MATH 304 - Linear Algebra, Section 6 // Exam 2

YOU MUST SHOW ALL WORK AND YOU MUST WRITE COMPLETE SENTENCES EXPLAINING WHAT YOU ARE DOING TO GET FULL CREDIT.

AT THE END OF THE EXAM THERE ARE SOME MATRICES LISTED TOGETHER WITH THEIR REDUCED ROW-ECHELON FORMS. YOU MAY USE THIS WHEN SOLVING SOME OF THE PROBLEMS BELOW. IN YOUR SOLUTIONS EXPLAIN WHICH MATRIX YOU ARE USING AND WHAT IS ITS REDUCED ROW-ECHELON FORM.

1. (4 points each) Complete each definition. Make sure that you write a complete meaningful sentences containing all necessary conditions.

   a) The span $\text{span}\{w_1, \ldots, w_n\}$ is ... .

   b) A function $T : U \longrightarrow W$ between vector spaces is called a linear transformation if ... .

   c) $u_1, u_2, \ldots, u_m$ is a basis of a vector space V if ... .

   d) The kernel (null space) of a linear transformation $L : W \longrightarrow V$ is ... .

   e) Vectors $w_1, \ldots, w_k$ are linearly independent if ....

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2. (20 points) Let $w_1 = (1, -1, 0, 2)$, $w_2 = (1, 0, 1, 0)$, $w_3 = (2, -5, -2, 7)$, $w_4 = (1, 1, 1, 1)$, $w_5 = (1, -3, -2, 6)$. Among these vectors find a basis of $\text{span}\{w_1, w_2, w_3, w_4, w_5\}$. Express each vector w_i as a linear combination of vectors in this basis. What is the dimension of $\text{span}\{w_1, w_2, w_3, w_4, w_5\}$?

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3. A linear transformation $L_A : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ has matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & -10 & -6 \\ 2 & -3 & 4 & 12 & 14 \\ 3 & -4 & 6 & 13 & 17 \\ 5 & -6 & 10 & 15 & 23 \end{bmatrix}$$

   a) (10 points) Find a basis of the image of L_A.

   b) (10 points) Find a basis of the kernel of L_A. Verify that the vectors you found are indeed in the kernel.

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4. (14 points) Let $u_1 = (0, 1, 0, 1)$, $u_2 = (1, -1, -1, 1)$, $u_3 = (-1, 1, 1, 0)$ and $U = \text{span}\{u_1, u_2, u_3\}$. Furthermore, let $v = (2, 1, 0, 4)$ $w = (1, 1, -1, 4)$.

   a) Does v belong to U? If yes, express it as a linear combination of u_1, u_2, u_3. If not, explain why.

   b) Does w belong to U? If yes, express it as a linear combination of u_1, u_2, u_3. If not, explain why.

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5. (10 points) Let $L_A : \mathbb{R}^9 \longrightarrow \mathbb{R}^9$ be a linear transformation given by a matrix A and such that the composition $L_A \circ L_A$ is the zero map (i.e. maps every vector of $\mathbb{R}^9$ to 0).

   a) Show that every vector v in the image of L_A belongs to the the kernel $\ker(L_A)$ of L_A.

   b) Can both the kernel of L_A and the image of L_A have dimension bigger than 4? Carefully justify your answer.

   c) Show that the rank of the matrix of A does not exceed 4.

6. Answer true or false (2 points each).

   a) Any 8 vectors which span $\mathbb{R}^8$ are linearly independent.

   b) There exist a 6×4 matrix M and a 4×7 matrix N such that MN has rank 5.

   c) The collection of all non-pivot columns of a matrix is always linearly dependent.

   d) There are 8 linearly independent vectors in $\mathbb{R}^7$.

   e) There is a linear transformation $L : \mathbb{R}^7 \longrightarrow \mathbb{R}^7$ such that the kernel of L is equal to the image of L.

   f) $\operatorname{rank}(M^T) + \operatorname{rank}(M)$ is even for every matrix M.

   g) The dimension of the row space of a matrix B is equal to the number of pivot columns of B.

   h) There is a 7×8 matrix A which has 4 pivot columns and such that every 4 rows of A are linearly dependent.

The following problem is optional. You can earn 10 extra points if you solve it, but work on it only if you are done with all the other problems

7. Let $T : V \longrightarrow V$ be a linear transformation. Suppose v is a vector in V such that $w = T(v) \neq 0$ but $T(w) = 0$. Show that v and w are linearly independent.

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ -1 & 0 & -5 & 1 & -3 \\ 0 & 1 & -2 & 1 & -2 \\ 2 & 0 & 7 & 1 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 4 & 3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \qquad \begin{bmatrix} 1 & 0 & 2 & -10 & -6 \\ 2 & -3 & 4 & 12 & 14 \\ 3 & -4 & 6 & 13 & 17 \\ 5 & -6 & 10 & 15 & 23 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 & 2 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$