -2 \\ 2 & 2 & -4 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. This matrix has rank 2, so our system has $3 - 2 = 1$-dimensional space of solutions. x_3 is the free varaible and the vector $v_3 = (1, 1, 1)$ forms a basis of the eigenspace corresponding to 1.

According to the results from class, vectors v_1, v_2, v_3 are linearly independent, hence form a basis of $\mathbb{R}^3$. This basis consists of eigenvectors of A, so A is diagonalizable. This solves part 1.

For part 2., we can take B to be the transition matrix from the basis v_1, v_2, v_3 to the standard basis. Then $B^{-1}AB = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (note that $-1, -1, 1$ are the eigenvalues corresponding to the vectors v_1, v_2, v_3 respectively).

We have

$$B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and a simple computations yields

$$B^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}.$$

We noe easily check that indeed

$$B^{-1}AB = \begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

QUIZ 9. a) Define the characteristic polynomial of a matrix A.

b) Compute the determinant of the matrix $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & 1 & 0 & 1 \\ 4 & 2 & 0 & 1 \end{bmatrix}$.

c) Find all eigenvalues of the matrix $\begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$.

Solution. a) The characteristic polynomial $p_A(t)$ of a matrix A is defined as $p_A(t) = \det(A - tI)$. It is a polynomial of degree n with leading coefficient $(-1)^n$, where n is the size of A.

b) Note that the third column of our matrix has only one non-zero entry. This suggests to start with the Laplace expansion by the third column:

$$\begin{vmatrix} 1 & 2 & 0 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & 1 & 0 & 1 \\ 4 & 2 & 0 & 1 \end{vmatrix} = (-1) \cdot 2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix}.$$

To compute the 3×3 determinant we first do the row operations $E_{2,1}(1)$, $E_{3,1}(1)$

and then do Laplace expansion by the third column:

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 4 & 3 & 0 \\ 5 & 4 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 4 & 3 \\ 5 & 4 \end{vmatrix} = -(16-15) = -1.$$

Thus the original determinant is equal to $-2 \cdot (-1) = 2$.

c) First we compute the characteristic polynomial of our matrix which is

$$p(t) = \begin{vmatrix} -t & 0 & 2 \\ 1 & -t & 2 \\ 0 & 1 & -1-t \end{vmatrix} = (-t)\begin{vmatrix} -t & 2 \\ 1 & -1-t \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2 \\ 1 & -1-t \end{vmatrix} =$$

$$= (-t)(t^2+t-2) - (-2) = -t^3 - t^2 + 2t + 2 = -(t+1)(t^2-2).$$

We see that the roots of $p(t)$ are $-1, \sqrt{2}, -\sqrt{2}$, so these are the eigenvalues of our matrix.

QUIZ 10. a) Define an inner product on a vector space V.

b) Consider the vector space $\mathbb{R}^4$ with the dort product as the inner product.

1. What is the angle between vectors $(1,1,0,0)$ and $(1,1,1,1)$?

2. Let $v_1 = (1,1,0,0)$, $v_2 = (1,1,1,0)$, $v_3 = (1,1,2,1)$. Use Gramm-Scmidt orthogonalization process to fina an orthogonal basis of span$\{v_1, v_2, v_3\}$.

Solution. a) An **inner product** on a vector space V is a function $V \times V \longrightarrow \mathbb{R}$ which to any pair of vectors u, w of V assigns a real number, denoted by $< u, w >$, such that the following conditions are satisfied:

1. $< u, w > = < w, u >$ for any two vectors $u, w \in V$.

2. $< au_1 + bu_2, w > = a < u_1, w > + b < u_2, w >$ for any vectors $u_1, u_2, w \in V$ and any numbers a, b.

3. $< u, u >$ is positive (i.e. $< u, u > > 0$) for any **non-zero** vector $u \in V$.

b) Recall that the **angle** $\angle(u,w)$ between two vectors $u, w \in V$ is the unique angle in the interval $[0,\pi]$ such that $\cos\angle(u,w) = \dfrac{<u,w>}{||u||\cdot||w||}$.

Since in our problem the inner product is the dot product, we have

$$<(1,1,0,0),(1,1,1,1)>=(1,1,0,0)\cdot(1,1,1,1)=2,$$

$$||(1,1,0,0)|| = \sqrt{<(1,1,0,0),(1,1,0,0)>} = \sqrt{(1,1,0,0)\cdot(1,1,0,0)} = \sqrt{2},$$

$$||(1,1,1,1)|| = \sqrt{<(1,1,1,1),(1,1,1,1)>} = \sqrt{(1,1,1,1)\cdot(1,1,1,1)} = \sqrt{4} = 2$$

Thus $\cos\angle(u,w) = 2/(2\cdot\sqrt{2}) = \sqrt{2}/2$, so $\angle(u,w) = \pi/4$.

c) Recall the Gramm-Schimidt orthogonalization process.

Gramm-Schmidt orthogonalization process. Let $<\ ,\ >$ be an inner product on a vector space V and let $v_1,\ldots,v_k$ be linearly independent. Define recursively vectors $w_1,\ldots,w_k$ as follows:

$$w_1 = v_1,\ \ w_{j+1} = v_{j+1} - \left(\frac{<v_{j+1},w_1>}{<w_1,w_1>}w_1 + \frac{<v_{j+1},w_2>}{<w_2,w_2>}w_2 + \ldots + \frac{<v_{j+1},w_j>}{<w_j,w_j>}w_j\right).$$

Then

1. $w_1, w_2, \ldots, w_k$ is orthogonal.
2. $\text{span}\{w_1,\ldots,w_j\} = \text{span}\{v_1,\ldots,v_j\}$ for $j = 1,\ldots,k$.

Let us apply the Gramm-Schmidt orthogonalization process to the vectors v_1, v_2, v_3. We have

<u>step 1:</u>

$$w_1 = v_1 = (1,1,0,0) \quad \text{and} \quad <w_1,w_1>=(1,1,0,0)\cdot(1,1,0,0)=2,$$

<u>step 2:</u>

$$<v_2,w_1>=(1,1,1,0)\cdot(1,1,0,0)=2$$

so

$$w_2 = v_2 - \frac{<v_2,w_1>}{<w_1,w_1>}w_1 = (1,1,1,0) - \frac{2}{2}(1,1,0,0) = (0,0,1,0),$$

and

$$<w_2,w_2>=(0,0,1,0)\cdot(0,0,1,0)=1.$$

step 3:

$$< v_3, w_1 >= (1,1,2,1)\cdot(1,1,0,0) = 2, \quad < v_3, w_2 >= (1,1,2,1)\cdot(0,0,1,0) = 2$$

so

$$w_3 = v_3 - \frac{< v_3, w_1 >}{< w_1, w_1 >} w_1 - \frac{< v_3, w_2 >}{< w_2, w_2 >} w_2 = (1,1,2,1) - \frac{2}{2}(1,1,0,0) - \frac{2}{1}(0,0,1,0) = (0,0,0,1).$$

We see that w_1, w_2, w_3 is an orthogonal basis of $\mathrm{span}\{v_1, v_2, v_3\}$(verify this!).

QUIZ 11. Consider $\mathbb{R}^4$ with the dot product as an inner product. Let $v_1 = (1,1,1,1)$, $v_2 = (1,2,3,2)$, $W = \mathrm{span}\{v_1, v_2\}$.

1. Find the orthogonal projection of $v = (1,3,5,7)$ onto W.
2. What is the orthogonal projection of v onto $W^\perp$ (THINK!)?
3. Find a basis of $W^\perp$

Solution. 1. First we need to find an orthogonal basis of W. We apply Gram-Schmidt process to v_1, v_2:

$$w_1 = v_1 = (1,1,1,1), \ < w_1, w_1 >= v_1 \cdot v_1 = 4,$$

$$w_2 = v_2 - \frac{< v_2, w_1 >}{< w_1, w_1 >} w_1 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = (1,2,3,2) - \frac{8}{4}(1,1,1,1) = (-1,0,1,0).$$

Once we have an othogonal basis w_1, w_2 of W, we can apply the formula for the projection $P_W(v)$ of v onto W:

$$P_W(v) = \frac{< v, w_1 >}{< w_1, w_1 >} w_1 + \frac{< v, w_2 >}{< w_2, w_2 >} w_2 = \frac{v \cdot w_1}{w_1 \cdot w_1} w_1 + \frac{v \cdot w_2}{w_2 \cdot w_2} w_2 =$$

$$= \frac{16}{4}(1,1,1,1) + \frac{4}{2}(-1,0,1,0) = (2,4,6,4).$$

2. Recall, that if W is a subspace of an inner-product space V and $v \in V$ then $v = w + u$ for unique $w \in W$ and $u \in W^\perp$. In other words, $v = P_W(v) + P_{W^\perp}(v)$. Thus

$$P_{W^\perp}(v) = v - P_W(v) = (1,3,5,7) - (2,4,6,4) = (-1,-1,-1,3).$$

3. Recall that a vector $x = (x_1, x_2, x_3, x_4)$ belongs to $W^{\perp}$ iff it is orthogonal to every vector in W, which is equivalent to beeing orthogonal to every vector of some spanning set of W. In our case, $x \in W^{\perp}$ if and only if $x \cdot v_1 = 0 = x \cdot v_2$. In other words, $W^{\perp}$ is the set of solutions to the system $x \cdot v_1 = 0,\ x \cdot v_2 = 0$, which is a homogeneous system of linear equations:

$$x \cdot v_1 = x_1 + x_2 + x_3 + x_4 = 0, \quad x \cdot v_2 = x_1 + 2x_2 + 3x_3 + 2x_4 = 0.$$

The coefficient matrix of this system is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \end{bmatrix}$ which has the reduced row-echelon form $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$. There are two free variables x_3, x_4 and a basis of solutions is $u_1 = (1, -2, 1, 0)$, $u_2 = (0, -1, 0, 1)$. Thus u_1, u_2 is a basis of $W^{\perp}$.